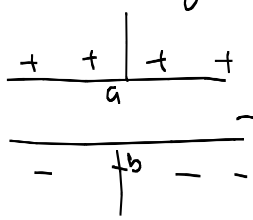
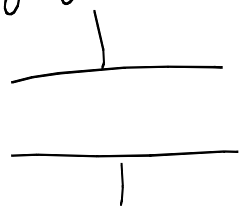
Note Title 10/17/2012

Capacitance and Dielectrics

- Two conductors seperated by an insulator or vacuum form a capacitor

Initially zero charge ⇒ then you put − charges on to one of them (charging it) ⇒ the other conductor gets + charge.





→ Remember $\frac{\sigma}{\varepsilon_0} = E$

∴ V_{ab} depends on σ

If we double the charge Q ⇒ σ ⇒ doubled ⇒ V_{ab} is doubled.

∴ $\frac{Q}{V_{ab}}$ is const ⇒ we call this ratio capacitance

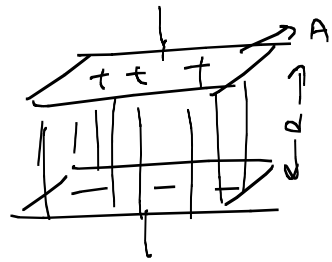
$$\boxed{C = \frac{Q}{V_{ab}}}$$ ⇒ unit of this is called Farad

$$1 \text{ Farad} = \frac{1\,C}{1\,V}$$

* Greater C means greater Q for given V.

∴ Capacitance is a measure of the ability of a capacitor to store el. energy.

Calculating Capacitance:



Remember that $E = \frac{\sigma}{\varepsilon_0}$

Using Gauss' Law.

$\sigma \Rightarrow$ surface charge density $\frac{Q}{A} = \sigma$

$V = E \cdot d \Rightarrow |E| = \frac{V}{d} = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0} = \frac{V}{d}$

$\therefore \quad V = Q\frac{d}{A\varepsilon_0} \Rightarrow \frac{Q}{V} = C = \varepsilon_0 \frac{A}{d}$

Depends on the area A, the distance between plates d, and the $\varepsilon_0 \Rightarrow$ the const. for vacuum.

Makes sense since bigger $A \Rightarrow$ easier to store charge and bigger $d \Rightarrow$ harder to control the storing of charges.

Example: Calculate, C, Q, E for a parallel plate capacitor of $d = 3\,mm$, $A = 6\,m^2$ and $V = 50\,kV$.

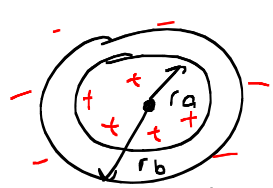
$$C = \varepsilon_0 \frac{A}{d} = \frac{8.85 \times 10^{-12} \times \cancel{6}^{\,2}}{\cancel{3} \times 10^{-3}} = 17.7 \times 10^{-9}\,F$$

$\Rightarrow C = 17.7\,nF$

$$Q = CV = 17.7 \times 10^{-9} \times 50 \times 10^{3} = 8.85 \times 10^{-4}\,C.$$

$$E = \frac{\sigma}{\varepsilon_0} \text{ or } E = V/d = 50 \times \cancel{10^3} / 3 \times \cancel{10^{-3}} = 16.6 \times 10^{6}\,\frac{N}{C}$$

A Spherical Capacitance:



First calculate the E-field between then $V \Rightarrow$ capacitance

Choose a gaussian surface between the shells

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q}{\varepsilon_0} \Rightarrow$$ a sphere between r_a, r_b

$$\Rightarrow E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}$$; $|Q|$ is same inside and outside

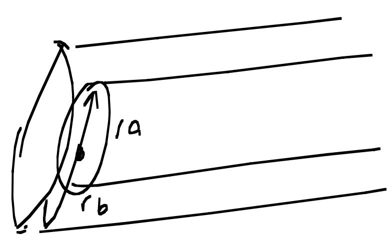
$$\Rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2} \Rightarrow V_{ab} = V_a - V_b$$

$$E \cdot r = V = \frac{Q}{4\pi\varepsilon_0 r} \Rightarrow V_{ab} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$$\frac{Q}{V_{ab}} = C = 4\pi\varepsilon_0\left(\frac{1}{r_a} - \frac{1}{r_b}\right)^{-1} = 4\pi\varepsilon_0\left(\frac{r_a r_b}{r_b - r_a}\right)$$

* This is related to $C = \varepsilon_0 \frac{A}{d}$ since $4\pi r_a r_b$ is like Area and d is distance between r_b and r_a.

Cylindrical Conductors: (Long)



Between these two shells same as a long (charged) rod

$$V = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r_0}{r}$$ where $V(r_0) = 0$ for some $r_0 \neq \infty$

$$\ln \frac{r_0}{r_0} = \ln 1 = 0$$

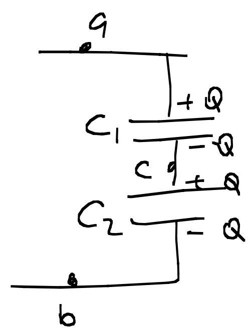
$$V_{ab} = V_a - V_b = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_b}{r_a}\right)$$

$$\frac{Q}{V_{ab}} = C = \lambda L \Big/ \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r_b}{r_a}$$

$$\therefore \; C = 2\pi\varepsilon_0 L \Big/ \ln \frac{r_b}{r_a} \Rightarrow$$ Capacitance per unit L

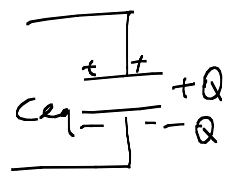
$$\rightarrow \frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln(r_b/r_a)}$$

Capacitance in Series Connection



} since 0 net charge

Same charge stored in both (must be)



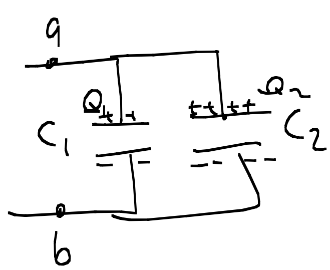
$$V_{ab} = V_{ac} + V_{cb} = \frac{Q}{C_1} + \frac{Q}{C_2} = \frac{Q}{C_{eq}}$$

$$\therefore \quad \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

* In series connection $\quad \dfrac{1}{C_{eq}} = \dfrac{1}{C_1} + \dfrac{1}{C_2} + \cdots$

$V_{eq} = V_1 + V_2 + \cdots \quad$; Q is same for each.

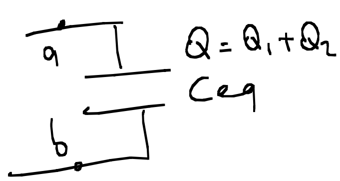
Capacitors In Paralel:



V_{ab} is same for both

$$\therefore \quad V_{C_1} = V_{ab} = \frac{Q_1}{C_1}$$

$$V_{C_2} = V_{ab} = \frac{Q_2}{C_2}$$



$$V_{ab} = \frac{Q}{C_{eq}}$$

$$\Rightarrow \quad Q = Q_1 + Q_2 = V_{ab} C_1 + V_{ab} C_2 = V_{ab} C_{eq}$$

$\therefore \quad C_{eq} = C_1 + C_2$

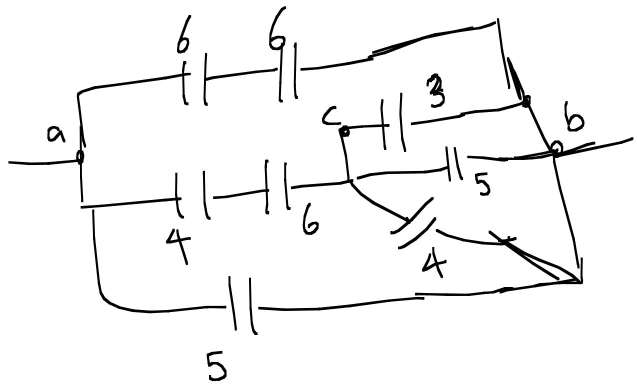
In paralel connection

$Q_T = Q_1 + Q_2 + \cdots$

V is same.

$C_{eq} = C_1 + C_2 + \cdots$

One can make various combinations of these connections



$C_{ob} = ?$

$3 + 4 + 5 = 12$

$\frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2}$

$\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$; $3 + 5 + 2 = 10$.

Then one can go ahead and ask for other things like If $V_{ab} = 30\,V$ $V_{cb} = ?$ ...

$Q = C \cdot V = 2 * 30 = 60$
for $4, 6, 12 = 2$

$Q_{cb} = C_{cb} V_{cb} = 12 * V_{cb} = 60$

$V = 5\,V.$

Energy Stored in a Capacitor

$Q = CV$ for a capacitor fully charged.

At an intermediate stage $V = \frac{q}{C}$

$dW = V\,dq$: remember $\frac{F}{q} = E$; $\frac{W}{q} = V$

for small charge dq $\Rightarrow$ Total work $= \int dW = \int_0^Q V\,dq$

$\Rightarrow W = \int_0^Q \frac{q\,dq}{C} = \frac{Q^2}{2C}$ work done by us.

$\therefore$ Potential energy stored inside a capacitor is $\Rightarrow U = \frac{Q^2}{2C}$

or in terms of other quantities $\Rightarrow$ $U = \frac{1}{2}CV^2 = \frac{1}{2}QV$

It's like the spring $\frac{1}{2}kx^2 \Rightarrow$ $x \sim Q$ $\quad k = \frac{1}{C}$

k is a property of the metal spring (length, thickness...)

C is a property of the conductor (area, d etc...)

* For the parallel plates we can define electrical energy density $u = \frac{U}{V}$ $\to$ volume

$$\Rightarrow u = \frac{1}{2}CV^2 \frac{1}{Ad} = \frac{1}{2}\varepsilon_0 \frac{\cancel{A}}{d} * V^2 \frac{1}{\cancel{A}d}$$

$$\Rightarrow \text{also } V = Ed \Rightarrow u = \frac{1}{2}\varepsilon_0 \frac{(Ed)^2}{d^2} \Rightarrow \boxed{u = \frac{1}{2}\varepsilon_0 E^2}$$

$\Rightarrow$ This is electric energy density in vacuum. And although this looks like a specific characteristic of parallel plates it's true for vacuum in general.

Spherical Capacitor, Again:

$$\Rightarrow C = 4\pi\varepsilon_0 \frac{r_a r_b}{r_b - r_a} \Rightarrow U = \frac{Q^2}{2C} = \frac{Q^2}{8\pi\varepsilon_0} \frac{r_b - r_a}{r_a r_b}$$

also using u (energy density) we can get it.

$$E_{\text{between}} = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$u = \frac{1}{2}\varepsilon_0 E^2$ and it's a func. of r

$U = uV$ can't be just multiplication (with which r?)

$$\Rightarrow U = \int u\, dV = \int_{r_a}^{r_b} \frac{1}{2}\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$\Rightarrow U = \frac{Q^2}{8\pi\epsilon_0} * \left(\frac{1}{r^2} \Big|_{r_a}^{r_b} \right) = \frac{Q^2}{8\pi\epsilon_0} \frac{r_b - r_a}{r_a r_b}$$

$$-\frac{1}{r} \Rightarrow \frac{1}{r_a} - \frac{1}{r_b} = \frac{r_b - r_a}{r_a r_b}$$

So we get the same thing.

Dielectrics

* Placing a nonconducting material between two conducting plates. ⇒ Diel. What's the use?

1) Helps to solve a practical problem: how to keep two conducting plates seperately
2) Helps to keep high voltage diff. across
3) Increases the capacitance.

charge being fixed → $K = \frac{C}{C_0} \Rightarrow \frac{V_0}{V} = K$

$\Rightarrow K > 1 \Rightarrow C \uparrow, V \downarrow$.

Table 24.1 Values of Dielectric Constant K at 20°C

Material	K	Material	K
Vacuum	1	Polyvinyl chloride	3.18
Air (1 atm)	1.00059	Plexiglas	3.40
Air (100 atm)	1.0548	Glass	5–10
Teflon	2.1	Neoprene	6.70
Polyethylene	2.25	Germanium	16
Benzene	2.28	Glycerin	42.5
Mica	3–6	Water	80.4
Mylar	3.1	Strontium titanate	310

Dielectric breakdown: every insulator might become ionized and let the charge to flow out after certain amount of e-field. This point is called diel. breakdown. Therefore capacitors might still be partially ionized and we could have a leak. But we ignore that.

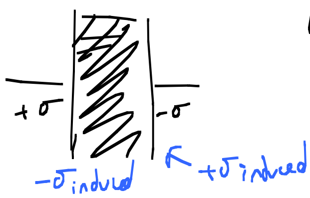
The point where diel. breakdown happens is called diel. strength of that material.

Table 24.2 Dielectric Constant and Dielectric Strength of Some Insulating Materials

Material	Dielectric Constant, K	Dielectric Strength, E_m (V/m)
Polycarbonate	2.8	3×10^7
Polyester	3.3	6×10^7
Polypropylene	2.2	7×10^7
Polystyrene	2.6	2×10^7
Pyrex glass	4.7	1×10^7

Where diel. strength of air is 3×10^6 V/m.

Induced Charge and Polarization



When we put a diel. material btwn.

$$E = \frac{E_0}{K} \quad \text{since} \quad V = \frac{V_0}{K}$$

Since Gauss' Law relates E to Q that is $\sigma \cdot A$ and because of the fact that σ per the conductor is same we effectively have a charge dist. induced by redistribution of $+,-$ charges on the surface of the diel. material: polarization

$$E_0 = \frac{\sigma}{\varepsilon_0}; \quad E = \frac{\sigma - \sigma_i}{\varepsilon_0} \quad \text{and} \quad E = \frac{E_0}{K}$$

$$\sigma - \sigma_i = \frac{\sigma}{K} \Rightarrow \sigma_i = \sigma - \frac{\sigma}{K} \Rightarrow \sigma_i = \sigma\left(1 - \frac{1}{K}\right)$$

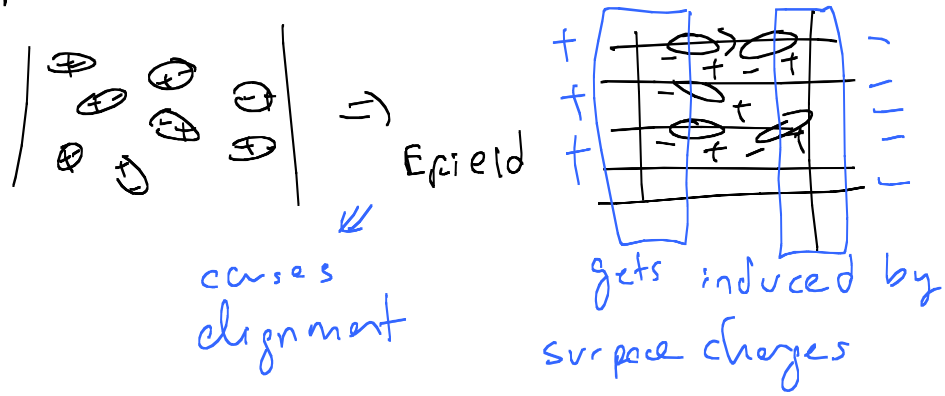
$K = 1 \Rightarrow$ (air) $\quad \sigma_i \to 0$.

We can call $\epsilon = K\epsilon_0$: the permittivity

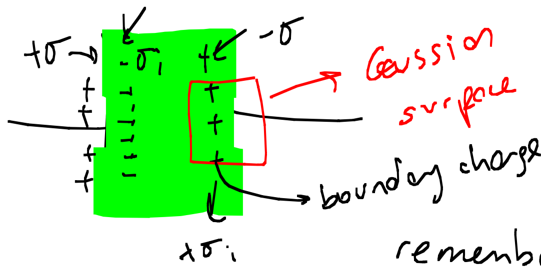
$$\Rightarrow E = \frac{\sigma}{\epsilon}; \quad C = \epsilon\frac{A}{d}; \quad u = \frac{1}{2}\epsilon E^2$$

$\therefore \epsilon_0$ can be called permittivity of free space (vacuum)

* Non conductors become polarized under an applied e-field. => Molecular structure allows it. (H_2O, N_2O)



Generalized Gauss' Law



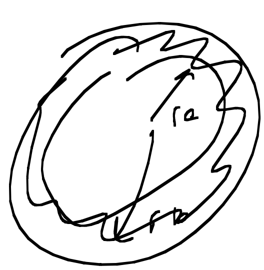
$$E = \frac{(\sigma - \sigma_i) A}{\varepsilon_0}$$

remember $\sigma - \sigma_i = \frac{\sigma}{K}$

$$\therefore \quad E = \frac{1}{K} \frac{\sigma A}{\varepsilon_0} \quad \Rightarrow \quad \frac{\sigma A}{\varepsilon_0} = KE$$

$$\therefore \quad \oint K \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

Sphere with non-conduc. material between:



$$\int K E \cdot dA = \frac{Q}{\varepsilon_0}$$

$$K E \int dA = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow \quad KE = \frac{Q}{4\pi \varepsilon_0 r^2}$$

$$\Rightarrow KE = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \Rightarrow \quad V_{ab} = V_a - V_b$$

$$E \cdot r \Rightarrow V = \frac{Q}{K 4\pi \varepsilon_0 r} \quad \Rightarrow \quad V_{ab} = \frac{Q}{K 4\pi \varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$$\Rightarrow \quad C = \frac{Q}{V_{ab}} = 4\pi\varepsilon_0 K \left(\frac{r_a r_b}{r_b - r_a}\right).$$