CHAPTER 30: INDUCTANCE

Inductance = coupling of conducting coils through changing magnetic field or flux

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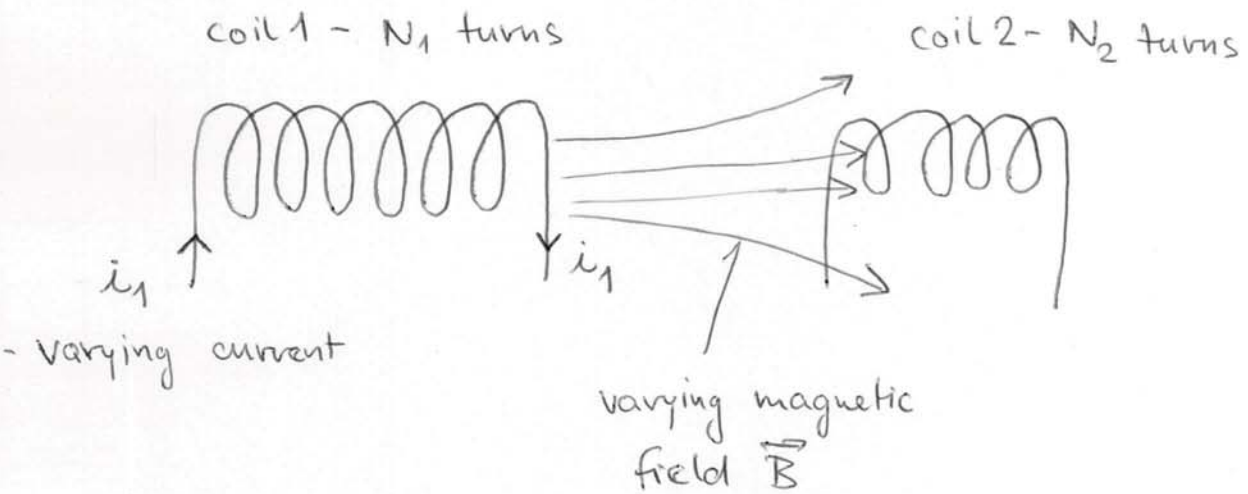
Coils are inductors

Section 30.1: Mutual Inductance

- Consider two neighboring conducting coils



- current i_1 flowing through coil 1 produces magnetic field $\vec{B}$ and magnetic flux Φ_{B2} through coil 2

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if the current i_1 changes, the flux Φ_{B2} also changes

⇓

electromotive force (emf) is induced in coil 2, following Faraday's law

- If $\underline{\underline{\Phi_{B2}}}$ denotes average magnetic flux through coil 2, Faraday's law gives the induced emf as:

$$\boxed{\mathcal{E}_2 = -N_2 \frac{d\Phi_{B2}}{dt}}$$

- Let us now define the <u>mutual inductance</u> of coils 1 and 2 as:

$$\boxed{M_{21} = \frac{N_2 \Phi_{B2}}{i_1} \Longrightarrow M_{21} i_1 = N_2 \Phi_{B2}}$$

M_{21} - constant of proportionality

$\downarrow$

With this definition of $\underline{\underline{M_{21}}}$, we can rewrite Faraday's law as:

$$\boxed{\mathcal{E}_2 = -M_{21} \frac{di_1}{dt}}$$

$\rightarrow$ induced emf in coil 2 is directly linked to the change of current in coil 1

M_{21} - a constant depending on the geometry of both coils (size, shape, number of turns), orientation and separation of coils, and magnetic properties of material between the coils

- for linear magnetic response of the material between the coils, $\underline{\underline{M_{21}}}$ is independent of $\underline{\underline{i_1}}$

- If we reverse the experiment →

changing current in coil 2 induces emf in coil 1

→ experiments show that $M_{12} = M_{21}$ for all configurations of coils 1 and 2

→ common mutual inductance $\underline{\underline{M}}$ characterizing induced-emf interactions of two coils

$$\boxed{\mathcal{E}_2 = -M \frac{di_1}{dt} \quad \text{and} \quad \mathcal{E}_1 = -M \frac{di_2}{dt}}$$

where

$$\boxed{M = \frac{N_2 \Phi_{B2}}{i_1} = \frac{N_1 \Phi_{B1}}{i_2}}$$

- Negative sign in induced emf equation reflects Lenz's law

- Only time-varying current induces an emf !!!

Units of M: $\quad$ Henry $\equiv 1\,H = 1 \frac{Wb}{A} = 1 \frac{J}{A^2}$

- Mutual inductance can lead to unwanted coupling in electric circuits

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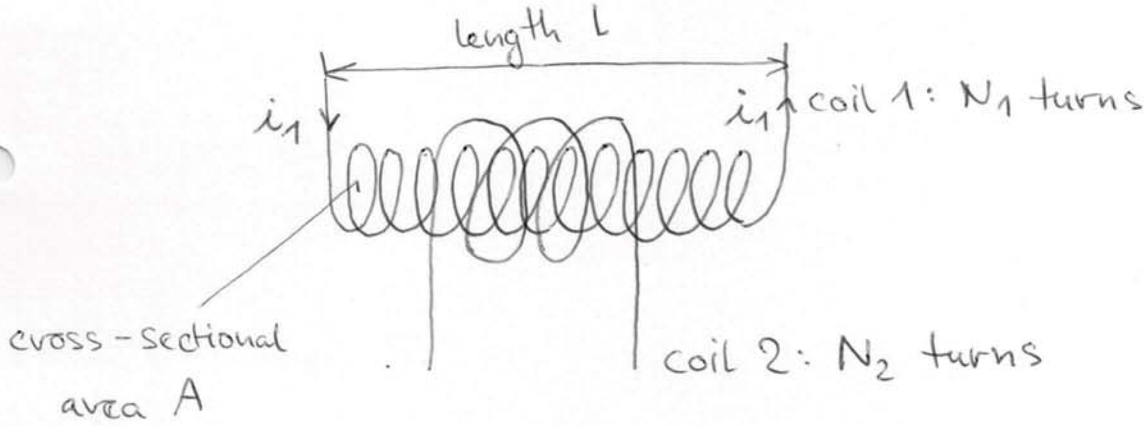
mutual inductance is useful in transformers for stepping up / down the voltage

Ex 30.1: Calculating mutual inductance

A long solenoid with length L and cross-sectional area A is closely wound with N_1 turns of wire. A coil with N_2 turns surrounds it at its center. Find the mutual inductance



- Magnetic field $\vec{B}_1$ of a solenoid close to its center is uniform and points along the axis of the solenoid:

$$B_1 = \mu_0 n_1 i_1 = \mu_0 \frac{N_1}{L} i_1$$

i_1 ... current through the solenoid

n_1 ... number of solenoid turns per unit length

- Flux through the cross-section of the solenoid: $B_1 A$

$\downarrow$

Since there is no field outside solenoid 1, we can write for the flux through solenoid 2 $\Phi_{B2} = B_1 \cdot A$

- From definition $M = \dfrac{N_2 \Phi_{B2}}{i_1} =$

$$= \frac{N_2 B_1 A}{i_1} = \frac{N_2}{\cancel{i_1}} \frac{\mu_0 N_1 \cancel{i_1}}{l} \cdot A =$$

$$= \underline{\underline{\frac{\mu_0 A N_1 N_2}{l}}}$$

→ The mutual inductance of two coils is proportional to the product $N_1 N_2$ of their turns.

It is independent of the current.

Section 30.2: Self-Inductance and Inductors

② What happens in a single isolated circuit with changing current?

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The current sets up a magnetic field that causes a magnetic flux through the same circuit.

This flux changes when the current changes

↓

Emf is induced in the circuit by the variation of its own magnetic field → self-induced emf

- From Lenz's law, self-induced emf always opposes the change in the current that caused the emf
  → current variations are more difficult!

- Self-induced emfs can occur in any current-carrying circuit
  → it is especially pronounced in coils with N turns of wire

- We can define self-inductance L as:

$$\boxed{L = \frac{N\Phi_B}{i}}$$

Φ_B - average magnetic flux through each turn of the coil

i - current through the coil

- Units of self-inductance are identical to the units of mutual inductance → 1 henry ≡ 1H

- Using definition of self-inductance, we can rearrange Faraday's law as:

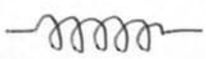
$$\boxed{\mathcal{E} = -N\frac{d\Phi_B}{dt} = -L\frac{di}{dt}}$$

→ negative sign reflects Lenz's law
⇓
induced emf opposes any current changes in the circuit

- Faraday's law can be used to measure unknown self-inductance $\underline{\underline{L}}$ from a known rate di/dt and measured induced emf $\underline{\underline{\mathcal{E}}}$

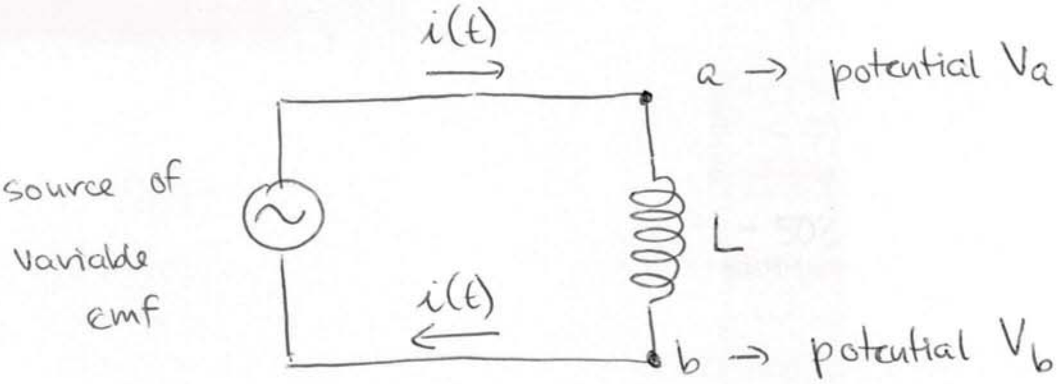
Inductors as circuit elements

Inductor ... device with a particular (self-) inductance

symbol 

- Inductors are used to oppose any variations in the current through the circuit

  → current stabilization



- Due to induced emf, charge accumulates on the terminals of the inductor
  - → this generates potential difference $V_{ab} = V_a - V_b$ across the inductor

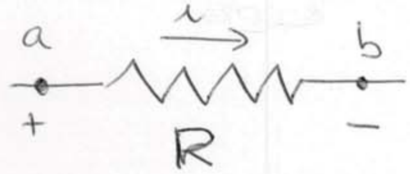
$$\boxed{V_{ab} = L\frac{di}{dt}}$$

→ non-conservative induced emf produces conservative electric field with potential

→ We can apply Kirchhoff's loop rule to analyze currents and voltage drops in circuits containing inductors

Potential difference across resistor vs. inductor

Resistor



$V_{ab} = iR > 0$

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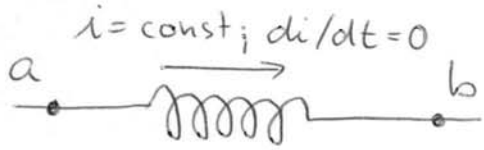
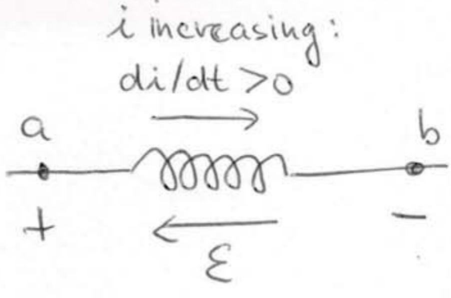
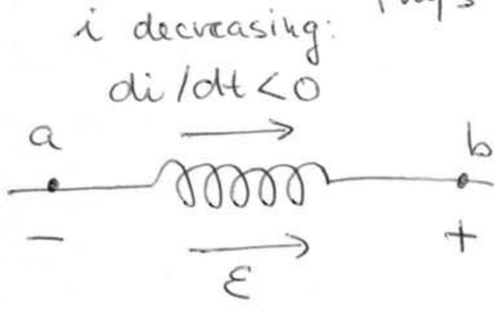
Inductor



$\mathcal{E} = 0$

$V_{ab} = L\frac{di}{dt} = 0$



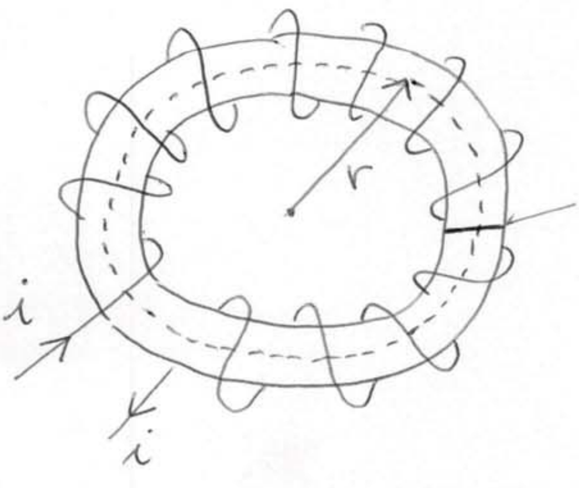


$$V_{ab} = L\frac{di}{dt} > 0 \qquad\qquad V_{ab} = L\frac{di}{dt} < 0$$

Ex 30.3 Calculating self-inductance

A toroidal selenoid with cross-sectional area $\underline{\underline{A}}$ and mean radius $\underline{\underline{r}}$ is closely wound with $\underline{\underline{N}}$ turns of wire. The toroid is wound on a nonmagnetic core. Determine its self-inductance $\underline{\underline{L}}$, assuming that magnetic field $\underline{\underline{B}}$ is uniform across a cross-section.

number of turns = N



cross-sectional area A

assumption of uniform field

- From definition, self-inductance $L = \dfrac{N\Phi_B}{i} \approx \dfrac{N.A.B}{i}$

- Magnetic field at a distance $\underline{\underline{r}}$ from the toroid axis is calculated from Ampere's law

$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$

$I_{encl} = N \cdot i$

$\vec{B}$ is tangential to the toroid $\rightarrow \oint \vec{B} d\vec{l} = \oint B dl =$

$= B \oint dl = 2\pi r B$

$\Downarrow$

$2\pi r B = \mu_0 N i \rightarrow B = \dfrac{\mu_0 N i}{2\pi r}$

• Inserting $\underline{\underline{B}}$ into the definition of self-inductance gives

$$L = \frac{N \cdot A \cdot B}{i} = \frac{NA}{\cancel{i}} \frac{\mu_0 N \cancel{i}}{2\pi r} = \frac{\mu_0 A N^2}{2\pi r}$$

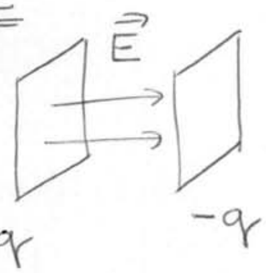
$\rightarrow$ Self-inductance is proportional to the square of the number of turns in the coil

(for closely wound coils)

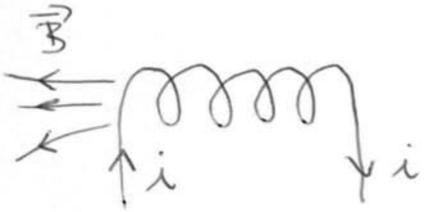
Section 30.3: Magnetic-Field Energy

- An inductor carrying a current has energy stored in it

In a capacitor, stored energy is used to build up electric field $\vec{E}$



In an inductor, stored energy is used to build up magnetic field $\vec{B}$

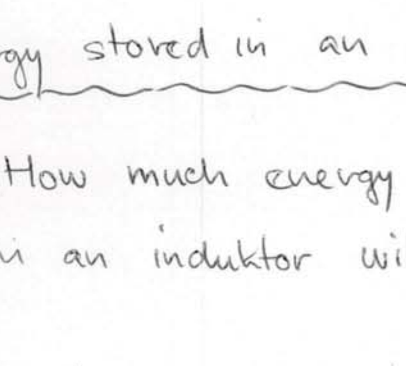


Energy stored in an inductor

(?) How much energy is needed to establish a final current I in an induktor with induktance L if the initial current is zero?

- Let us assume an inductor with zero resistance
  → Zero energy dissipation

$i(t) \equiv i$ → time-varying current with rate $\frac{di}{dt} > 0$



↓

Voltage drop across the inductor

$$V_{ab} = L \frac{di}{dt}$$

↓

Power is delivered to the inductor at the rate $P = V_{ab} \cdot i = L i \frac{di}{dt}$

- Energy supplied to the inductor in time $\underline{\underline{dt}}$ is $dU = P \cdot dt = \underline{\underline{L i \, di}}$

$$\Downarrow$$

The total energy $\underline{\underline{U}}$ supplied while the current increases from zero to the final value $\underline{\underline{I}}$ is

$$\boxed{U = \int_0^I dU = L \int_0^I i \, di = \frac{LI^2}{2}}$$

- After reaching the final steady value of current $\underline{\underline{I}}$, energy input into the inductor stops → energy is stored in the magnetic field of the inductor

- When the current decreases from $\underline{\underline{I}}$ to zero, the inductor acts as a source that supplies a total energy of $\frac{LI^2}{2}$ to the circuit

<u>NOTE</u> Resistor → energy flows <u>into it</u> whenever current passes through it (steady or time-varying)
→ energy is dissipated as heat

×

Inductor → energy flows <u>into it</u> only when current passing through it <u>increases</u>
→ energy flows <u>out of it</u> when current passing through it <u>decreases</u>

→ for a steady current in an inductor, there is no energy flow in or out of the inductor

Magnetic energy density

- Energy in an inductor is stored in its magnetic field
- Let us consider an ideal toroidal solenoid
  → magnetic field is completely confined to a finite space within the toroid core
  → magnetic field is ~uniform across the toroid cross-section
  → total volume enclosed by the toroid is approximately

$$V = 2\pi r \cdot A$$

$$\Downarrow$$

Total stored energy $U = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\mu_0 A N^2}{2\pi r} \cdot I^2$

is localized in the volume $\underline{\underline{V}}$

$$\Downarrow$$

Energy density $u \equiv \frac{U}{V} = \frac{1}{2} \frac{\mu_0 A N^2}{2\pi r} I^2 \cdot \frac{1}{2\pi r A}$

$$= \underline{\underline{\frac{1}{2} \mu_0 \frac{N^2 I^2}{(2\pi r)^2}}}$$

- If we realize that the magnetic field of the toroid is $B = \mu_0 \frac{NI}{2\pi r}$, we can rearrange to obtain $\frac{B}{\mu_0} = \frac{NI}{2\pi r}$

→ substituting this into the definition of magnetic energy density gives

$$\boxed{u = \frac{1}{2}\mu_0\left(\frac{B^2}{\mu_0^2}\right) = \frac{1}{2}\frac{B^2}{\mu_0}}$$

– magnetic energy density in vacuum

• If the material inside the toroid has a constant magnetic permeability μ, the energy density is

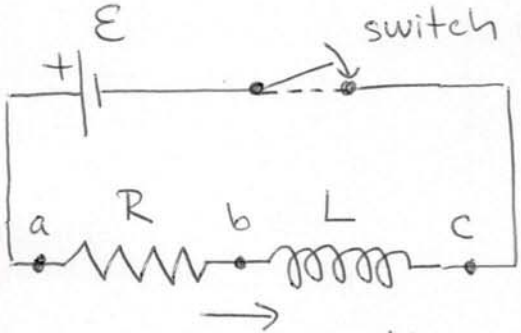
$$\boxed{u = \frac{1}{2}\frac{B^2}{\mu}}$$

→ this expression is valid generally for any magnetic field configuration in a material with constant permeability

Section 30.4: The R-L Circuit

- An electric circuit that includes a resistor and an inductor

a) Current growth in an R-L circuit



$i = i(t) \rightarrow \frac{di}{dt} > 0$

- switch is closed at time $t=0 \Rightarrow$ current starts flowing through the circuit

$\Downarrow$

induced emf appears across inductor that opposes the current growth

- potential difference across the resistor: $v_{ab} = iR > 0$
- potential difference across the inductor : $v_{bc} = L\frac{di}{dt} > 0$

Kirchhoff's loop rule for the circuit in the counter-clockwise direction:

$$+\mathcal{E} - iR - L\frac{di}{dt} = 0$$

$$\Downarrow$$

Rate of current increase : $\frac{di}{dt} = \frac{\mathcal{E}}{L} - \frac{R}{L} \cdot i$

- When the switch is first closed, $i=0$

$\Rightarrow$ initial rate of current change $\left(\frac{di}{dt}\right)_{initial} = \frac{\mathcal{E}}{L}$

- After a long time, current approaches a steady value I

$$\Rightarrow \left(\frac{di}{dt}\right)_{final} = 0 \Rightarrow \frac{\mathcal{E}}{L} - \frac{R}{L} I = 0 \Rightarrow I = \frac{\mathcal{E}}{R}$$

- final current does not depend on the inductance, only on resistance

Current evolution with time

- Let us rearrange the Kirchhoff's loop rule $\frac{di}{dt} = \frac{\mathcal{E}}{L} - \frac{R}{L} i$

as $$\frac{di}{i - \frac{\mathcal{E}}{R}} = -\frac{R}{L} dt$$ and integrate both sides:

$$\int_0^i \frac{di'}{i' - \frac{\mathcal{E}}{R}} = -\int_0^t \frac{R}{L} dt' \Rightarrow \left[\ln\left(i' - \frac{\mathcal{E}}{R}\right)\right]_0^i = -\frac{R}{L}\left[t'\right]_0^t$$

$$\ln\left(\frac{i - \mathcal{E}/R}{-\mathcal{E}/R}\right) = -\frac{R}{L} t$$

- If we take exponentials of both sides, we obtain

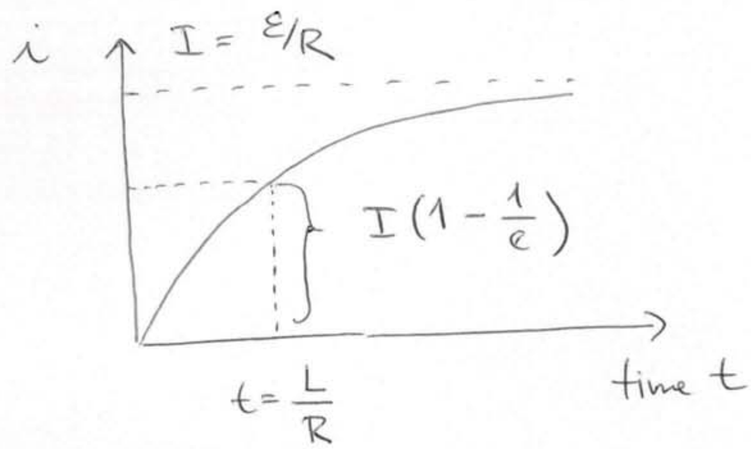
$$\frac{i - \mathcal{E}/R}{-\mathcal{E}/R} = e^{-\frac{R}{L}t}$$ which can be rearranged as $\hookrightarrow$

$$\boxed{i = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$$

time-varying current in R-L circuit when the current goes from zero to a finite steady value $\frac{\mathcal{E}}{R}$

- Quantity $\boxed{\tau = \frac{L}{R}}$ defines the time constant of R-L circuit → it characterizes how quickly the current builds up towards its final value

NOTE Remember time constant for R-C circuit $\tau = R \cdot C$ that describes dynamics of capacitor charging



→ for $t \to \infty$, inductor behaves as a simple conducting element with zero resistance

Energy considerations SKIP

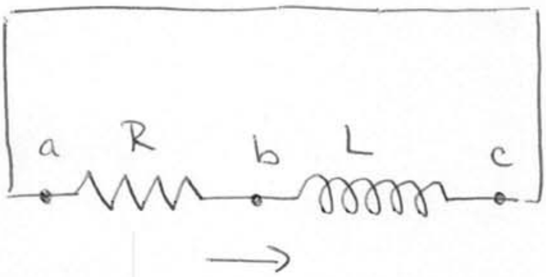
- power delivered to the circuit: $P_{in} = \mathcal{E} \cdot i$
- power dissipated in the resistor: $P_R = i^2 R$
- power stored in the inductor: $P_L = \frac{d}{dt}\left(\frac{1}{2}Li^2\right) = L i \frac{di}{dt}$

Energy balance requires $P_{in} = P_R + P_L$

$$\Downarrow$$

$$\boxed{\mathcal{E} i = i^2 R + L i \frac{di}{dt}}$$

b) Current decay in an R-L circuit



$i \equiv i(t) \rightarrow \frac{di}{dt} < 0$

Initial current I_0 through the circuits decays as the energy is dissipated in the resistor

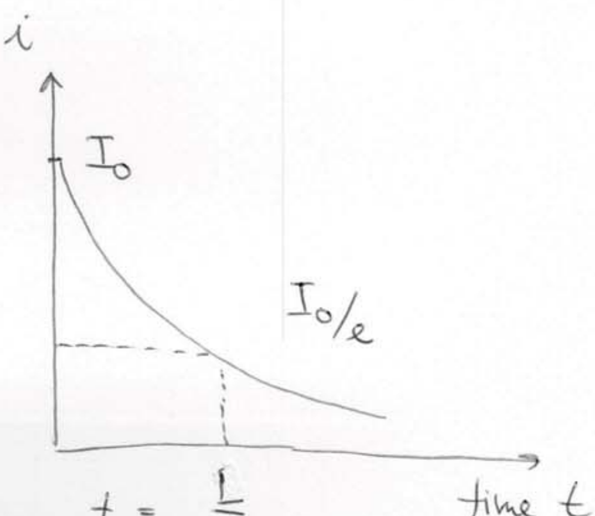
Kirchhoff's loop rule requires $iR + L\frac{di}{dt} = 0$

$$\downarrow$$

$$\int_{I_0}^{i} \frac{di'}{i'} = -\frac{R}{L}\int_0^t dt' \Rightarrow \ln\frac{i}{I_0} = -\frac{R}{L}t$$

$$\Downarrow$$

$$\boxed{i = I_0 e^{-\frac{R}{L}t}}$$



Energy balance:

$$0 = i^2 R + L i \frac{di}{dt}$$

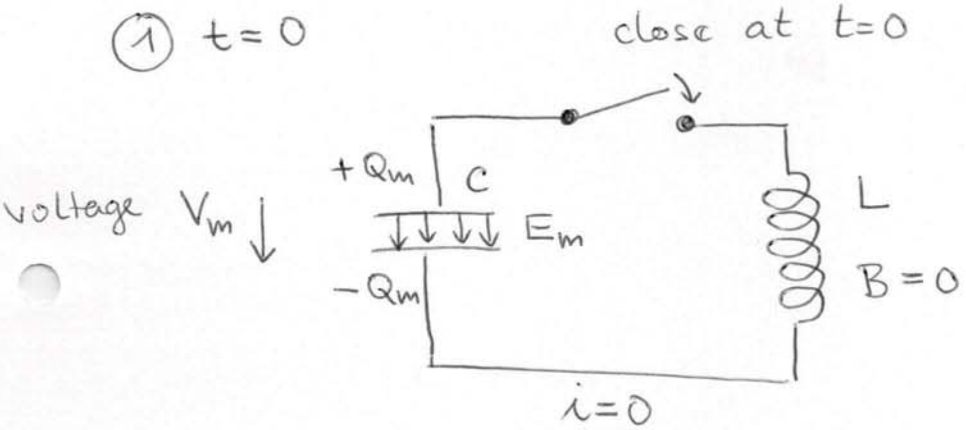
$$\downarrow$$

energy stored in the inductor is dissipated in the resistor

Section 30.5: The L-C Circuit

- A circuit containing an inductor and a capacitor shows an entirely new mode of behavior
  → charge and current oscillate in time

(1) $t = 0$

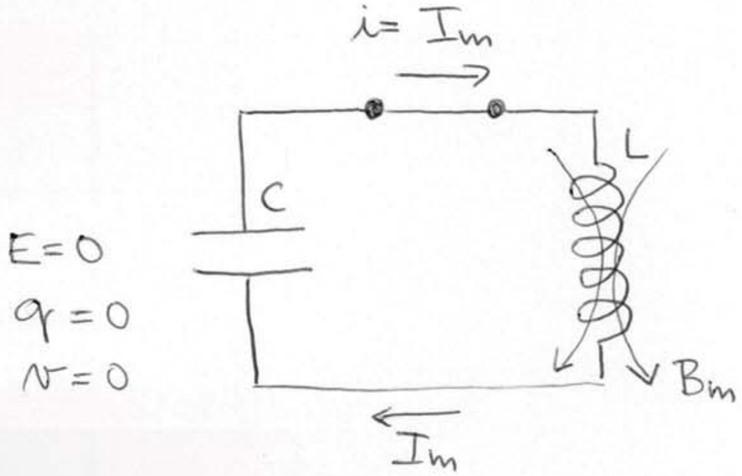


At time $t=0$, capacitor is fully charged → charge $Q_m = C \cdot V_m$, electric field E_m are maximal

- initial current is zero → no magnetic field
  → all energy stored in the electric field of the capacitor

↓ capacitor discharges, current builds up to a maximum value $\underline{\underline{I_m}}$

(2) $t = \frac{T}{4}$

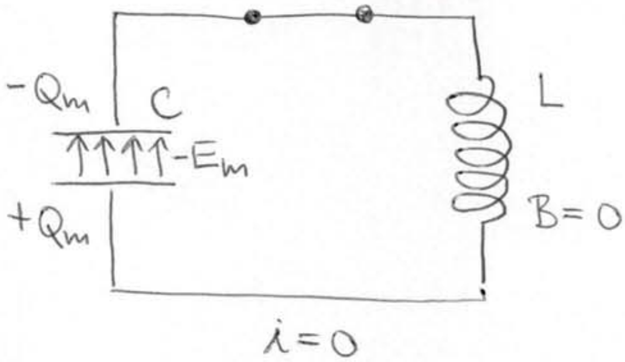


At time $t = \frac{T}{4}$, capacitor is fully discharged
→ zero voltage $\underline{\underline{v}}$ and electric field $\underline{\underline{E}}$

→ all energy stored in the magnetic field of the inductor carrying maximal current $\underline{\underline{I_m}}$

current starts decaying, induced emf tries to keep it going, capacitor charges with opposite polarity

③ $t = \frac{T}{2}$



voltage $-V_m$

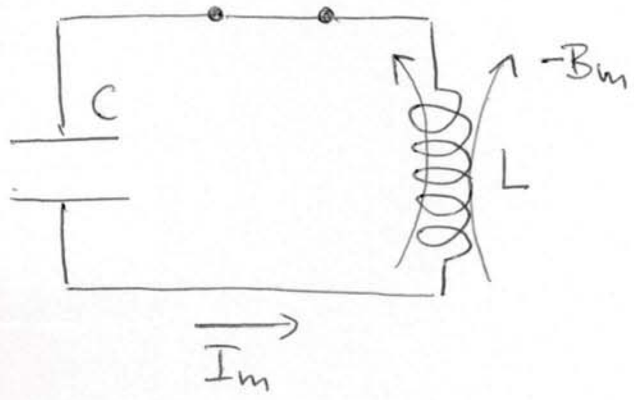
At time $t = \frac{T}{2}$, capacitor is fully charged with reversed polarity
→ maximal charge $-Q_m$, voltage $-V_m$, and electric field $-E_m$

- current is again zero → no magnetic field in the inductor → all energy stored in the electric field of the capacitor

capacitor discharges, current with reversed polarity builds up to maximal value $-\underline{\underline{I_m}}$

④ $t = \frac{3T}{4}$

$E=0$
$q=0$
$v=0$



At time $t = \frac{3T}{4}$, capacitor is again fully discharged
→ zero voltage and electric field

- maximal reversed current $-\underline{\underline{I_m}}$ → all energy stored in the magnetic field of the inductor

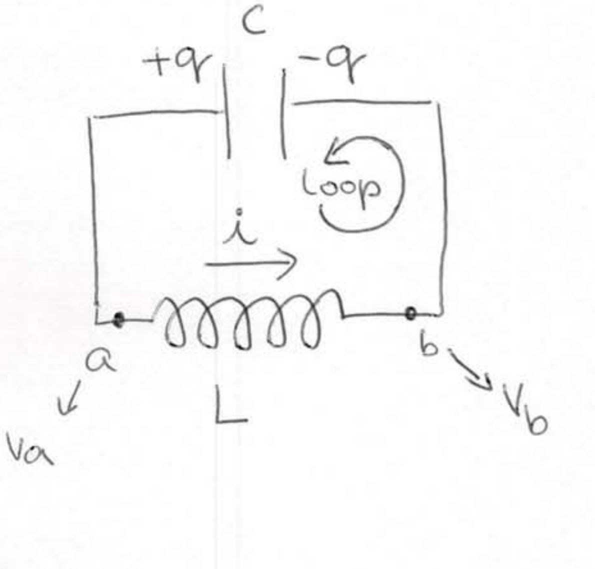
- Full oscillation cycle is completed by full charging of the capacitor to the original value of charge at time $t=0$ → the whole process repeats again, in the absence of dissipation indefinitely

⇓

<u>electrical oscillation</u>

- During oscillations, energy transfers between capacitor's electric field and inductor's magnetic field
  - → the total energy remains constant
  - → analogy to the transfer between potential and kinetic energy during mechanical oscillations

Electrical oscillations in an L-C circuit



Consider a circuit containing an (ideal) inductor and capacitor

↓

No energy dissipation in the circuit

↓

Energy oscillates between the capacitor and inductor

- Let's assume the charge is decreasing → Kirchhoff's loop rule gives $\frac{q}{C} - L\frac{di}{dt} = 0$ (current grows with time)

- Current $\underline{\underline{i}}$ with positive direction indicated in the figure is linked to the time rate of change of the charge q on the <u>left</u> capacitor plate by

$$\boxed{i = -\frac{dq}{dt}}$$ $\rightarrow$ decreasing charge on the left plate causes positive current

$$\downarrow$$

Kirchhoff's loop rule: $\frac{q}{C} - L\frac{d}{dt}\left(-\frac{dq}{dt}\right) = 0$

Dynamics of the capacitor charge

$$\boxed{\frac{d^2q}{dt^2} + \frac{1}{LC}\cdot q = 0}$$

$\Updownarrow$ analogy with mechanical oscillations

Dynamics of the oscillator displacement

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

	Electrical oscillation		Mechanical oscillation	
charge	q	$\longleftrightarrow$	x (displacement)	
current	$i = \frac{dq}{dt}$	$\longleftrightarrow$	velocity	$v = \frac{dx}{dt}$
inductance	L	$\longleftrightarrow$	mass	m
capacitance	C	$\longleftrightarrow$	reciprocal force constant	$\frac{1}{k}$

- Solutions of differential equation for charge q has the form

$$\boxed{q(t) = Q\cos(\omega t + \varphi)}$$ ← harmonic oscillation

with frequency $\boxed{\omega = \sqrt{\frac{1}{LC}}}$

⇓

Current $$\boxed{i(t) = -\frac{dq}{dt} = \omega Q \sin(\omega t + \varphi)}$$

- Constants Q and φ are determined by initial conditions

EG $t = 0 \rightarrow$ q is maximal, i is zero $\Big\} \Rightarrow Q = Q_{max}$, $\varphi = 0$

Energy in an L-C circuit

- Inductor energy $$U_L = \frac{1}{2}Li^2 = \frac{1}{2}L\omega^2 Q_{max}^2 \sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$= \frac{1}{2}\cancel{L}\frac{1}{\cancel{L}C}Q_{max}^2 \sin^2\left(\frac{t}{\sqrt{LC}}\right) =$$

$$= \frac{Q_{max}^2}{2C}\sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

- Capacitor energy $$U_C = \frac{q^2}{2C} = \frac{Q_{max}^2}{2C}\cos^2\left(\frac{t}{\sqrt{LC}}\right)$$

- Total energy $U_T = U_L + U_C =$

$$= \frac{Q_{max}^2}{2C}\sin^2\left(\frac{t}{\sqrt{LC}}\right) + \frac{Q_{max}^2}{2C}\cos^2\left(\frac{t}{\sqrt{LC}}\right) =$$

$$= \underline{\underline{\frac{Q_{max}^2}{2C}}} = \text{const}$$

Ex 30.9: An oscillating circuit

A 300 V dc power supply is used to charge a 25 μF capacitor. After the capacitor is fully charged, it is disconnected from the power supply and connected across a 10 mH inductor.

a) Find frequency and period of circuit oscillation

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10\cdot 10^{-3}\cdot 25\cdot 10^{-6}}} = \underline{\underline{2000\ \frac{rad}{s}}}$$

$$\omega = \frac{2\pi}{T} \longrightarrow T = \frac{2\pi}{\omega} = \underline{\underline{3{,}1\ ms}}$$

b) Find the capacitor charge and circuit current 1,2 ms after the inductor and capacitor are connected

$$q = Q_{max}\cos\omega t$$

$$i = -\frac{dq}{dt} = \omega Q_{max}\sin\omega t$$

$$Q_{max} = C\cdot V = 25\cdot 10^{-6}\ \mu F\cdot 300\ V = \underline{\underline{7{,}5\cdot 10^{-3}\ C}}$$

Phys 102

$$q = 7{,}5 \cdot 10^{-3} \cdot \cos\left(2000 \cdot 1{,}2 \cdot 10^{-3}\right) = -\underline{\underline{5{,}5 \cdot 10^{-3}\,C}}$$

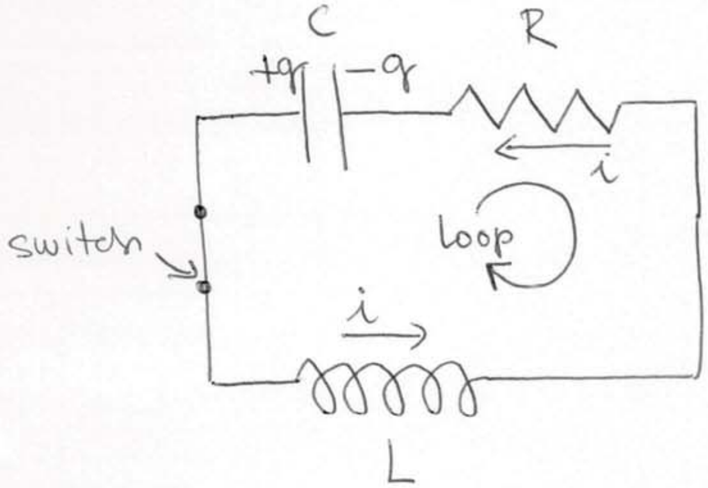
$$i = 2000 \cdot 7{,}5 \cdot 10^{-3} \cdot \sin\left(2000 \cdot 1{,}2 \cdot 10^{-3}\right) = \underline{\underline{10{,}1\,A}}$$

Section 30.6: The L-R-C Circuit - Series

- Including resistance R in an oscillating LC circuit causes energy dissipation → total stored energy decays towards zero for long times

↓

- For small resistance, circuit displays damped harmonic oscillation
- For increasing resistance, circuit behavior loses oscillatory character



At time $t=0$, capacitor is fully charged to charge $Q = \mathcal{E} \cdot C$

↓

closing the switch causes capacitor discharging

Kirchhoff's loop rule:

$$-\frac{q}{C} + iR + L\frac{di}{dt} = 0$$

Current $i = -\frac{dq}{dt}$ → rate of change of the charge on the left capacitor plate

$$\Downarrow$$

$$-L\frac{d^2q}{dt^2} - R\frac{dq}{dt} - \frac{q}{C} = 0 \rightarrow \boxed{\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0}$$

- second-order differential equation for charge q

- solutions have the general form $q(t) = A \cdot e^{kt}$

$\downarrow$ insert into differential equation

$$A k^2 e^{kt} + \frac{R}{L} A k e^{kt} + \frac{A}{LC} e^{kt} = 0$$

$$A e^{kt} \left(k^2 + \frac{R}{L} k + \frac{1}{LC} \right) = 0 \rightarrow \text{valid for all times}$$

$$\Downarrow$$

$$k^2 + \frac{R}{L} k + \frac{1}{LC} = 0$$

Two solutions of the quadratic equation

$$\boxed{k_{1,2} = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}$$

General solution:

$$q(t) = A e^{k_1 t} + B e^{k_2 t} =$$

$$= e^{-\frac{R}{2L}t} \cdot \left\{ A e^{+\frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\, t} + B e^{-\frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\, t} \right\}$$

- There are three possible regimes of operation depending on the values of $\underline{\underline{R}}$, $\underline{\underline{L}}$, and $\underline{\underline{C}}$

a) $\frac{R^2}{L^2} = \frac{4}{LC} \rightarrow q(t) = (A+B)\, e^{-\frac{R}{2L}t}$

- critically damped case with exponential decay of the charge

b) $\frac{R^2}{L^2} > \frac{4}{LC} \rightarrow$ q(t) given by the sum of two decreasing exponentials with different time constants

- overdamped case

c) $\frac{R^2}{L^2} < \frac{4}{LC} \rightarrow$ underdamped case

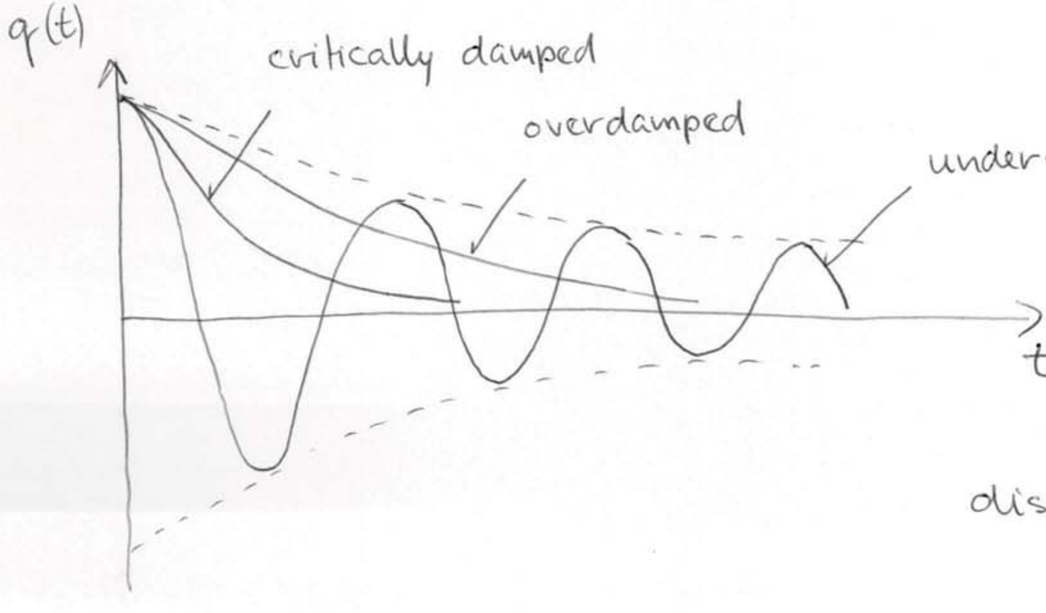
$$q(t) = e^{-\frac{R}{2L}t} \underbrace{\left\{ A e^{+\frac{i}{2}\sqrt{\frac{4}{LC}-\frac{R^2}{L^2}}\,t} + B e^{-\frac{i}{2}\sqrt{\frac{4}{LC}-\frac{R^2}{L^2}}\,t} \right\}}_{\text{oscillatory motion}}$$

exponential damping

oscillatory motion

$$K \cdot \cos\left[\frac{1}{2}\sqrt{\frac{4}{LC}-\frac{R^2}{L^2}}\, t + \phi \right]$$

$$q(t) = K\, e^{-\frac{R}{2L}t} \cdot \cos\left[\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\, t + \phi \right]$$



$\omega' = \sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$

$\downarrow$

smaller than dissipation-free $\omega = \sqrt{\frac{1}{LC}}$

Ex 30.11: Underdamped L-R-C series circuit

What resistance R is required to give an L-R-C circuit a frequency that is one-half the undamped frequency?

Underdamped frequency $\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

Undamped frequency $\omega = \sqrt{\frac{1}{LC}}$

$$\omega' = \frac{1}{2}\omega \rightarrow \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{1}{2}\sqrt{\frac{1}{LC}}$$

$$\frac{1}{LC} - \frac{R^2}{4L^2} = \frac{1}{4LC}$$

$$\frac{3}{4LC} = \frac{R^2}{4L^2} \Rightarrow \underline{\underline{R = \sqrt{\frac{3L}{C}}}}$$