CHAPTER 21: ELECTRIC CHARGE AND ELECTRIC FIELD

Section 21.1: Electric charge

- Electric charge is a fundamental property of matter, analogical to mass

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objects with mass are accelerated by gravitational forces,
objects with charge are accelerated by electric forces

- Electromagnetic interactions are omnipresent in the world around us
  - → they hold together atoms to form solid matter
  - → they are fundamental for function of electronic devices
  - → they control chemical reactions
  - → they allow us to see the world around us
  - ...

- The word "electric" is derived from Greek "elektron" meaning amber (yellowish solidified resin from ancient trees)

↓

amber objects rubbed with fur can attract other objects at a distance

- Numerous experiments have shown there are exactly two kinds of charges → Benjamin Franklin (1706 – 1790) suggested to call them "positive" and "negative"

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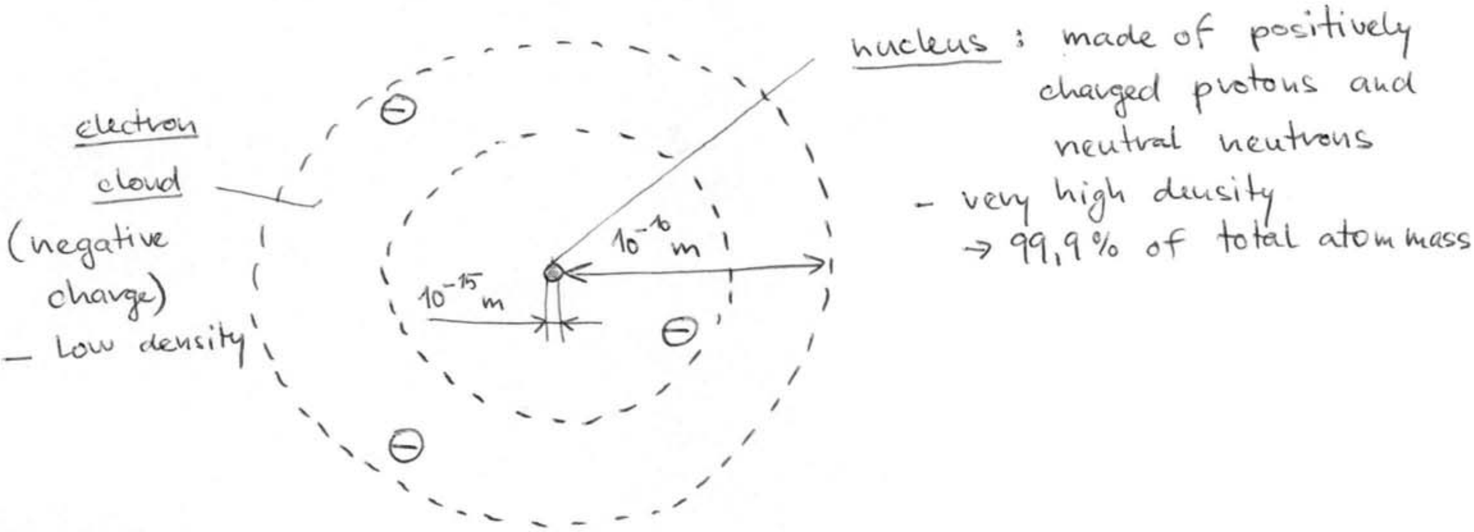
Two positive charges or two negative charges repel each other.

Positive and negative charge attract each other.

The structure of an atom ("planetary model")



- negatively charged electrons are held within atom by attractive electric forces exerted by positively charged nucleus
- nucleus itself is stabilized by the strong nuclear force
  → attractive interaction that overcomes electrostatic repulsion
  → short range → only effective within nucleus

- Electron (e) - mass $\sim 10^{-30}$ kg
- Proton (p) and neutron (n) - mass $\sim 10^{-27}$ kg
- Negative charge of electron has exactly the same magnitude as positive charge of proton

  ↓

  Neutral atoms with identical numbers of electrons and protons have no <u>net charge</u> (algebraic sum of all charges); number of e/p in a neutral atom = atomic number

- Acquisition of net charge by adding or losing electrons is called ionization

EG

neutral Li	positive Li ion	negative Li ion
3p → 3+	3p → 3+	3p → 3+
4n → 0	4n → 0	4n → 0
3e → 3−	2e → 2−	4e → 4−
net charge 0	net charge +1	net charge −1

- Net charge can be acquired by changing numbers of both electrons and protons

↓

Typically, it is much easier to add / remove light and very mobile electrons

↓

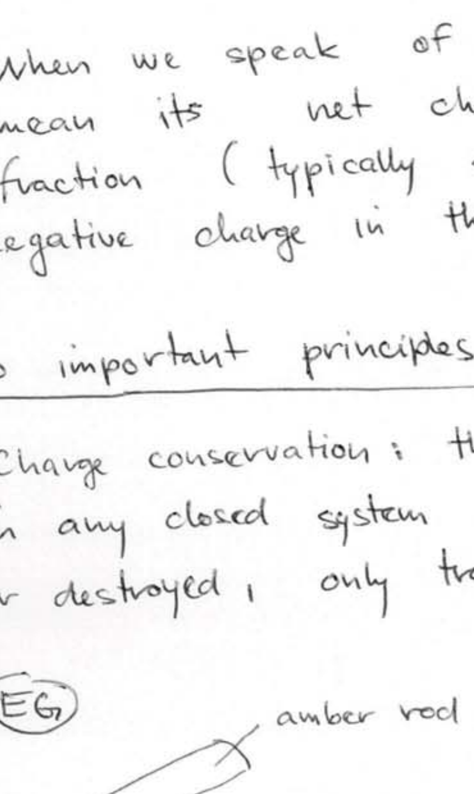
Net positive charge $\Leftrightarrow$ loss of electrons
Net negative charge $\Leftrightarrow$ gain of electrons

- When we speak of the charge of a body, we always mean its net charge which is only a very small fraction (typically $< 10^{-12}$) of the total positive or negative charge in the body

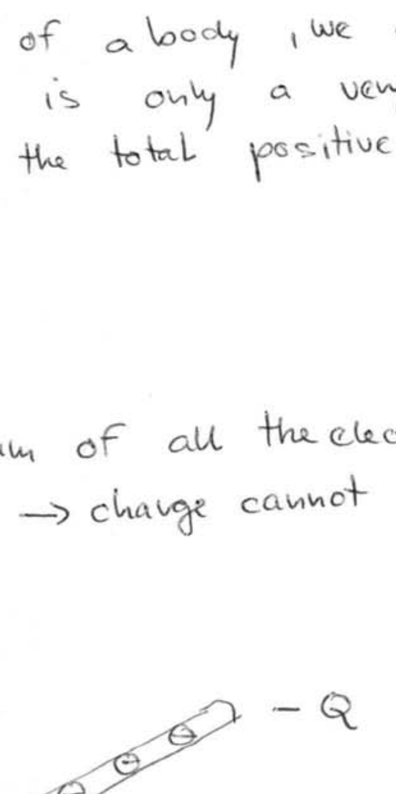
Two important principles

(1) Charge conservation: the algebraic sum of all the electric charges in any closed system is constant → charge cannot be created or destroyed, only transferred

(EG)



before rubbing - net charge of both amber rod and fur is zero



after rubbing - magnitudes of net charges of amber rod & fur are identical, charge sign is opposite

(2) The magnitude of charge of the electron or proton is a natural unit of charge

→ every observable amount of charge is always an integer multiple of this basic unit

→ charge is quantized (similar to money with basic unit of 1 kuruş)

charge $|Q| = n \cdot e$

integer

natural unit of charge
$e = 1{,}602 \cdot 10^{-19}\,C$

Section 21.2: Conductors, Isolators, and Induced Charges

Conductors: materials which permit electric charge to move easily from one region to another
e.g. copper and most metals

Insulators: materials which do not permit easy movement of charge through them
e.g. plastics, wood, ceramics, glass

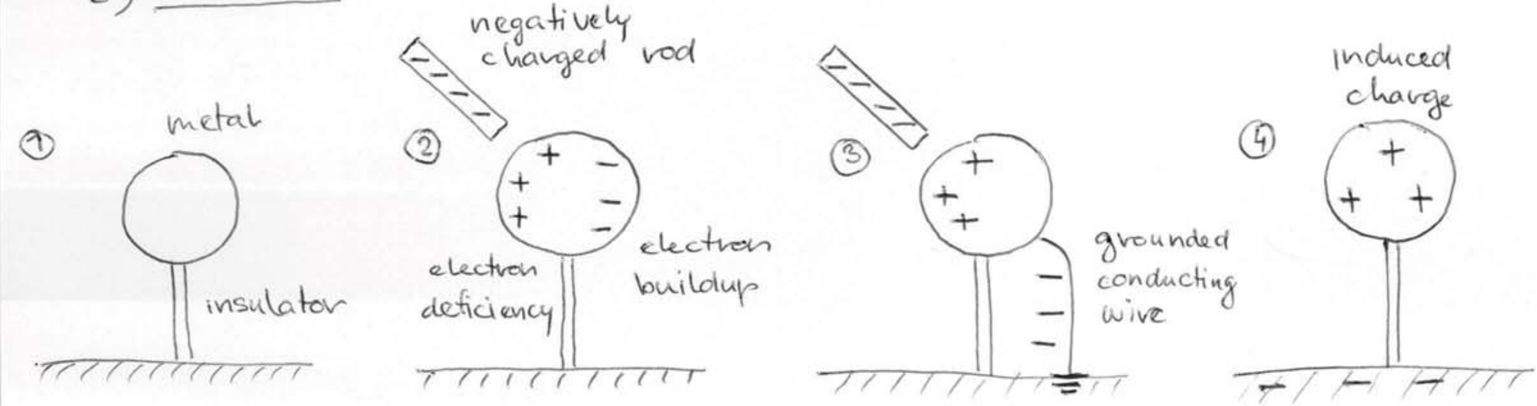
Semiconductors: materials whose charge conducting properties can be tuned by external stimulus or manipulation of atomic structure
e.g. Si, GaAs → electronic devices

- Conductivity of metals is caused by large mobility of outer electrons that become detached from their "parent" atoms and move freely through the material; inner electrons remind bound to the nuclei that are themselves nearly fixed within the material.

Charging of conductors

a) direct contact with another charged object, e.g. amber rod rubbed with fur → electron transfer between the rod and conductor

b) induction



① Initially, metal ball has no net charge

② Negatively charged rod in the vicinity of the ball repels electrons within the ball → zones of electron deficiency and buildup are created

③ Built-up electrons (induced negative charge) are moved through a grounded wire to earth – a good, large conductor that can serve as practically infinite source or sink of electrons

④ After removing the wire & rod, net positive charge is left on the metal ball. Due to electrostatic repulsion, this charge distributes uniformly within the ball.

- Charged body can exert forces even on neutral objects that are not charged themselves

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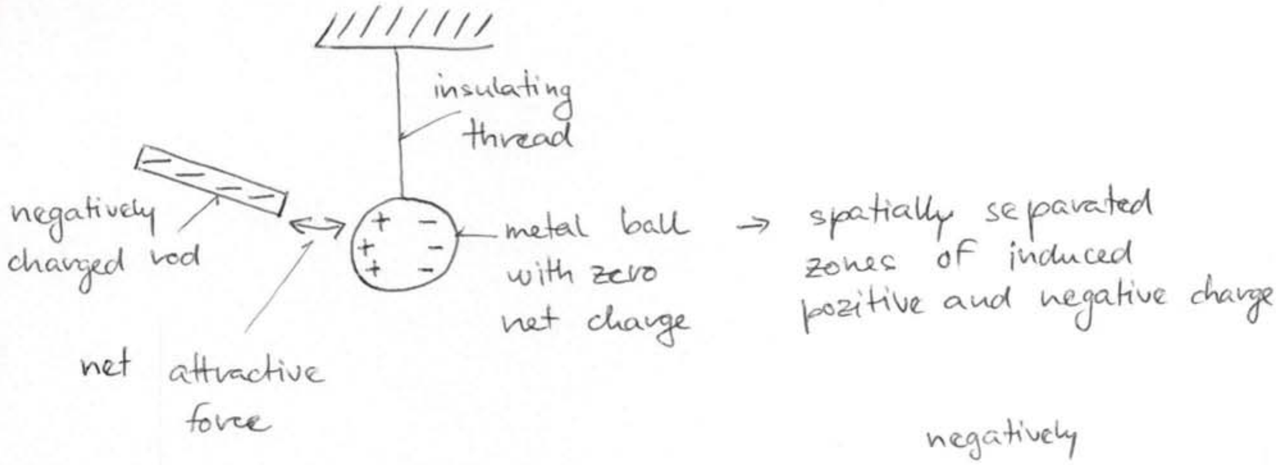
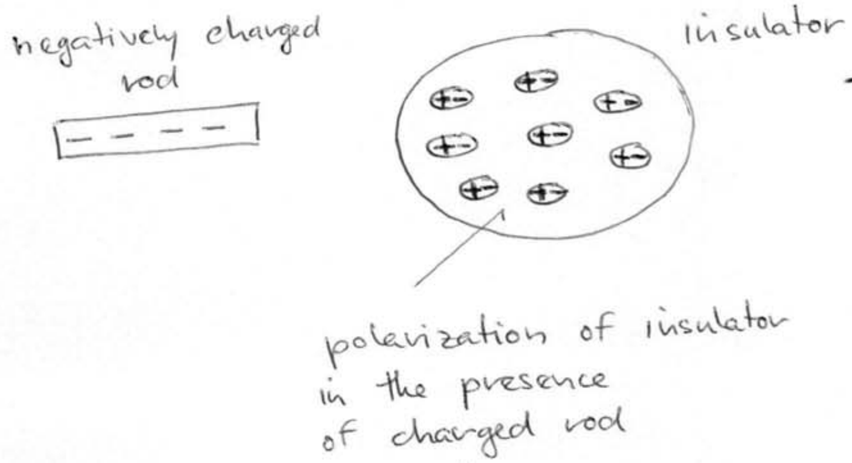
induced charge effect



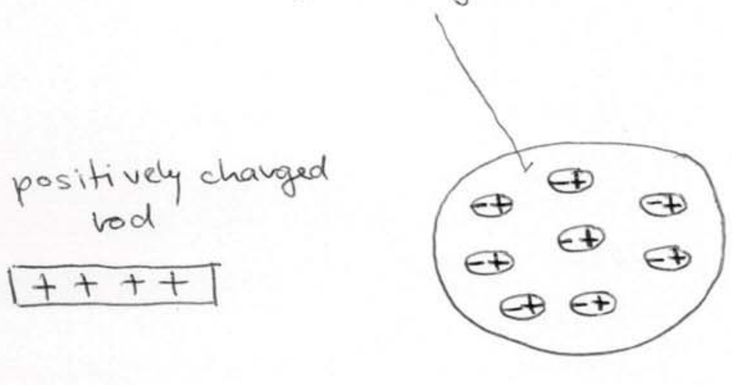
– because induced positive charge is closer to the negatively charged rod than induced negative charge in the metal ball, attraction between the ⊖ rod and ⊕ induced charge is stronger than repulsion between the ⊖ rod and ⊖ induced charge

→ net attractive force between the two bodies

- In an insulator, charges cannot move freely; however, they can still shift slightly from their rest position in the presence of another charged body



- center of mass of positive charge induced in insulator is closer to the negatively charged rod than the center of mass of negative induced charge → net attraction is stronger than net repulsion



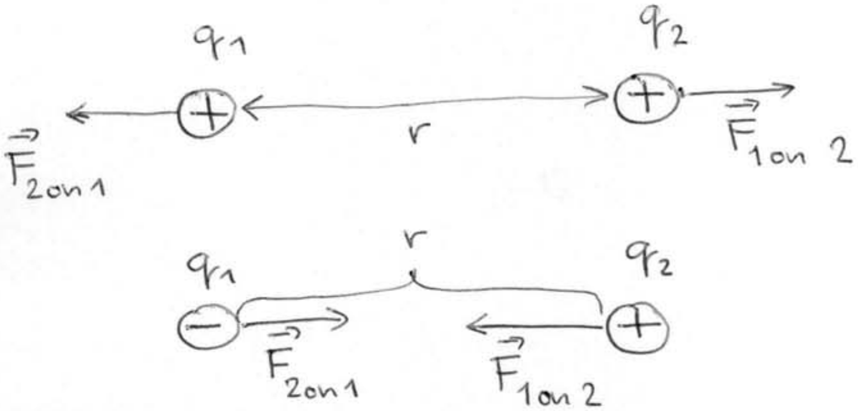
- in the presence of a positively charged rod, polarization of insulator is reversed → net attraction is again stronger than net repulsion

⇓

Uncharged insulator is always attracted to a charged object of any charge sign

Section 21.3: Coulomb's Law

- Coulomb's law = fundamental rule describing electrostatic interaction between charged particles, found & verified experimentally



Consider two point charges q_1, q_2 which are very small in comparison with distance r between them

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"The magnitude of the electric force between q_1 and q_2 is directly proportional to the product of the charges and indirectly proportional to the square of the distance between them"

$$|\vec{F}_{2on1}| = |\vec{F}_{1on2}| \equiv k \cdot \frac{|q_1 q_2|}{r^2}$$

- Direction of the electric force is along the line joining the two point charges
- When q_1 and q_2 have the same sign, the electric force is repulsive, when they have opposite signs, the force is attractive
- 3rd Newton's law applies: $\vec{F}_{2on1} = -\vec{F}_{1on2}$

↓

mutual forces are equal in magnitude and opposite in direction

- Constant $\underline{\underline{k}}$ in Coulomb's law depends on the system of units used

$\downarrow$

In SI unit system $k = 8{,}9876 \cdot 10^9 \ N\frac{m^2}{C^2} \approx 9 \cdot 10^9 \ N\frac{m^2}{C^2}$

C ... Coulomb = unit of electric charge ; $1C = 6 \cdot 10^{18}$ elementary charges

Value of $\underline{\underline{k}}$ is related to the speed of light in vacuum $\underline{\underline{c}}$ as

$$k = 10^{-7} \ \frac{N \cdot s^2}{C^2} \cdot c^2$$

speed of light

Coulomb

- Alternative expression for constant $\underline{\underline{k}}$

$$k = \frac{1}{4\pi\varepsilon_0}$$

; ε_0 = dielectric permittivity of vacuum

$$\varepsilon_0 = 8{,}854 \cdot 10^{-12} \ \frac{C^2}{N \cdot m^2}$$

Force between two point charges in vacuum

$\Downarrow$

$$\boxed{F = \frac{1}{4\pi\varepsilon_0} \frac{|q_1 q_2|}{r^2}}$$

We will use Coulomb's Law in this form

- Elementary charge $e = 1{,}602 \cdot 10^{-19}$ C
- Typical charges range from $nC = 10^{-9}$ C to $\mu C = 10^{-6}$ C

$\Downarrow$

1C is a huge charge !!!

$q_1 = 1C \xleftrightarrow{r = 1m} q_2 = 1C$

$F \approx 9 \cdot 10^9 \ N \sim 1$ million tons

NOTE : Functional form of Coulomb's law is very similar to Newton's law of gravitation

$$|\vec{F}_g| = G \frac{m_1 m_2}{r^2}$$

m_1, m_2 ... masses of interacting bodies

r ... separation of inter. bodies

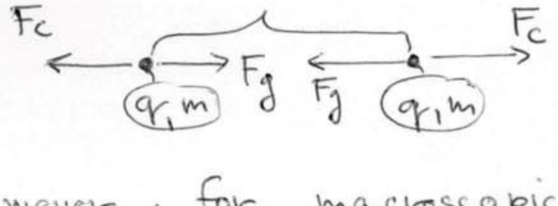
×

Gravitational Force is ALWAYS attractive while electrostatic force can be attractive OR repulsive

Ex 21.1 : An α-particle is the nucleus of a He atom. It has mass $m \approx 6{,}64 \cdot 10^{-27}$ kg and charge $q = +2e \approx 3{,}2 \cdot 10^{-19}$ C. Compare magnitude of electrostatic repulsion and gravitational attraction between two α-particles

$$F_c = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{r^2} \qquad F_g = G \frac{m^2}{r^2} \; ; \; G = 6{,}67 \cdot 10^{-11} \, N \frac{m^2}{kg^2}$$

$$\frac{F_c}{F_g} = \frac{1}{4\pi\varepsilon_0 G} \frac{q^2}{m^2} \approx 3{,}1 \cdot 10^{35} \quad !!!$$



However, for macroscopic objects (persons, planets) with almost zero net charge, electric force is usually much smaller than gravity

Superposition of Forces

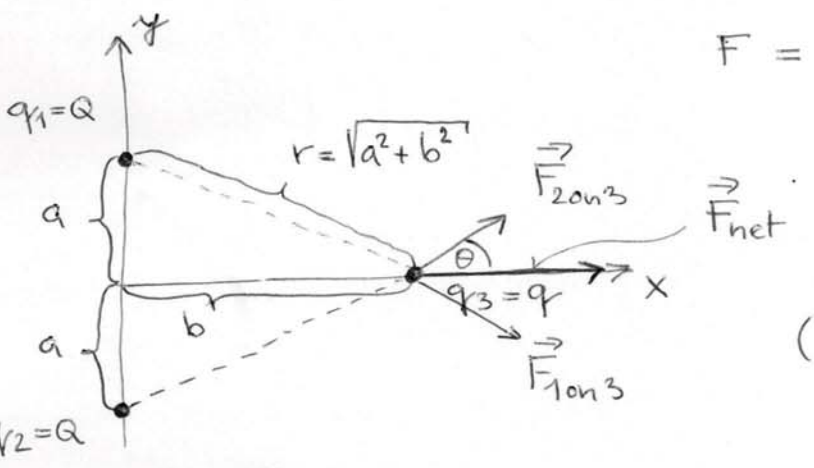
- Coulomb's law only gives force between two point charges
- From experiments, it follows that if two charges exert forces simultaneously on a third charge, the total force acting on that charge is the vector sum of the forces that the two charges would exert individually

$\downarrow$

Principle of superposition of forces that can be extended to an arbitrary number of charges

Ex 21.4: Two equal positive charges $q_1 = q_2 \equiv Q > 0$ are located at $[0, a]$ and $[0, -a]$. What is the force that the two charges q_1, q_2 exert on a third charge $q_3 \equiv q > 0$ located at $[b, 0]$?

(general)



$$F = |\vec{F}_{1on3}| = |\vec{F}_{2on3}| = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{(a^2+b^2)}$$

Force components

$$(F_{1on3})_x = F \cdot \cos\theta = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{(a^2+b^2)} \frac{b}{\sqrt{a^2+b^2}}$$

$$(F_{1on3})_y = -F \sin\theta = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{(a^2+b^2)} \frac{a}{\sqrt{a^2+b^2}}$$

symmetry $\Rightarrow (F_{2on3})_x = (F_{1on3})_x$

$(F_{2on3})_y = -(F_{1on3})_y$

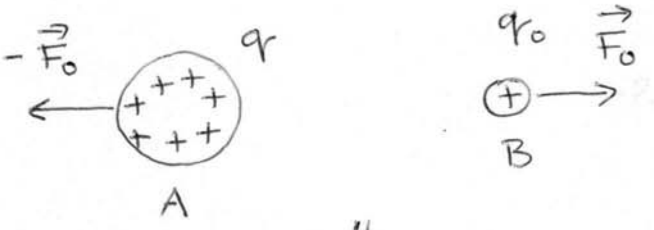
Total force $\vec{F}_{net}$

$$\boxed{(F_{net})_x = 2 \cdot (F_{1on3})_x = \frac{2}{4\pi\varepsilon_0} \frac{Q \cdot q \cdot b}{(a^2+b^2)^{3/2}} \qquad (F_{net})_y = 0}$$

$\Longleftarrow$

(?) What happens if $q_2 = -Q$?

Section 21.4 : Electric field and Electric Forces



A and B exert electric forces on each other

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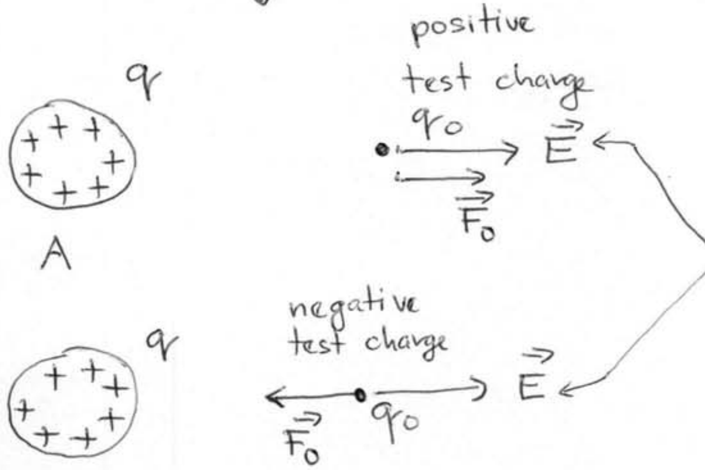


$\vec{E}$ … electric field set up by charged body A at the location of test charge q_0

$$\boxed{\vec{E} = \frac{\vec{F}_0}{q_0}} \quad [\vec{E}] = \frac{N}{C}$$

„The electric force on a charged body is exerted by the electric field created by other charged bodies"

- The electric field of body A exists everywhere in the neighborhood of A independently of the presence of test charge q_0
- Each charged body generates its own electric field that acts on other charged bodies in the neighborhood but not on itself → a body cannot exert a net force on itself !!!
- Electric field ≡ electric force per unit charge (units – $\frac{N}{C}$)

$\Downarrow$

If the field is known at a point, then the force exerted on a point charge q_0 at this point is $\vec{F} = q_0 \vec{E}$

$q_0 > 0 \Rightarrow \vec{F} \parallel \vec{E}$ and $q_0 < 0 \Rightarrow \vec{F} \parallel -\vec{E}$

- Analogy between electric and gravitational forces

$$\vec{F}_c = q_0 \vec{E} \Rightarrow \vec{E} = \frac{\vec{F}_c}{q_0}$$

q_0 ... point test charge

m_0 ... point test mass

$$\vec{F}_g = m_0 \vec{g} \Rightarrow \vec{g} = \frac{F_g}{m_0}$$

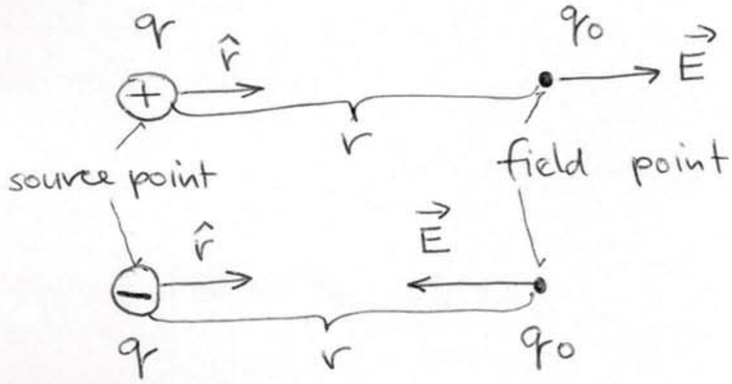
→ acceleration due to gravity can be viewed as <u>gravitational field</u>

- Field at a given point does not depend on the probe mass/charge but only on the generating mass/charge

- The above simple relationship between the force and the field holds only for <u>point test objects</u>

  ⇒ if the test object has <u>finite size</u>, field can <u>differ at different locations</u> within the object !!!

<u>Electric field of a point charge q</u>



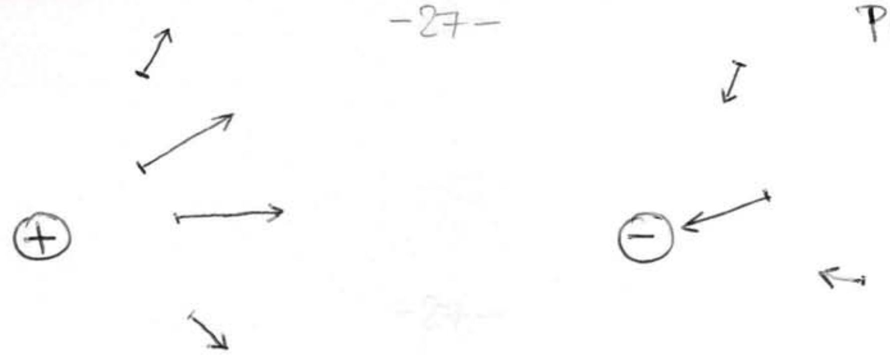
$$\vec{F}_{q \text{ on } q_0} = \frac{1}{4\pi\varepsilon_0} \frac{q q_0}{r^2} \hat{r}$$

$$\Downarrow$$

$$\vec{E} = \frac{\vec{F}_{q \text{ on } q_0}}{q_0} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

$\hat{r}$ - unit vector pointing from source point to field point

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$$



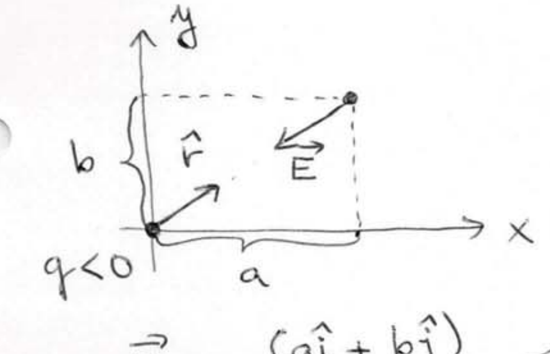
- Electric field of a point charge always points away from a positive charge ($\hat{r}$ - direction) and towards a negative charge ($-\hat{r}$ - direction)

- In general, electric field is an example of vector field that is typically inhomogeneous in space

$$\vec{E}(x,y,z) = E_x(x,y,z)\hat{i} + E_y(x,y,z)\hat{j} + E_z(x,y,z)\hat{k}$$

Ex 21.6 : (general) A point charge $q<0$ is located at the origin. Find the electric-field vector at a field point $[a,b]$.

$$|\vec{E}| = \frac{1}{4\pi\varepsilon_0} \frac{|q|}{(a^2+b^2)}$$



$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{(a^2+b^2)} \hat{r}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{(a\hat{i} + b\hat{j})}{\sqrt{a^2+b^2}}$$

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{(a^2+b^2)^{3/2}} \cdot (a\hat{i} + b\hat{j})}$$

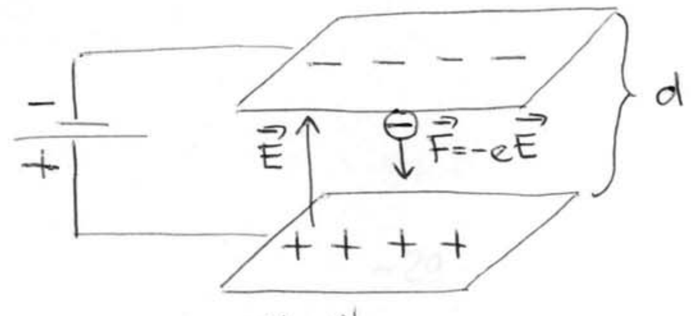
If $q = -8{,}0\,nC$, $a = 1{,}2\,m$, and $b = -1{,}6\,m$

$$\vec{E} = 9\cdot 10^9 \cdot \frac{(-8\cdot 10^{-9})}{(1{,}2^2 + (-1{,}6)^2)^{3/2}} \cdot (1{,}2\hat{i} - 1{,}6\hat{j}) =$$

$$= (-10{,}8\hat{i} + 14{,}4\hat{j})\ \frac{N}{C}$$

Ex 21.7:

(general) 100V



capacitor

Consider electric field $\vec{E}$ between two large parallel conducting plates connected to terminals of a 100 V battery and separated by distance $\underline{d}$. This Field is practically uniform with field vector $\vec{E} = E \cdot \hat{j}$

ⓐ If an electron is released from rest at the upper plate, what is its acceleration? Neglect the gravitational force.

$$\vec{F} = -e\vec{E} \Rightarrow F_y = -eE = m_e a_y$$

electron charge is negative

electron mass

$$\Downarrow$$

acceleration $\boxed{a_y = \frac{-eE}{m_e}}$ ← constant

For $E = 10^4 \frac{N}{C}$, $a_y = -1{,}76 \cdot 10^{15} \frac{m}{s^2}$

ⓑ What speed and kinetic energy does the electron acquire after traveling distance $\underline{d}$ to the lower plate?

Motion with constant acceleration:

$$\left.\begin{aligned} y(t) &= y_0 + v_{0y} t + \frac{1}{2} a_y t^2 \\ v(t) &= \frac{dy(t)}{dt} = v_{0y} + a_y t \end{aligned}\right\} \quad y_0 = 0, \ v_{0y} = 0$$

$$\Downarrow$$

$$y(t) = \frac{1}{2} a_y t^2$$

$$v(t) = a_y t$$

$$y(t_d) = d \Rightarrow d = \frac{1}{2} a_y t_d^2 \Rightarrow t_d = \sqrt{\frac{2d}{a_y}}$$

$$\boxed{v_d = a_y t_d = a_y \sqrt{\frac{2d}{a_y}}}$$

For $a_y = -1{,}76 \cdot 10^{+15} \frac{m}{s}$, $d = -1 \cdot 10^{-2} m$

$$v_d = -5{,}9 \cdot 10^6 \frac{m}{s}$$

Kinetic energy

$$K = \frac{1}{2} m_e v_d^2 = \frac{1}{2} m_e \left(\sqrt{2 d a_y}\right)^2 = \frac{1}{2} \cancel{m_e} \cdot 2d \cdot \frac{eE}{\cancel{m_e}} =$$

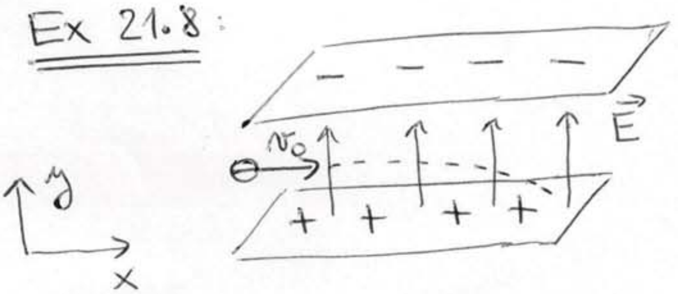
$$= eE \cdot d = |\vec{F}| \cdot d$$ → work done by constant electric field along the electron path

For $E = 10^4 \, \frac{N}{C}$, $d = 10^{-2}\, m$, $K \approx 1{,}6 \cdot 10^{-17}\, J$

(c) How much time is needed for the electron to travel distance $\underline{d}$?

$$t_d = \sqrt{\frac{2d}{a_y}}$$; for our specific conditions $t_d \approx 3{,}4 \cdot 10^{-9}\, s$

<u>Ex 21.8</u>:



What is the trajectory of an electron launched into the uniform vertical field $\vec{E}$ with an initial horizontal velocity v_0?

$$\vec{a} = \left(0 ; -\frac{eE}{m_e}\right) \Rightarrow \vec{v}(t) = \left(v_{0x} ; v_{0y} - \frac{eE}{m_e} t\right)$$

$$\Rightarrow \vec{r}(t) = \left(x_0 + v_{0x} t ; y_0 + v_{0y} t - \frac{1}{2} \frac{eE}{m_e} t^2\right)$$

Assuming $x_0 = 0$, $y_0 = 0$, $v_{0x} = v_0$, $v_{0y} = 0$, we obtain

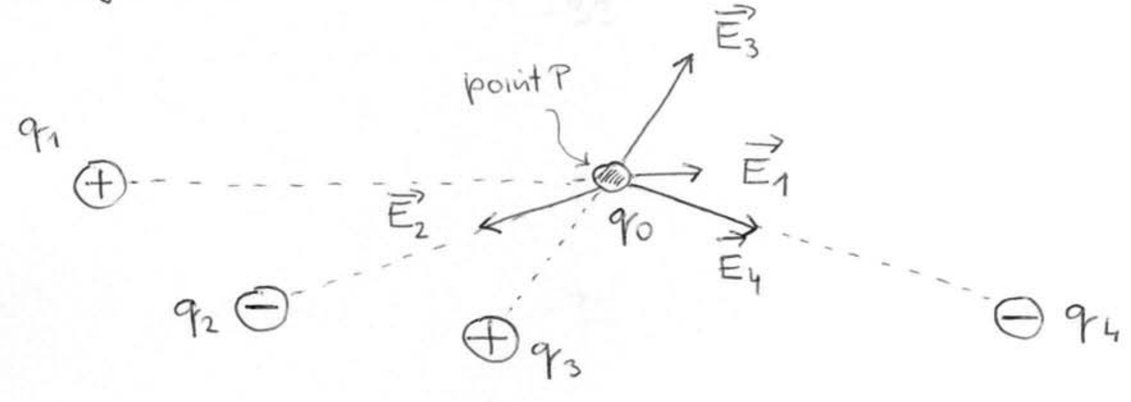
$$\vec{r}(t) = \Big(\underbrace{v_0 t}_{x} ; \underbrace{-\frac{1}{2} \frac{eE}{m_e} t^2}_{y}\Big)$$

$$t = \frac{x}{v_0} \Rightarrow y = -\frac{1}{2} \frac{eE}{m_e} t^2 = -\frac{1}{2} \frac{eE}{m_e v_0^2} x^2$$ → equation of a parabola

(?) What happens if the charge sign is reversed?

Section 21.5: Electric - Field Calculations

- In order to calculate the electric field caused by a distribution of charges, we use the superposition of electric forces



The total force acting on a test charge $\underline{q_0}$ located at point P due to all other point charges $q_1, q_2, q_3 \dots$ is given by:

$$\vec{F}_0 = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \dots = q_0\vec{E}_1 + q_0\vec{E}_2 + q_0\vec{E}_3 + q_0\vec{E}_4 + \dots$$

Thus, the total electric field $\underline{\vec{E}}$ at point P due to the distribution of charges can be expressed as:

$$\boxed{\vec{E} = \frac{\vec{F}_0}{q_0} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 + \dots}$$

total electric field $\equiv$ vector sum of individual electric fields

- In general, charge distribution can be continuous

$\Downarrow$

summation $\longrightarrow$ integration (summation over infinitezimally small charges)

charge Q $\longrightarrow$ charge density

Linear - λ ($\lambda = \frac{Q}{L}$ for uniform distribution)

surface - σ ($\sigma = \frac{Q}{A}$ for uniform distribution)

volume - ρ ($\rho = \frac{Q}{V}$ for uniform distrib.)

illustrate by pictures !!!

Ex 21.9: Calculate the electric field caused by two point charges $q_1 = Q$, $q_2 = -Q$, at locations a, b, c

$Q = 12\,nC$

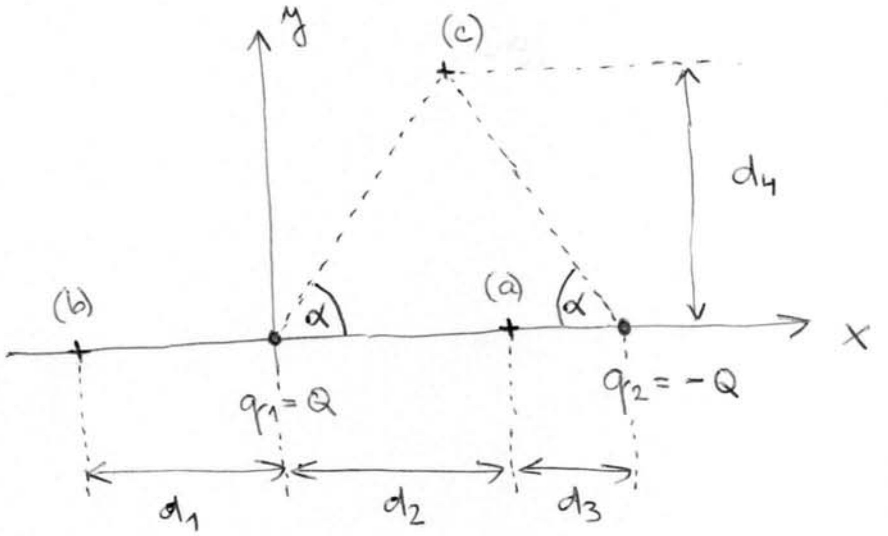
$d_1 = 4\,cm$

$d_2 = 6\,cm$

$d_3 = 4\,cm$

$d_4 = 12\,cm$

$Q > 0$



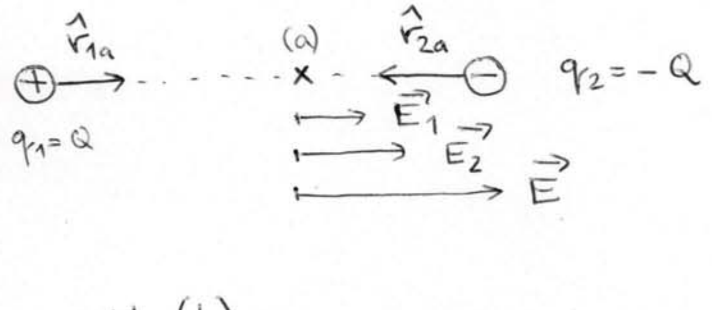
a) at point (a)

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{d_2^2}\hat{r}_{1a} \qquad \vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{(-Q)}{d_3^2}\hat{r}_{2a}$$

$$\hat{r}_{1a} = \hat{i} \qquad \hat{r}_{2a} = -\hat{i}$$

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{d_2^2}\hat{i} \qquad \vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{d_3^2}\hat{i}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{d_2^2} + \frac{1}{d_3^2}\right)\hat{i} = \left(9{,}75 \cdot 10^4 \frac{N}{C}\right)\hat{i}$$



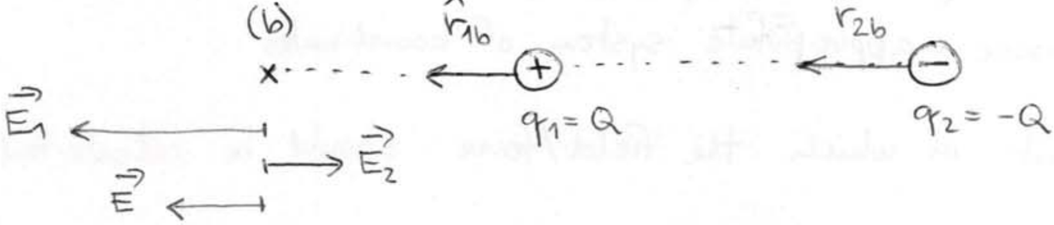
b) at point (b)

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{d_1^2}\hat{r}_{1b} \qquad \vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{(d_1+d_2+d_3)^2}\hat{r}_{2b}$$

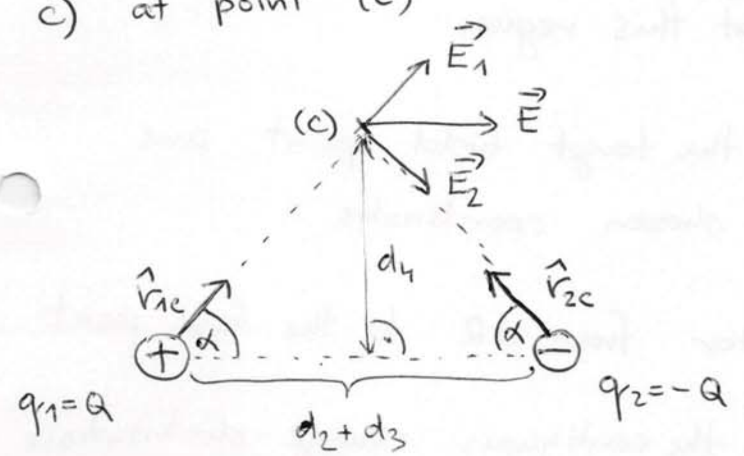
$$\hat{r}_{1b} = -\hat{i} \qquad \hat{r}_{2b} = -\hat{i}$$

$$\vec{E}_1 = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{d_1^2}\hat{i} \qquad \vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{(d_1+d_2+d_3)^2}\hat{i}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{d_1^2} + \frac{1}{(d_1+d_2+d_3)^2}\right)\hat{i} = \left(-6{,}2\cdot 10^4\ \frac{N}{C}\right)\hat{i}$$



c) at point (c)



distance $q_1 - (c)$ = distance $q_2 - (c)$ =

$$= \sqrt{\left(\frac{d_2+d_3}{2}\right)^2 + d_4^2}$$

$$\cos\alpha = \frac{(d_2+d_3)/2}{\sqrt{\left(\frac{d_2+d_3}{2}\right)^2 + d_4^2}}$$

– using problem symmetry, we can see that $|\vec{E}_1| = |\vec{E}_2| = E_0$

$$E_{1x} = E_{2x} = E_0 \cdot \cos\alpha \,, \quad E_{1y} = -E_{2y}$$

$$E_0 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\left[\left(\frac{d_2+d_3}{2}\right)^2 + d_4^2\right]}$$

$$E_x = E_{1x} + E_{2x} = 2E_0\cos\alpha = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{\left[\left(\frac{d_2+d_3}{2}\right)^2 + d_4^2\right]^{3/2}}\cdot\frac{(d_2+d_3)}{2} =$$

$$= 4{,}9\cdot 10^3\ \frac{N}{C}$$

$$E_y = E_{1y} + E_{2y} = 0$$

$$\Downarrow$$

Total electric field is horizontal

Problem-solving strategy

① Make up a drawing that clearly shows individual involved objects and choose appropriate system of coordinates

② Identify the point in which the field/force should be calculated

③ Choose a general region within the charged object(s) and express the value of local charge dQ at this region

④ Express the distance between the target field point and general local charge dQ using chosen coordinates.

⑤ Find direction of unit vector from dQ to the field point

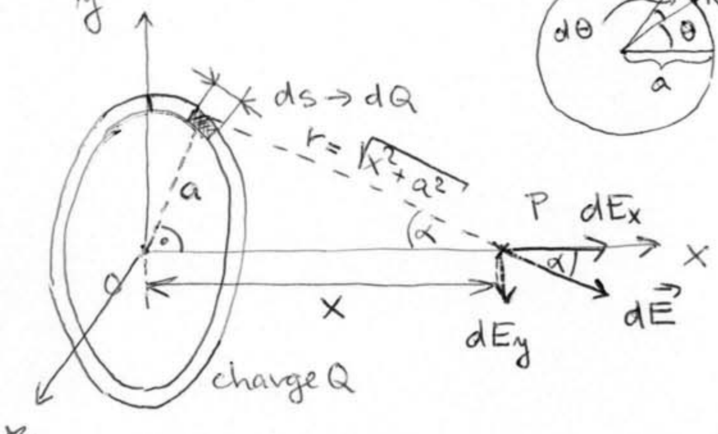
⑥ Carry out integration over the continuous charge distribution to find the total electric field at the target point.

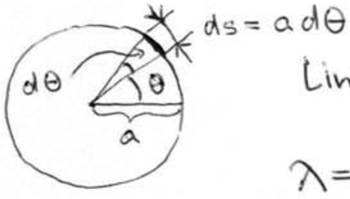
Use correct integration limits and make use of symmetry of the problem to simplify the solution.

Remember that electric field is a VECTOR →

use vector addition/integration

Ex 21.10 : A ring-shaped conductor with radius a carries a total charge Q uniformly distributed around it. Find the electric field $\vec{E}$ at point P that lies on the ring axis at a distance x from its center.





Linear charge density

$$\lambda = \frac{Q}{2\pi a}$$

$$\Downarrow$$

$$dQ = \lambda\, ds = \frac{Q\, ds}{2\pi a}$$

$$|d\vec{E}| = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{x^2+a^2} \longrightarrow \begin{aligned} dE_x &= |d\vec{E}|\cdot\cos\alpha = |d\vec{E}|\frac{x}{\sqrt{x^2+a^2}} \\ dE_y &= |d\vec{E}|\cdot\sin\alpha = |d\vec{E}|\cdot\frac{a}{\sqrt{x^2+a^2}} \end{aligned}$$

Using symmetry, we see that for two segments ds of the ring lying on the opposite sides of the ring, dE_x is identical but dE_y has opposite sign → y-components of the field cancel, the total field only has x-component E_x

$$E_x = \int_{ring} dE_x = \int_{ring} \frac{1}{4\pi\varepsilon_0}\frac{dQ}{x^2+a^2}\underbrace{\frac{x}{\sqrt{x^2+a^2}}}_{\cos\alpha}$$

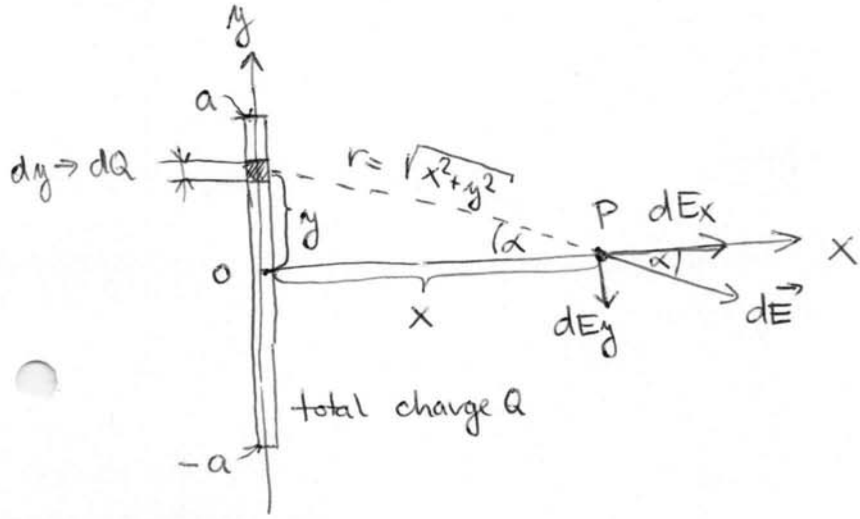
x is constant for all points on the ring → we can take it outside of the integral

$$\Downarrow$$

$$\vec{E} = E_x\cdot\hat{\imath} = \frac{1}{4\pi\varepsilon_0}\frac{x}{(x^2+a^2)^{3/2}}\overbrace{\int_{ring} dQ}^{\text{total charge } Q}\,\hat{\imath} = \frac{1}{4\pi\varepsilon_0}\frac{Qx}{(x^2+a^2)^{3/2}}\hat{\imath}$$

For $x \gg a$, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{x^2}\hat{\imath}$ → point-charge-like field (shape of the object does not matter)

Ex 21.11 Find the field of a line with total length $\underline{2a}$ and positive electric charge $\underline{Q}$ distributed uniformly along it at point P located on the axis perpendicular to the line, going through the line center, at distance $\underline{x}$



linear charge density

$$\lambda = \frac{Q}{2a} = \text{constant}$$

$$\Downarrow$$

$$dQ = \lambda dy = \frac{Q}{2a}\,dy$$

$$|d\vec{E}| = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r^2} = \frac{Q}{4\pi\varepsilon_0}\frac{dy}{2a\,(x^2+y^2)} \rightarrow dE_x = |d\vec{E}|\cos\alpha = |d\vec{E}|\frac{x}{\sqrt{x^2+y^2}}$$

$$dE_y = |d\vec{E}|\sin\alpha$$

Using symmetry (as with the ring), we see that dE_y components of symmetrically located line segments cancel → the total field $\vec{E}$ only has E_x - component

$$dE_x = \frac{Q}{4\pi\varepsilon_0}\frac{x\,dy}{2a\,(x^2+y^2)^{3/2}}$$

$$E_x = \int_{line} dE_x = \frac{Q}{4\pi\varepsilon_0}\frac{x}{2a}\int_{line}\frac{dy}{(x^2+y^2)^{3/2}} = \frac{Q}{4\pi\varepsilon_0}\frac{x}{2a}\int_{-a}^{a}\frac{dy}{(x^2+y^2)^{3/2}}$$

– using substitution $y = x\cdot\tan\alpha$, $dy = \frac{x}{\cos^2\alpha}d\alpha$, we can evaluate the field as

$$\boxed{E_x = \frac{Q}{4\pi\varepsilon_0}\frac{\cancel{x}}{\cancel{2a}}\cdot\frac{\cancel{2a}}{x^{\cancel{2}}\sqrt{a^2+x^2}} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{x\sqrt{a^2+x^2}}}$$

Asymptotic forms of the field

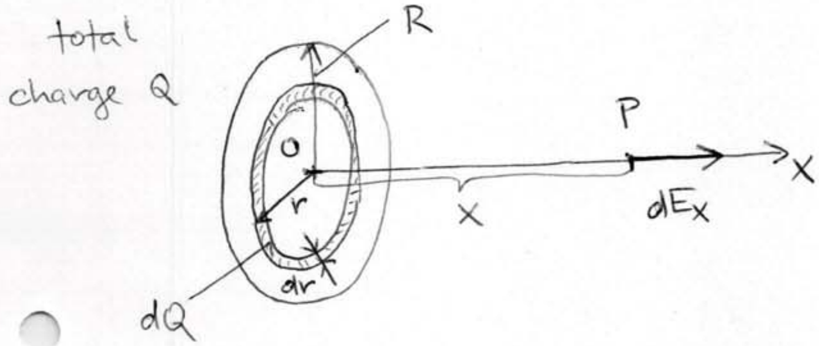
a) $x \gg a \rightarrow \vec{E} = \frac{Q}{4\pi\varepsilon_0} \frac{1}{x^2} \hat{i}$ point-charge-like field (compare with ring)

b) $a \gg x \rightarrow \vec{E} = \frac{Q}{4\pi\varepsilon_0} \frac{1}{x \cdot a} \hat{i} = \frac{\lambda}{2\pi\varepsilon_0 x} \hat{i}$

- field decreases with distance slower than for a point charge

• In a general case, both $E_x \neq 0$, $E_y \neq 0$

Ex 21.12: Find the field of a disk of radius R with a uniform positive surface charge density σ at a point P along the axis of the disk at a positive distance x from its center.



surface charge density
$\sigma = \frac{Q}{\pi R^2} = \text{constant}$

Disk ≡ system of concentric rings with radius r varying from 0 to R

↓

each of the rings carries charge $dQ = dA \cdot \sigma$ where dA is the ring area

Assuming ring inner radius r and outer radius $r+dr$

$$dA = \pi(r+dr)^2 - \pi r^2 = \cancel{\pi r^2} + 2\pi r\,dr + \underbrace{\cancel{\pi dr^2}}_{\text{negligible}} - \cancel{\pi r^2} \approx 2\pi r\,dr$$

$$\boxed{dQ = 2\pi\sigma r\,dr}$$

Using the solution of 21.10, we can write for the field of a ring with radius $\underline{r}$

$$dE_x = \frac{1}{4\pi\varepsilon_0}\,\frac{dQ\,x}{(x^2+r^2)^{3/2}} = \frac{1}{4\pi\varepsilon_0}\,\frac{(2\pi\sigma r\,dr)\cdot x}{(x^2+r^2)^{3/2}}$$

$$\Downarrow$$

x is constant for all rings!!!

$$E_x^{disk} = \int_{\text{all rings}} dE_x = \int_0^R \frac{1}{4\pi\varepsilon_0}\,\frac{(2\pi\sigma r\,dr)\cdot x}{(x^2+r^2)^{3/2}} = \frac{\sigma x}{2\varepsilon_0}\int_0^R \frac{r\,dr}{(x^2+r^2)^{3/2}}$$

change of variables : $z = x^2 + r^2 \Rightarrow dz = 2r\,dr$

$$\Downarrow$$

$$E_x^{disk} = \frac{\sigma x}{2\varepsilon_0}\int_{x^2}^{x^2+R^2}\frac{dz}{2z^{3/2}} = \frac{\sigma x}{2\varepsilon_0}\left[-\frac{1}{\sqrt{x^2+R^2}} + \frac{1}{x}\right] =$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{1}{\sqrt{1+\left(\frac{R}{x}\right)^2}}\right]$$

Asymptotic forms of the field

a) $x \gg R \longrightarrow E_x^{disk} \approx \frac{\sigma}{2\varepsilon_0}\left[1-\left(1-\frac{1}{2}\left(\frac{R}{x}\right)^2\right)\right] =$

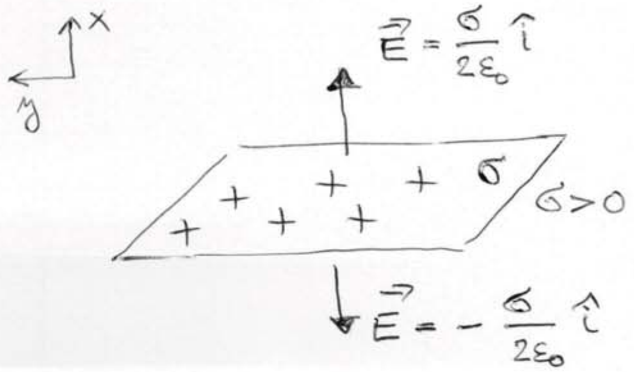
$$= \frac{\sigma}{4\varepsilon_0}\,\frac{R^2}{x^2} = \frac{Q}{4\varepsilon_0\pi R^2}\,\frac{R^2}{x^2} = \frac{1}{4\pi\varepsilon_0}\,\frac{Q}{x^2}$$

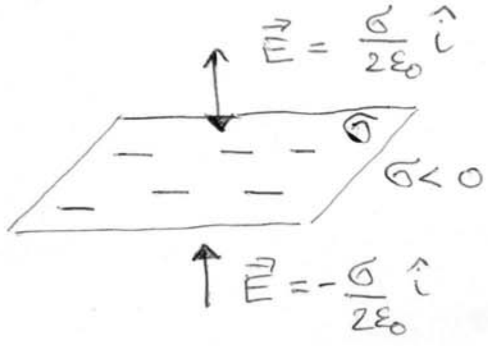
point charge

b) $R \gg x \longrightarrow E_x^{disk} = \frac{\sigma}{2\varepsilon_0}$ = constant

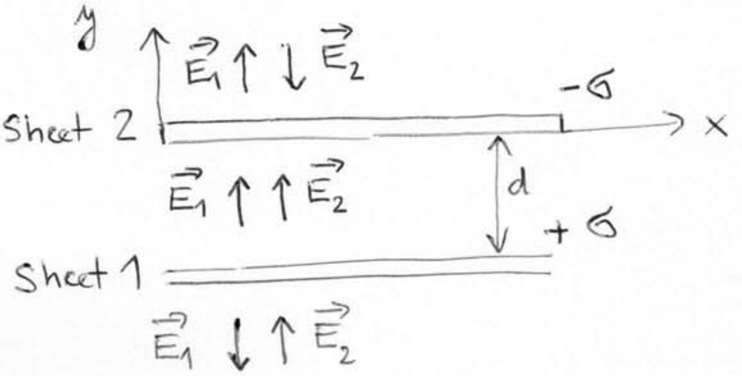
(infinite plane)

uniform electric field independent of the distance from the disk





Ex 21.13: Find the field of two oppositely charged, parallel infinite plane sheets separated by a distance d

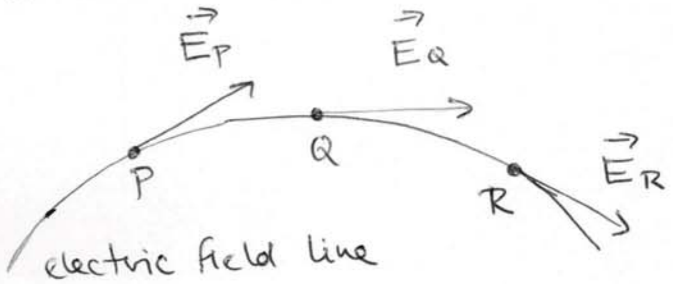


field superposition

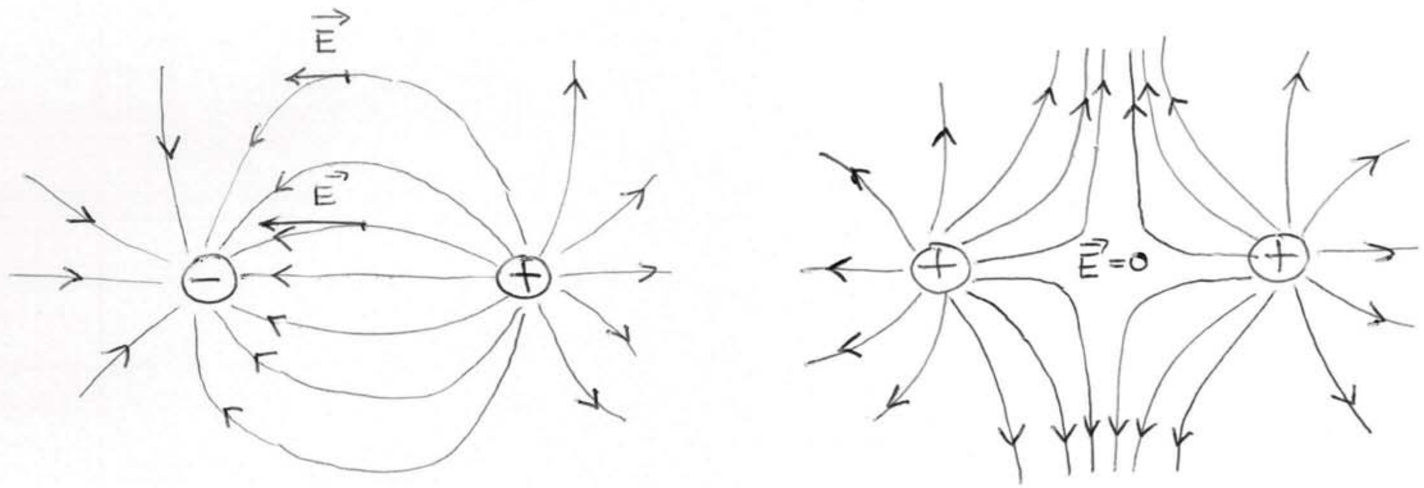
$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \begin{cases} 0 & \text{above sheet 2} \\ \dfrac{\sigma}{\varepsilon_0}\,\hat{j} & \text{between the sheets} \\ 0 & \text{below sheet 1} \end{cases}$$

Section 21.6: Electric Field Lines

- Field lines help visualizing electric field
- An electric field line is an imaginary curve drawn through a region of space so that its tangent at any point is in the direction of the electric field vector at that point

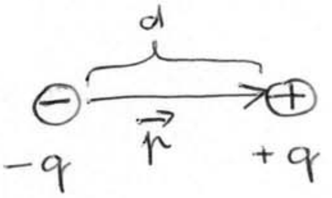


Electric field is unique in each point without a charge
→ only one field line can pass through any point of the field
⇓
Field lines never intersect !!!



- Field lines are close together where the field is strong and farther apart where the field is weaker
- Arrows on field lines indicate field direction
- Field lines are NOT the curves of the constant field magnitude !!!

Section 21.7: Electric Dipoles



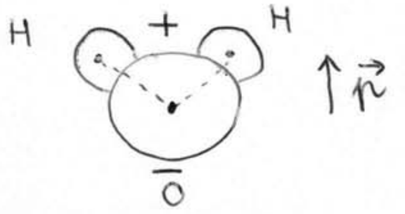
An electric dipole is a pair of point charges of equal magnitude q and opposite sign separated by a distance d

Definition: electric dipole moment $\vec{p}$

magnitude $|\vec{p}| = q \cdot d$ + direction from ⊖ to ⊕ charge

(EG) water molecule - H_2O : no net charge

×



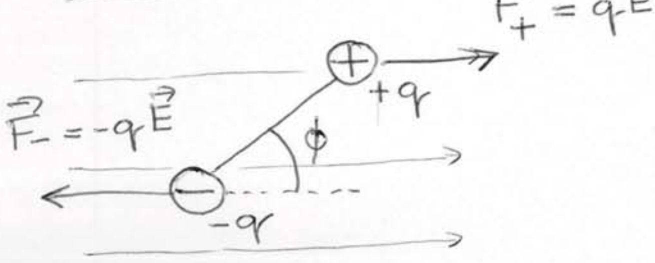
centers of mass of the positive and negative charges are displaced by distance d_{H_2O}

⇓

a dipole is formed; this dipole is permanent, as opposed to induced dipoles we discussed earlier → it does not need electric field to exist

- permanent dipole makes water an excellent solvent for charged particles (ions, proteins, nucleic acids ...) that interact with oppositely charged end of water molecules

Force and torque on an electric dipole



Consider a dipole in a uniform electric field

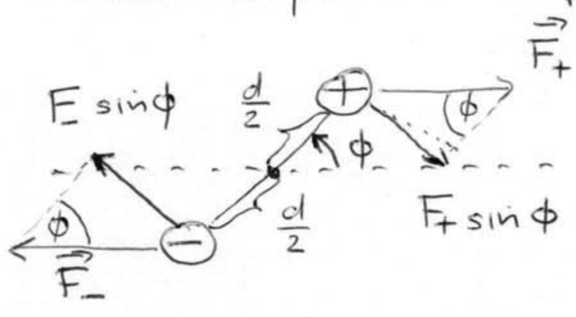
Net force acting on the dipole in the field

$$\vec{F} = \vec{F}_+ + \vec{F}_- = (q-q)\vec{E} = 0$$

↓

Center of mass of a dipole does not move in a uniform electric field

Net torque with respect to the dipole center



$$|\vec{\tau}| = 2 \cdot \frac{d}{2}\, qE \cdot \sin\phi =$$
$$= (qE) \cdot (d \sin\phi)$$

$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$ where $\vec{\tau}_{1,2}$ have the same magnitude and rotation sense (they point into the board)

$E = |\vec{E}|$
$\tau = |\vec{\tau}|$

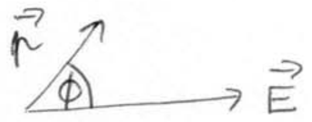
$$|\vec{\tau}| = (qd)\, E \sin\phi = p \cdot E \cdot \sin\phi$$

$\downarrow$

this is the magnitude of vector product $\vec{p} \times \vec{E}$

$\Downarrow$

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$



$\otimes\, \vec{\tau}$

The torque is largest for $\phi = \pi/2$

Potential energy of an electric dipole

- When electric dipole changes direction in an external electric field, the electric-field torque does work on it
- Work through infinitezimal rotation $d\phi$

$$dW = \tau\, d\phi$$

The torque is in the direction of decreasing ϕ

$\longrightarrow dW = (-pE \sin\phi)\, d\phi$ — positive work is done while decreasing ϕ

$$W = \int_{\phi_1}^{\phi_2} (-pE \sin\phi)\, d\phi = pE\cos\phi_2 - pE\cos\phi_1 = U_1 - U_2$$

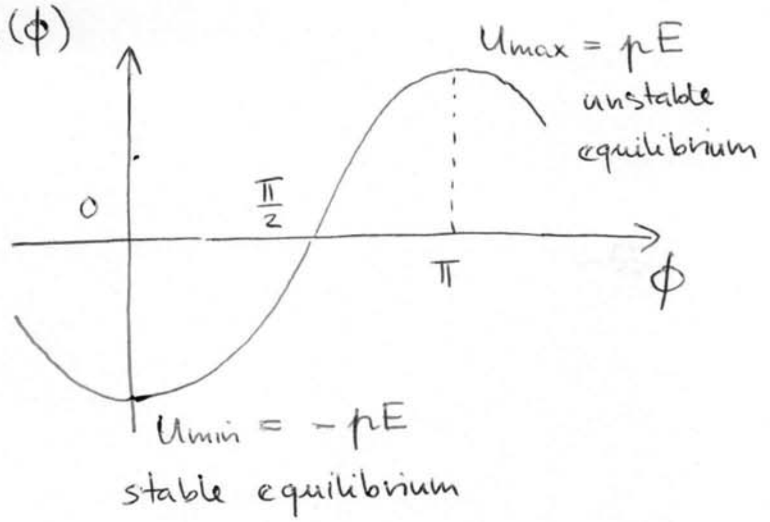
$\Downarrow$

$$U(\phi) = -pE\cos\phi$$

$\Downarrow$

$$\boxed{U = -\vec{p} \cdot \vec{E}}$$

potential energy of a dipole in an electric field

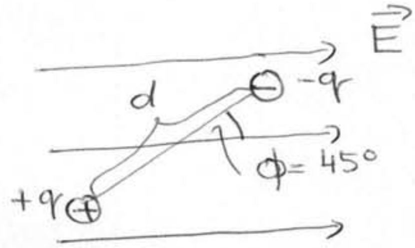


- potential energy is minimal for $\phi = 0$

$\Downarrow$

dipoles orient themselves parallel to the applied external electric field

(EG) Torque and potential energy for a molecular dipole



$|\vec{E}| = 5 \cdot 10^5 \frac{N}{C}$

$q = 1{,}6 \cdot 10^{-19}\,C$

$d = 0{,}125\,nm$

Torque $\tau = q \cdot d \cdot E \cdot \sin(\pi - \phi) = 7{,}1 \cdot 10^{-24}\,N \cdot m$

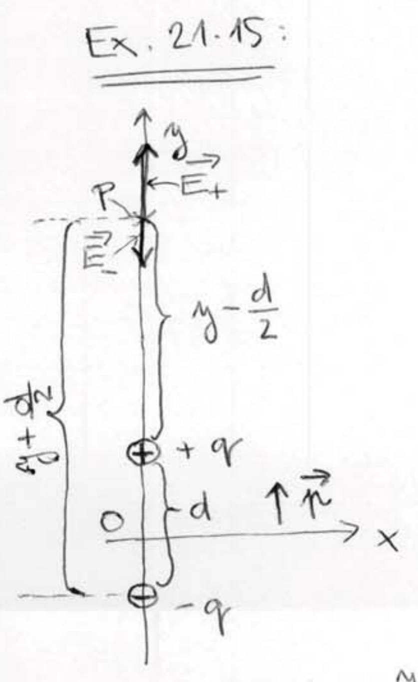
Potential energy $U = -qdE\cos(\pi - \phi) = 7{,}1 \cdot 10^{-24}\,J$

$\downarrow$

very small values characteristic of molecular scale

Field of an electric dipole

Ex. 21.15:



$$\vec{E}_p = \vec{E}_+ + \vec{E}_- = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(y-\frac{d}{2})^2} - \frac{1}{(y+\frac{d}{2})^2}\right]\hat{j} =$$

$$= \frac{q}{4\pi\varepsilon_0 y^2}\left[\left(1-\frac{d}{2y}\right)^{-2} - \left(1+\frac{d}{2y}\right)^{-2}\right]\hat{j}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2} + \dots$$

for $|x| < 1$

$\Downarrow$

$$\left(1-\frac{d}{2y}\right)^{-2} \approx 1 + \frac{d}{y} \; ; \; \left(1+\frac{d}{2y}\right)^{-2} \approx 1 - \frac{d}{y}$$

$y \gg \frac{d}{2} \longrightarrow$ higher order terms are neglected

$$\vec{E}_p \approx \frac{q}{4\pi\varepsilon_0 y^2} \cdot \left[1 + \frac{d}{y} - \left(1 - \frac{d}{y}\right)\right] =$$

$$= \frac{q}{4\pi\varepsilon_0 y^2} \, \frac{2d}{y} = \frac{qd}{2\pi\varepsilon_0 y^3} = \frac{p}{2\pi\varepsilon_0 y^3}$$

- dipole electric field decays with distance as $\frac{1}{r^3}$, faster than for a point charge with $\frac{1}{r^2}$ dependence