Introduction to Topology
Quiz 1, February 23rd, 2016

Name and Last Name: ______________________________

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1. Assume $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences of real numbers. Assume $(a_n)_{n\in\mathbb{N}}$ converges to a number $a \in \mathbb{R}$, and that the difference sequence $(a_n - b_n)_{n\in\mathbb{N}}$ converges to 0. Prove that $(b_n)_{n\in\mathbb{N}}$ is also convergent, and that it converges to the same number $a \in \mathbb{R}$.

Solution: We know the following:

$$\forall\alpha > 0, \exists A \in \mathbb{N}, \ \forall a > A, \ |a_n - a| < \alpha$$
$$\forall\beta > 0, \exists B \in \mathbb{N}, \ \forall b > B, \ |a_n - b_n| < \beta$$

Given any $\epsilon > 0$, take $\alpha = \beta = \epsilon/2$. We have two integers A and B corresponding to the α and β we have just chosen. Let $N = \max\{A, B\}$. Then for every $n > N$ we have

$$|b_n - a| = |b_n - a_n + a_n - a| \leq |b_n - a_n| + |a_n - a| < \epsilon$$

i.e. $(b_n)_{n\in\mathbb{N}}$ converges to a.

2. Show that the infinite union

$$\bigcup_{n\in\mathbb{N}} [-1, \sin(n)]$$

is a connected subset in $\mathbb{R}$.

Solution: Let $X = \bigcup_{n\in\mathbb{N}} [-1, \sin(n)]$. We need to show that

$$\forall a, b, x \in \mathbb{R}, \ \text{ if } a, b \in X \text{ and } x \in [a, b] \text{ then } x \in X$$

So, take $a, b, x \in \mathbb{R}$ such that $a, b \in X$ and $a \leq x \leq b$. Since $a, b \in X$, there are two integers $n, m \in \mathbb{N}$ such that $a \in [-1, \sin(n)]$ and $b \in [-1, \sin(m)]$. Then

$$x \in [a, b] \subseteq [-1, \max\{\sin(n), \sin(m)\}] = [-1, \sin(n)] \cup [-1, \sin(m)]$$

and therefore, $x \in [-1, \sin(n)] \cup [-1, \sin(m)]$.