Introduction to Topology Quiz 2, March 15th, 2016

Name:____________________

Number:__________________

1. (a) Calculate the union $\bigcup_{n\in\mathbb{N}}\left[-1, \frac{n}{1+n}\right]$. Briefly explain your answer.

Solution: The sequence $\frac{n}{n+1}$ can also be written as $1 - \frac{1}{n+1}$. So the elements $\frac{n}{n+1}$ go to 1 increasingly as n goes to ∞. Our claim is that $\bigcup_{n\in\mathbb{N}}\left[-1, \frac{n}{n+1}\right)$ is $[-1, 1)$. It is clear that 1 is not in the union since it is not in any of the sets $[-1, n/(n+1))$. On the other hand, if we take any $-1 \leq \alpha < 1$, since $\frac{n}{n+1}$ approaches to 1 on the left, there is going to be a $N \in \mathbb{N}$ such that for every $n \geq N$ we would have $\alpha \leq \frac{n}{n+1} < 1$. In other words $\alpha \in [-1, N/(N+1))$, i.e.

$$\bigcup_{n=0}^{\infty}\left[-1, \frac{n}{n+1}\right) = [-1, 1)$$

(b) Calculate the intersection $\bigcap_{\epsilon\in[0,\infty]}[-1, 1+\epsilon)$. Briefly explain your answer.

Solution: We claim that the answer is

$$\bigcap_{\epsilon\in[0,\infty)}[-1, 1+\epsilon) = [-1, 1]$$

The fact that 1 is in this set is obvious since $1 \in [-1, 1+\epsilon)$ for every $\epsilon \in [0, \infty)$. On the other hand if we take any $1 < \alpha$ we see that $\alpha \notin \left[-1, 1 + \frac{\alpha-1}{2}\right)$.

2. Let $f(x, y)$ be any real-valued function of two variables such that $f(x, y) \neq 0$ for every $(x, y) \in \mathbb{R}^2$. Define a relation $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting

$$S = \{((x, y), (x', y')) \in \mathbb{R}^2 \times \mathbb{R}^2 |\ f(x, y)f(x', y') \geq 0\}$$

Show that S is an equivalence relation.

Solution: In the version I gave I forgot to add $f(x, y) \neq 0$. So, if you got the reflexivity and symmetricity right, you'll get full points.

i) Reflexivity: for every $(x, y) \in \mathbb{R}^2$, we have $f(x, y)f(x, y) = f(x, y)^2 > 0$ since every square real number is either 0 or positive.

ii) Symmetricity: for every $(x, y), (x', y') \in \mathbb{R}^2$, we have $f(x, y)f(x', y') \geq 0$ if and only if $f(x', y')f(x, y) \geq 0$ since the multiplication in $\mathbb{R}$ is commutative.

iii) Transitivity: for every $(x, y), (x', y'), (x'', y'') \in \mathbb{R}^2$, if $f(x, y)f(x', y') \geq 0$ and $f(x', y')f(x'', y'') \geq 0$ then $f(x, y)f(x', y')^2 f(x'', y'') \geq 0$. Since $f(x'y')^2 \neq 0$, we see that $f(x, y)f(x'', y'') \geq 0$.