Introduction to Topology Quiz 4, May 2nd, 2016

Name:____________________

Number:__________________

1. Consider the square determined by the points $(1,1), (-1,1), (1,-1)$ and $(-1,1)$ on the xy-plane. Consider the region Ω inside the square without the boundary. Let us use the ordinary distance function

$$d((x,y),(a,b)) = \sqrt{(x-a)^2 + (y-b)^2}$$

(a) For each point $(x,y) \in \Omega$, find an explicit expression for a real number $\delta > 0$ such that

$$B_\delta(x,y) = \{(a,b) \in \mathbb{R}^2 | \ d((x,y),(a,b)) < \delta\}$$

lies completely inside Ω. [Hint: What is the shape of the open ball $B_\delta(x,y)$? Calculate the distances of (x,y) to all sides of the square. How does this help to find δ?]

Solution: The shape of the open ball centered at a point (x,y) with radius δ with respect to the Euclidean metric is a disk centered at (x,y) with radius δ. So, given a point (x,y) in our square, its distance to the sides of the square are $|1+x|$, $|1-x|$, $|1+y|$ and $|1-y|$. The δ we are looking for is the minimum of all of these distances

$$\delta = \min\{|1+x|, |1-x|, |1+y|, |1-y|\}$$

(b) Show that Ω is open.

Solution: We already showed for all (x,y) in our square there exists $\delta > 0$ such that $B_\delta(x,y)$ lies completely in our square. This already proves that the set is open.

(c) What if I changed my distance function on $\mathbb{R}^2$? The ball you considered uses the ordinary distance. If we use the taxicab distance $\ell((x,y),(a,b)) = |x-a| + |y-b|$, what is the shape of $B_\delta(x,y)$? In that case, find an explicit expression for $\delta > 0$ such that $B_\delta(x,y) \subseteq \Omega$.

Solution: The open ball centered at (x,y) with radius δ with respect to the taxi-cab distance is the square whose corners are at points $(x-\delta,y)$, $(x,y+\delta)$, $(x+\delta,y)$ and $(x,y-\delta)$. This square lies completely in the disk centered at (x,y) of radius δ. So, the same δ works in this case too.

$$\delta = \min\{|1+x|, |1-x|, |1+y|, |1-y|\}$$

2. Let (X, d) be a discrete metric space, i.e. $d(x, y) = 1$ whenever $x \neq y$ and $d(x, y) = 0$ whenever $x = y$. Assume (Y, g) is another metric space and let $f : X \to Y$ is an arbitrary function. Show that f is continuous. [Hint: Consider a convergent sequence in (X, d). What happens when a sequence is convergent in a discrete metric space?]

Solution: In a discrete metric space a sequence (x_n) is convergent if and only if it is eventually constant. That is, (x_n) is converges to an element a iff there is an index N such that $x_n = a$ for all $n \geq N$. But then $f(x_n) = f(a)$ for the same index n. Then $g(f(x_n), f(a)) = 0 < \epsilon$ for every $\epsilon > 0$. In other words, $(f(x_n))$ converges to $f(a)$. Since (x_n) was arbitrary, f must be continuous.