Mathematical preliminaries

FIZ102E: Electricity & Magnetism

Yavuz Ekşi

İTÜ, Fizik Müh. Böl.

Contents

Taylor series expansion

A function $f(x)$ can be expanded into a Taylor series as

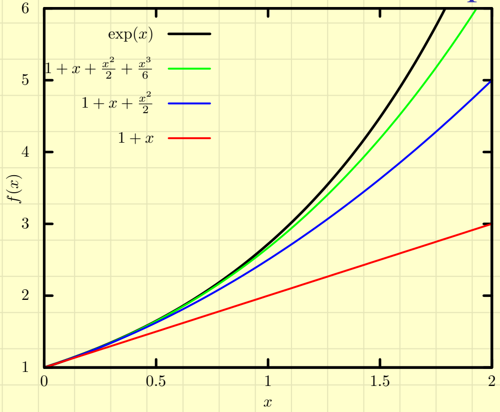
$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots \tag{1}$$

Ex: It can be shown that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{2}$$

(show it).

Exponential function



See that as terms are added to the series it becomes a better representation of the function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

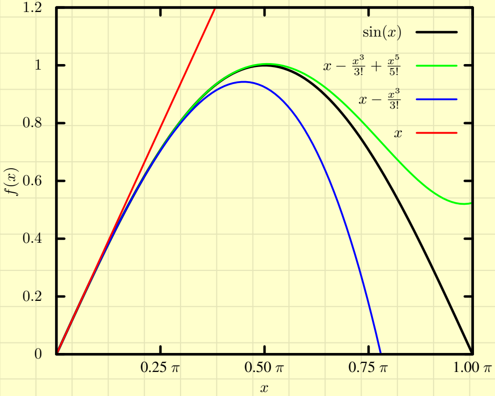
sinus function

It can be shown that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{3}$$

(show it).

sinus function



See that as terms are added to the series it becomes a better representation of the function

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

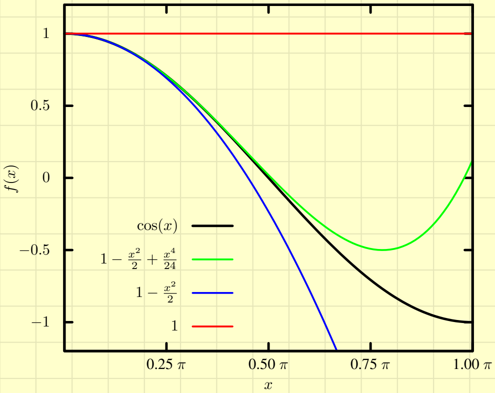
cosinus function

It can be shown that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{4}$$

(show it).

cosinus



See that as terms are added to the series it becomes a better representation of the function

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Attention!

- The series converge if $|x| < 1$.
- x is measured in radians, not degrees.

De Moivre's formula

Recall: Eqn. (2):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Plug $x = i\theta$ to this equation where $i^2 = -1$ to show that

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{5}$$

(show it).

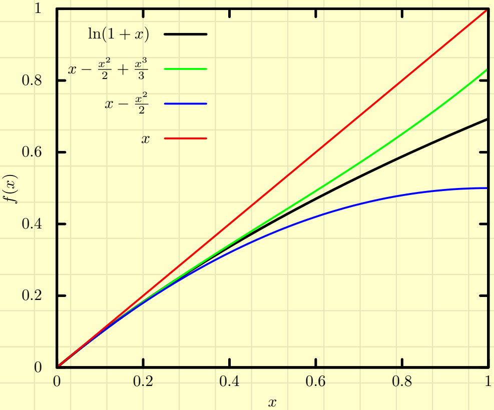
$\ln(1+x)$

Just another expansion we will need is

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots \tag{6}$$

(show it).

$\ln(1 + x)$



See that as terms are added to the series it becomes a better representation of the function

$$\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots$$

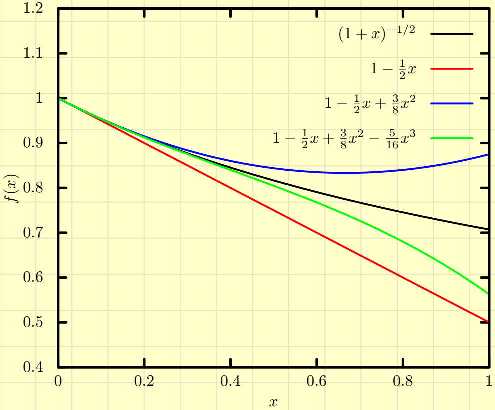
Binomial expansion

- As you well know $(1+x)^2 \equiv 1 + 2x + x^2$
- and $(1+x)^3 \equiv 1 + 3x + 3x^2 + x^3$
- What is the expansion for $(1+x)^n$? Can we have an expansion for non-integer n?
- By using Taylor series given in Eqn. (1) we obtain

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \qquad (7)$$

- Interestingly, the expansion is valid even when n is non-integer, but it has infinite terms then.
- Here x can be negative or positive but the series converges if $|x| < 1$.

$$(1+x)^{-1/2}$$



See that as terms are added to the series it becomes a better representation of the function

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 \cdots$$

Outline

Small x approximation

If $x \ll 1$ then higher order terms in the series are even smaller and can be neglected:

- Eqn. (2) implies
$$e^x \simeq 1 + x, \qquad (x \ll 1); \tag{8}$$

- Eqn. (3) implies
$$\sin(x) \simeq x, \qquad (x \ll 1); \tag{9}$$

- Eqn. (4) implies
$$\cos(x) \simeq 1 - \frac{x^2}{2}, \qquad (x \ll 1); \tag{10}$$

- Eqn. (6) implies
$$\ln(1 + x) \simeq x, \qquad (x \ll 1); \tag{11}$$

- Eqn. (7) implies
$$(1 + x)^n \simeq 1 + nx, \qquad (x \ll 1); \tag{12}$$

Check the figures again that these approximations are sufficient for $x \ll 1$.

Ex: Kinetic energy

Of course, $(1+x)^n$ under the condition that $x \ll 1$ is a number very close to 1. But sometimes we need a better approximation than that. This is where we use the approximations above.
In Einstein's special relativity the kinetic energy of a particle of mass m is

$$E_K = (\gamma - 1)mc^2, \quad \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{13}$$

Here $c = 3 \times 10^8$ m/s is the speed of light. In Newtonian mechanics the kinetic energy is

$$E_K^{NR} = \frac{1}{2}mv^2 \tag{14}$$

Ex: Kinetic energy

How are these theories related then?

- Experiment tells us that Newton's theory breaks down at speeds close to the speed of light whereas Einstein's theory remains valid.
- Remembering that the industrial revolution relied on Newton's theory, we know it should be valid at small speeds.
- So we expect that Einstein's theory should reduce to Newton's theory for small speeds; otherwise we would have two theories for this regime.
- Of course both equations give $E_K = 0$ for $v = 0$, but this is trivial. Let us show by series expansion that the Einstein's theory reduce to Newtonian theory for $v \ll c$.

Ex: Kinetic energy

Expanding $\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ into series

$$\gamma \simeq 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots \tag{15}$$

and plugging into Eqn. (13), $E_K = (\gamma - 1)mc^2$, we obtain

$$E_K \simeq \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \cdots \tag{16}$$

This explains why Newton's formula given in Eqn. (14) produces correct results at small speeds.

Ex: Kinetic energy

In order to simplify the comparison we define dimensionless quantities $\epsilon \equiv \frac{E_K}{mc^2}$ and $\beta \equiv \frac{v}{c}$. Accordingly $E_K = (\gamma - 1)mc^2$ becomes

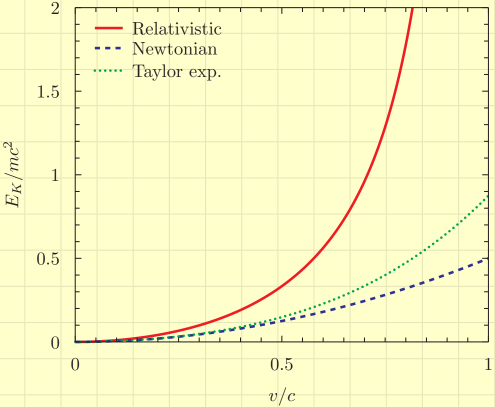
$$\epsilon = \frac{1}{\sqrt{1-\beta^2}} - 1 \tag{17}$$

and $E_K = \frac{1}{2}mv^2$ becomes

$$\epsilon = \frac{1}{2}\beta^2 \tag{18}$$

And the Taylor expansion becomes

$$\epsilon = \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 \cdots \tag{19}$$



The continuous curve (red) stands for Einstein's exact result, dashed curve (blue) Newtonian result and dashed dotted (green) curve shows the effect of the first correction term added to Newtonian theory.

All three curves coincide at small speeds ($\beta \ll 1$) as expected. At speeds close to the speed of light they differ and we have to refer to the experiment to see which model is correct. Experiments verify Einstein's formula and rejects Newton's formula at high speeds. Yet Newton's mechanics, as a theory with known limits, is still a good theory that we have to learn.

Exercises:

Expand these up to 3 terms

- $\frac{1}{(1+\epsilon)^2}$
- $\frac{1}{(1+\epsilon)^{3/2}}$
- $\frac{1}{(1+\epsilon)^{1/2}}$
- $\frac{1}{(1+\epsilon)^{1/2}} - \frac{1}{(1-\epsilon)^{1/2}}$
- (For $r > a$) $(r^2 + a^2 - 2ar\cos\theta)^{-1/2}$

Geometrical concepts

You most likely know the material presented in here from your high school education. But you should be fluent with them so that you can focus on the new concepts rather than being hindered.

Outline

Circumference and surface area of a circle

The circumference of a circle with radius a is

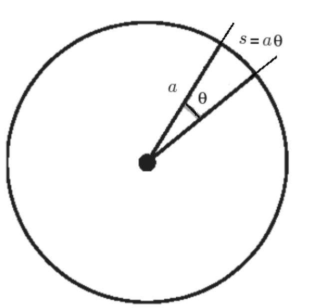
$$C = 2\pi a \tag{20}$$

The area of the same circle is

$$A = \pi a^2 \tag{21}$$

Note that circumference is proportional to the radius ($C \propto a$) while area is proportional to the square of radius ($A \propto a^2$).

Arc length



The arc length of a segment with angle θ has length $s = a\theta$.
Note here that the angle is measured in radians, and so one obtains the circumference for $\theta = 2\pi$.

The surface area and volume of a sphere

The surface area of a sphere of radius a is

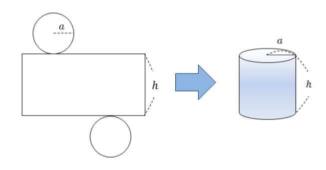
$$A = 4\pi a^2 \tag{22}$$

The volume of the same sphere is

$$V = \frac{4}{3}\pi a^3 \tag{23}$$

Note are goes with the square of radius ($A \propto a^2$), and volume with cube of radius ($V \propto a^3$).

Volume and surface area of a cylinder



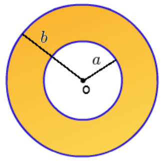
The volume of a cylinder is base area A multiplied by height h. If the base of the cylinder is a circle with radius a, the base area is $A = \pi a^2$. The volume is then

$$V = \pi a^2 h \tag{24}$$

If the cylinder is unfolded it is seen that the adjacent side is a rectangle with sides $2\pi a$ and h. Accordingly the area of the adjacent side is $A = 2\pi ah$.

Outline

Area of a hollow circle

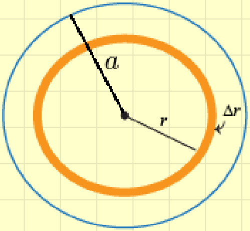


The area of a hollow circle with inner radius a and outer radius b is to found by subtracting the area of the smaller circle from the larger:

$$A = \pi(b^2 - a^2) \tag{25}$$

Note that the area in question is *not* $\pi(b-a)^2$!

Area of a circular strip



Consider a circular strip of inner radius r and outer radius $r + \Delta r$. Accordingly the area of the strip is

$$\Delta A = \pi \left[(r + \Delta r)^2 - r^2 \right]$$

Arranging this expression we get

$$\Delta A = \pi \left[r^2 \left(1 + \frac{\Delta r}{r} \right)^2 - r^2 \right] = \pi r^2 \left[\left(1 + \frac{\Delta r}{r} \right)^2 - 1 \right] \qquad (26)$$

If the strip is very narrow ($\epsilon \equiv \Delta r / r \ll 1$) we can use $(1+\epsilon)^2 \simeq 1 + 2\epsilon$ to obtain

$$\Delta A \simeq \pi r^2 \left[\left(1 + 2\frac{\Delta r}{r} \right) - 1 \right] = \pi r^2 \left(2\frac{\Delta r}{r} \right) = 2\pi r \Delta r \qquad (27)$$

The area of an infinitely thin circular strip

Recall

$$\Delta A \simeq 2\pi r \Delta r$$

For infinitely thin strip this becomes, by $\Delta r \to \mathrm{d}r$ and $\Delta A \to \mathrm{d}A$,

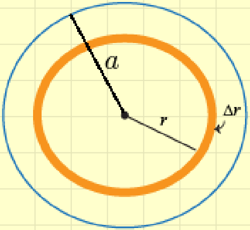
$$\mathrm{d}A = 2\pi r\,\mathrm{d}r \tag{28}$$

Note that this is the area of a rectangle with sides $2\pi r$ and $\mathrm{d}r$. Obviously we do not get exactly a rectangle when we unfold the strip, but a trapezoid. The area of the trapezoid is larger than the rectangle by $\frac{1}{2} \cdot dr \cdot 2\pi \mathrm{d}r$. But this difference which scales as $\mathrm{d}r^2$ is infinitely smaller than the area of the rectangle and so is neglected. The series expansion we used becomes an exact expression at the limit of infinitely small.

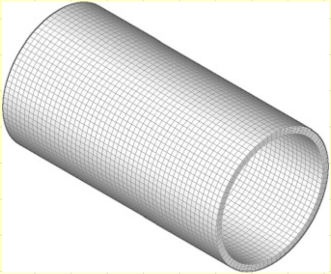
Check:

Let us check the result above, that the area of an infinitely narrow strip is $\mathrm{d}A = 2\pi r\,\mathrm{d}r$: As we can consider the circle as composed of infinite number of such circles which we can add up by integration:

$$A = \int_0^a 2\pi r\,\mathrm{d}r = \pi a^2. \tag{29}$$



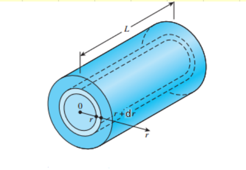
The volume of a cylindrical shell



A cylindrical shell with inner radius a outer radius b height h has the volume

$$V = \pi \left(b^2 - a^2\right) h \tag{30}$$

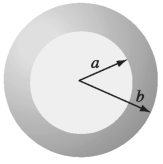
The volume of an infinitely thin cylindrical shell:



The volume would be found by multiplying the base area ($\mathrm{d}A = 2\pi r\,\mathrm{d}r$) with the height L

$$\mathrm{d}V = 2\pi r \mathrm{d}r\, L \tag{31}$$

Volume of a spherical shell:



The volume of a spherical shell with inner radius a outer radius b is found by subtracting the volume of the smaller sphere from the larger

$$V = \frac{4}{3}\pi\left(b^3 - a^3\right) \tag{32}$$

Note that the volume in question is *not* $\pi(b-a)^3$!

Volume of an infinitely thin spherical shell:

Consider a spherical shell with inner radius r and outer radius $r + \Delta r$. The volume would be found by using Eqn. (32)

$$\Delta V = \frac{4}{3}\pi \left[(r + \Delta r)^3 - r^3\right]$$

Arranging this expression we obtain

$$\Delta V = \frac{4}{3}\pi \left[r^3 \left(1 + \frac{\Delta r}{r}\right)^3 - r^3\right] = \frac{4}{3}\pi r^3 \left[\left(1 + \frac{\Delta r}{r}\right)^3 - 1\right] \quad (33)$$

If the shell is very thin $\Delta r / r \ll 1$ is valid and we can use the approximation $(1 + \epsilon)^3 \simeq 1 + 3\epsilon$ to obtain

$$\Delta V \simeq \frac{4}{3}\pi r^3 \left[\left(1 + 3\frac{\Delta r}{r}\right) - 1\right] = \frac{4}{3}\pi r^3 \left(3\frac{\Delta r}{r}\right) = 4\pi r^2 \,\Delta r \quad (34)$$

If the shell is infinitely thin $\Delta r \to \mathrm{d}r$ and $\Delta V \to \mathrm{d}V$:

$$\mathrm{d}V = 4\pi r^2 \,\mathrm{d}r \quad (35)$$

Check:

We can check the result by considering that a solid sphere is made up of shells of thickness $\mathrm{d}V = 4\pi r^2\,\mathrm{d}r$ and by integration:

$$V = \int_0^a 4\pi r^2\,\mathrm{d}r = \frac{4}{3}\pi a^3. \tag{36}$$

Density and mass

Later on we will refer to concepts of electric charge and electric charge density. Here I discuss similar concepts, mass and mass density, because they are more intuitive.

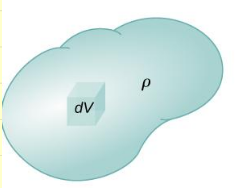
Average density

The Earth has mass $M = 6 \times 10^{24}$ kg and radius $a = 6300$ km. Accordingly the average density of the Earth is

$$\bar{\rho} = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi a^3} = 5.7 \times 10^3 \text{ kg/m}^3 \tag{37}$$

Obviously, the central density of the Earth is larger than this value, and the density of matter at the surface (mostly water of density 1.0×10^3 kg/m^3) is smaller. Surely somewhere inside the Earth there is a location with the same density as the average density of the Earth.

Volume element



If we know the density distribution we can find the mass of each volume element ($\mathrm{d}V$) by $\mathrm{d}m = \rho \mathrm{d}V$ and summation of all these mass elements over the volume gives the total mass

$$M = \int_V \rho \, \mathrm{d}V \tag{38}$$

In case the mass is uniformly distributed we can take ρ outside the integral and write $M = \rho \int_V \mathrm{d}V = \rho V$.

Volume average

Using $\bar{\rho} = \frac{M}{V}$ and $M = \int_V \rho \, \mathrm{d}V$ we write the average density as

$$\bar{\rho} = \frac{1}{V} \int_V \rho \, \mathrm{d}V \tag{39}$$

This is the volume averaged density: Note that we obtain it by integrating the quantity (ρ) over the volume and dividing by the total volume. So the average density is the answer for the question “what should be the density of a uniform object be so that it has the same mass and radius?” If the object is uniform $\rho = M/\frac{4}{3}\pi a^3 = \bar{\rho}$.

Attention!

The usually known $\rho = \frac{M}{V}$ only gives the average density. It would give the density only if the object is uniform.

Ex: Average pressure

If we wanted to find the average of any other quantity (say pressure) over the volume we would do it by

$$\bar{P} = \frac{1}{V} \int_V P \, \mathrm{d}V \tag{40}$$

Spherical symmetry

For a spherically symmetric density distribution the density depends only on the radial distance from the center:

$$\rho = \rho(r) \qquad \text{(Spherically symmetric distribution).} \tag{41}$$

We can assume the Earth to be spherically symmetric. In general density also depends on longitude and latitude and the Earth is not eactly spherically symmetric.
In the less symmetric *axial symmetry* case that quantities like density depend on radial distance and latitude but not on longitude. A rapidly rotating sphere loses its spherical symmetry, but may still be axially symmetric.

Spherical symmetry

- In a spherically symmetric object points at the same radial distance r has the same density.
- In this case a spherical shell of radius r and thickness $\mathrm{d}r$ has mass

$$\mathrm{d}m = \rho\,\mathrm{d}V = \rho(r)4\pi r^2\,\mathrm{d}r$$

- Adding the masses of all those spherical shells up to the radius a we obtain the total mass:

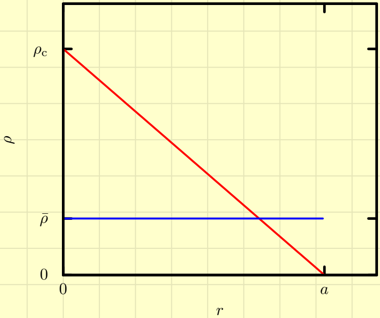
$$M = \int_0^a \rho(r)4\pi r^2\,\mathrm{d}r \tag{42}$$

Ex: A linearly decreasing mass distribution

Consider a sphere with total mass M, radius a. We are given that the density is distributed as

$$\rho = \rho_{\mathrm{c}} \left(1 - \frac{r}{a} \right)$$

within the sphere.



Question:
a) What is the central density ρ_{c}?
b) What is the mass within radial distance r?

Solution

The volume of a spherical shell with radius r' and thickness $\mathrm{d}r'$ is $dV = 4\pi r'^2\,\mathrm{d}r'$. Adding up the masses of such shells up to r:

$$m(r) = \int_0^r \rho(r') 4\pi r'^2\,\mathrm{d}r' = 4\pi\rho_c \int_0^r \left(1 - \frac{r'}{a}\right) r'^2\,\mathrm{d}r'$$

Evaluating this integral we get

$$m(r) = 4\pi\rho_c \int_0^r \left(r'^2 - \frac{r'^3}{a}\right) \mathrm{d}r' = 4\pi\rho_c \left(\frac{1}{3}r^3 - \frac{r^4}{4a}\right)$$

According to the givens (M and a) and using $m(a) = M$

$$M = 4\pi\rho_c a^3 \left(\frac{1}{3} - \frac{1}{4}\right) \Rightarrow \rho_c = \frac{3M}{\pi a^3}$$

is obtained.
Average density: total mass divided by total volume $\bar{\rho} = \frac{M}{\frac{4}{3}\pi a^3}$
Notice for this problem that $\rho_c = 4\bar{\rho}$. Plugging this value we get

$$m(r) = 4M \left(\frac{r}{a}\right)^3 \left(1 - \frac{3r}{4a}\right)$$

Check that this expression yields M when $r \to a$.

Outline

Time average

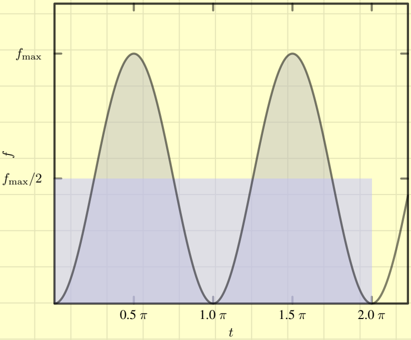
How can we find the average value of a function varying in time? In a time period from T_1 to T_2 the area under the function $f(t)$ is $\int_{T_1}^{T_2} f(t)\,\mathrm{d}t$. The average value of the function $\bar{f}$ is chosen such that the rectangle $\bar{f}(T_2 - T_1)$ has the same area. Thus

$$\bar{f} = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} f(t)\,\mathrm{d}t \tag{43}$$

Ex: $f(t) = f_{max} \sin^2 \omega t$

According to the definition above the average value of the function $f(t) = f_{max} \sin^2 \omega t$ in range $(0, T = 2\pi/\omega)$ is

$$\bar{f} = \frac{1}{T}\int_0^T f_{max} \sin^2 \omega t \, \mathrm{d}t = \frac{f_{max}}{2}$$



Hint!

To evaluate this integral refer to the trigonometrical identity $\sin^2 x \equiv \frac{1-\cos 2x}{2}$

Outline

1. Taylor series expansion
   - Small x approximation
2. Geometrical concepts
   - Circumference, surface area and volume
   - Strips and shells
3. Density and mass
4. Time average, effective value
   - Time average
   - Effective value: root mean square
5. Linear and surface density distributions
6. Vector fields
   - Flux
   - Circulation
   - Helmholtz' theorem

Effective value

Consider a function with average about zero: $I(t)$. For example,

$$I(t) = I_{\max} \sin \omega t \tag{44}$$

has zero average value. We know that the current provided in houses is of this form. Even then we get a bill at the end of the month! The electric providing company does not ask only for the positive part and return back the negative part. We need a quantity different than zero but still meaningful.

Effective value

We first need no get rid of the negative part. For this we can take the square of the value. Certainly what we find is a positive-definite quantity with a positive-definite average value, but has the dimension of not I but I^2. For finding a quantity of the dimension of I we need to take the square root. The value we find in this way—by taking the square and averaging and square rooting— is called the *effective value*:

$$I_{\text{eff}} \equiv \sqrt{\frac{1}{T}\int_0^T I^2 \, \mathrm{d}t} \tag{45}$$

Ex:

For the sinusoidal function $I(t) = I_{\max} \sin \omega t$ we obtain

$$I_{\mathrm{eff}} \equiv \sqrt{\frac{1}{T} \int_0^T I_{\max}^2 \sin^2 \omega t \, \mathrm{d}t} = I_{\max} \sqrt{\frac{1}{T} \int_0^T \sin^2 \omega t \, \mathrm{d}t} \tag{46}$$

by using the result given in Eqn. (50) we get $\bar{I^2} = I_{\max}^2/2$

$$I_{\mathrm{eff}} = \frac{I_{\max}}{\sqrt{2}} \tag{47}$$

The effective value of the sinusoidally changing current is obtained by dividing the maximum value by $\sqrt{2} \simeq 1.41$. Similar is true for the electric potential. A potential with effective value of $V_{\mathrm{eff}} = 110$ V has the amplitude $V_{\max} = \sqrt{2}\,110$ V.

Another application or RMS

What is the average speed of molecules in a vessel? The answer is O because the velocity vectors of every molecule which are moving randomly, by the law of large numbers, would be zero as there are almost equal number of particle moving in each direction and opposite. Yet zero is not a satisfactory answer; we do know that molecules move randomly and there should be typical velocity of particles independent of direction. In order to get rid of the cancellation of the vectors in opposite directions we can take the square of the vectors. We will have a finite value when we take the average now. Finally we can take the square root of this to find the *root mean square* value of the speed of molecules.

Linear mass density

We have a string of length $L = 100$ m. The mass of the string is $M = 1$ kg. The mass of a unit length string, i.e. the linear mass density, is $\bar{\lambda} = M/L = 0.01$ kg/m. The mass M and linear mass density λ in general are related as

$$M = \int_l \lambda \, \mathrm{d}l \tag{48}$$

For a homogeneous string λ does not depend on x and can be taken outside the integral yielding $M = \lambda L$. But if λ depends on x Eqn. (48) should be integrated.

Surface mass density

We have a sheet of area $A = 10 \text{ m}^2$ and mass $M = 2$ kg. The mass per unit area, i.e. surface mass density is then $\bar{\sigma} = M/A = 5 \text{ kg/m}^2$. In general mass M and surface mass density σ are related as

$$M = \int_A \sigma \, \mathrm{d}A \tag{49}$$

Vector fields

In this section we are going to overview vector calculus concepts needed to understand Maxwell's equations.

Field

The concept of field arises when a quantity is attributed to every point of space in describing continuous media. The quantity also could depend on time.

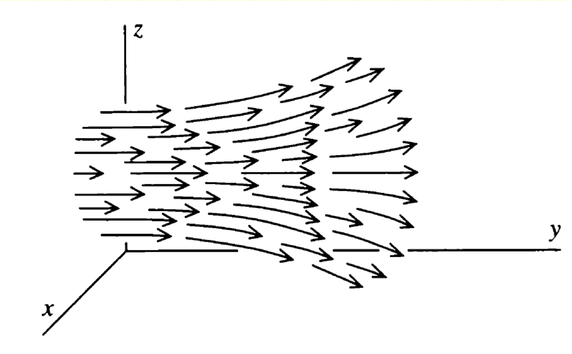
Scalar field

If the quantity in question is a scalar then we talk about a scalar field. Density and temperature are scalar fields: $\rho = \rho(x, y, z)$, $T = T(x, y, z, t)$

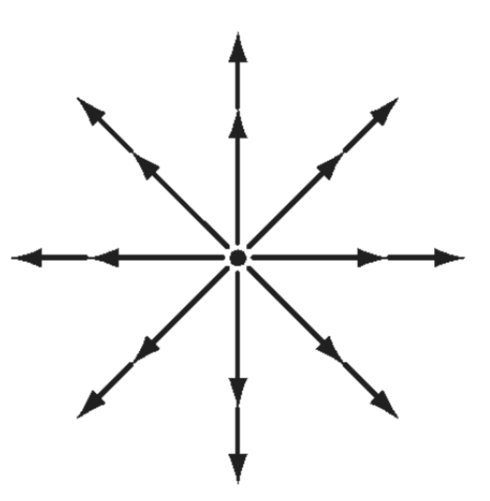
Vector fields

If the quantity in question is a vector then we talk about a vector field. Ex. velocity field of a fluid, electric and magnetic fields: $\vec{\mathbf{v}} = \vec{\mathbf{v}}(x, y, z)$, $\vec{\mathbf{E}} = \vec{\mathbf{E}}(x, y, z)$, $\vec{\mathbf{B}} = \vec{\mathbf{B}}(x, y, z)$

Velocity vector field



Electric vector field



for a positive charge.

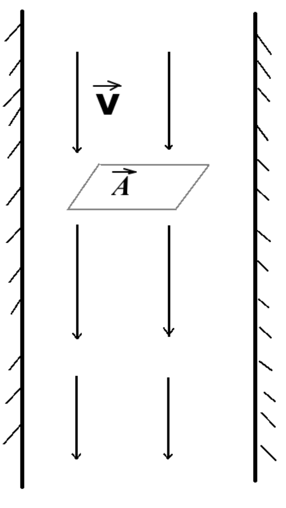
Outline

Flux

Imagine that, in order to quantify the amount of fluid flowing in a river, we submerge an imaginary rectangular frame the presence of which does not perturb the flow. Let us assume the velocity is uniform at all points on the surface: $\vec{\mathbf{v}}$. If the normal of the surface $\hat{\mathbf{n}}$ is parallel then the flux of fluid through the imaginary surface is

$$\Phi = vA \ .$$

Ex: If the velocity in the river is $v = 2$ m/s and the surface area is $A = 3$ m^2 then the flux is $\Phi = 6$ m^3/s.

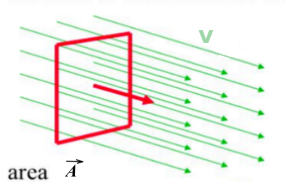


Flux

Imagine that, in order to quantify the amount of fluid flowing in a river, we submerge an imaginary rectangular frame the presence of which does not perturb the flow. Let us assume the velocity is uniform at all points on the surface: $\vec{\mathbf{v}}$. If the normal of the surface $\hat{\mathbf{n}}$ is parallel then the flux of fluid through the imaginary surface is

$$\Phi = vA \ .$$

Ex: If the velocity in the river is $v = 2\ \mathrm{m/s}$ and the surface area is $A = 3\ \mathrm{m}^2$ then the flux is $\Phi = 6\ \mathrm{m}^3/\mathrm{s}$.

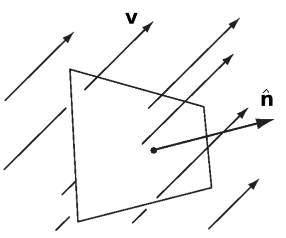


Flux

- If the angle between the normal of the surface and velocity is θ then the flux would be reduced to

$$\Phi = vA\cos\theta$$

.



- If we represent the surface with the vector $\vec{\mathbf{A}} \equiv A\hat{n}$ the flux can be written as a scalar product:

$$\Phi = \vec{\mathbf{v}} \cdot \vec{\mathbf{A}}$$

Flux

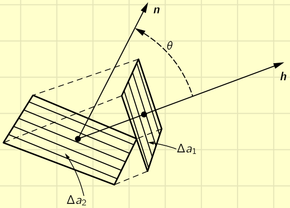
- If the angle between the normal of the surface and velocity is θ then the flux would be reduced to

$$\Phi = vA\cos\theta$$

.

- If we represent the surface with the vector $\vec{\mathbf{A}} \equiv A\hat{n}$ the flux can be written as a scalar product:

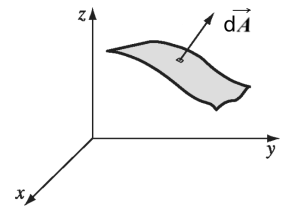
$$\Phi = \vec{\mathbf{v}} \cdot \vec{\mathbf{A}}$$



Flux

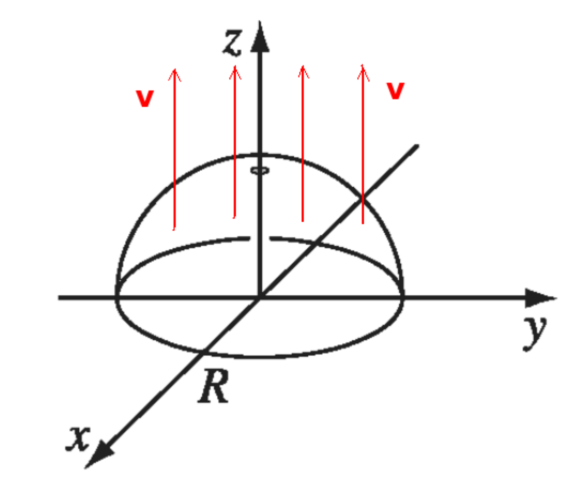
If the imaginary surface submerged in the fluid is not planar but curved, or if the velocity vector field is not uniform we consider flux through infinitesimally small surface elements $d\vec{\mathbf{A}}$ and calculate the flux through each of them $d\Phi = \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}}$. To find the total flux we finally integrate these over the surface:

$$\Phi = \int \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}} \qquad (50)$$



Ex: Flux through semi-sphere

What is the flux through a semi-sphere of radius R?

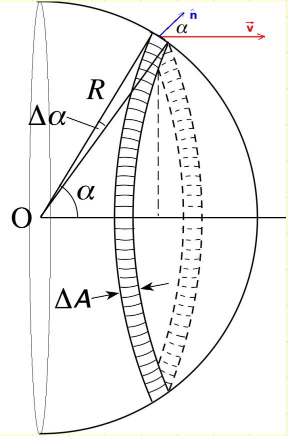


Ex: Flux through semi-sphere

- Area of the strip
  $\Delta A = 2\pi(R\sin\alpha)R\Delta\alpha$
- All points on the strip make the angle α with the velocity vector $\vec{\mathbf{v}}$.
- The flux through the strip
  $\Delta\Phi = v2\pi(R\sin\alpha)R\Delta\alpha\cos\alpha$
- Flux through the semi-sphere

$$\Phi = 2\pi R^2 v\int_0^{\pi/2}\sin\alpha\cos\alpha\,\mathrm{d}\alpha = \pi R^2 v$$

$u \equiv \sin\alpha,\ du = \cos\alpha\,\mathrm{d}\alpha$

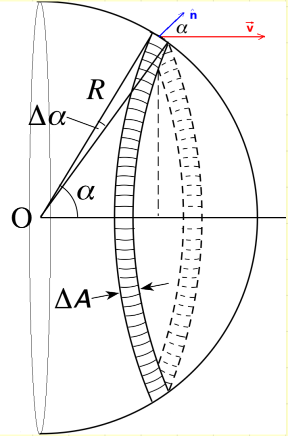


Ex: Flux through semi-sphere

- Area of the strip
  $\Delta A = 2\pi(R\sin\alpha)R\Delta\alpha$
- All points on the strip make the angle α with the velocity vector $\vec{\mathbf{v}}$.
- The flux through the strip
  $\Delta\Phi = v2\pi(R\sin\alpha)R\Delta\alpha\cos\alpha$
- Flux through the semi-sphere

$$\Phi = 2\pi R^2 v \int_0^{\pi/2} \sin\alpha\cos\alpha\,\mathrm{d}\alpha = \pi R^2 v$$

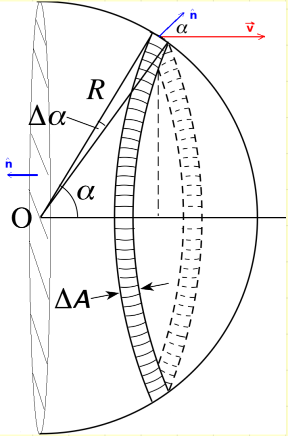
$u \equiv \sin\alpha$, $du = \cos\alpha\,\mathrm{d}\alpha$



Notice that this is exactly the flux that would pass though the base of the semi-sphere.

Ex: Flux through a *closed* semi-sphere

- Definition: *What we mean by a closed surface is one enclosing a volume.*
- Convention: *For a closed surface the normal vectors are always pointing outwards at all points.*
- We have already shown the flux through the curved surface is $\Phi_{\text{semi-sph}} = \pi R^2 v$.
- The flux through the planar base is: $\Phi_{\text{circle}} = \pi R^2 v \cos\pi = -\pi R^2 v$

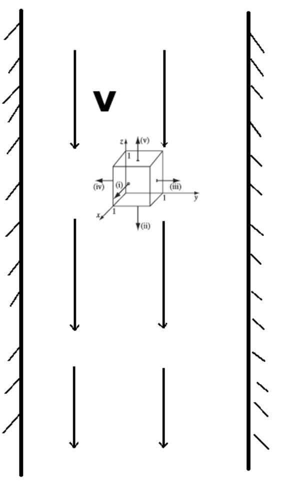


Total net flux:

$$\Phi = \oint \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}} = \Phi_{\text{semi-sph}} + \Phi_{\text{circle}} = \pi R^2 v - \pi R^2 v = 0!$$

Flux through a closed cube

- $\Phi = \oint \vec{\mathbf{v}} \cdot \mathrm{d}\vec{\mathbf{A}} = \sum_i \Phi_i = \sum_i \int_i \vec{\mathbf{v}} \cdot \mathrm{d}\vec{\mathbf{A}}$
- $\Phi_i = \int_i \vec{\mathbf{v}} \cdot \mathrm{d}\vec{\mathbf{A}} = vA\cos(\pi/2) = 0$
- $\Phi_{ii} = \int_{ii} \vec{\mathbf{v}} \cdot \mathrm{d}\vec{\mathbf{A}} = vA\cos(0) = vA$
- $\Phi_{iii} = \int_{iii} \vec{\mathbf{v}} \cdot \mathrm{d}\vec{\mathbf{A}} = vA\cos(\pi/2) = 0$
- $\Phi_{iv} = \int_{iv} \vec{\mathbf{v}} \cdot \mathrm{d}\vec{\mathbf{A}} = vA\cos(\pi/2) = 0$
- $\Phi_{v} = \int_{v} \vec{\mathbf{v}} \cdot \mathrm{d}\vec{\mathbf{A}} = vA\cos(\pi) = -vA$
- $\Phi_{vi} = \int_{vi} \vec{\mathbf{v}} \cdot \mathrm{d}\vec{\mathbf{A}} = vA\cos(\pi/2) = 0$
- $\Phi = 0 + vA + 0 + 0 - vA + 0 = 0$



Conservation

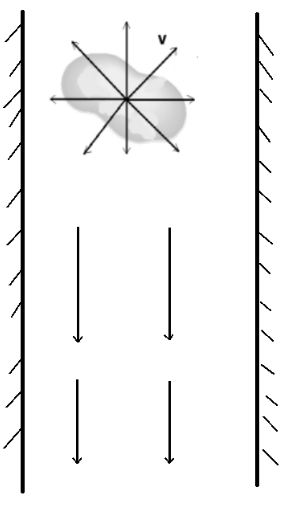
- The net flux through a closed surface is independent of the shape of the surface.
- It is then a wise idea to write the laws in terms of closed surfaces.
- We have found the flux through a closed surface to be zero as the amount of fluid entering and leaving are the same.
- Under what condition would it be that the flux through a closed surface is non-zero?

Source

- If the closed surface encloses the source of the vector field the net flux is equal to the source:

$$\Phi = \oint \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}} = \text{sources} \qquad (51)$$

- Ex: If the source provides $15\ \text{m}^3$ of water then the net flux through the surface is $15\ \text{m}^3$.

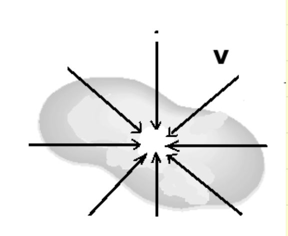


Sinks

- If the closed surface encloses a sink rather than source then the net flux would be negative.
- In general

$$\Phi = \oint \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}} = \text{sources} - \text{sinks} \tag{52}$$

- Whatever the source of the vector field we write it on the right hand side.



Gauss' law

In general we found that

$$\Phi = \oint \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}} = \text{sources} - \text{sinks}$$

Accordingly Gauss' law

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = Q_{\text{enclosed}}/\epsilon_0 \tag{53}$$

simply means that positive (negative) electric charges are the sources (sinks) of electric fields.
Gauss' law magnetic fields

$$\Phi_B = \oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0 \tag{54}$$

then simply means there is no magnetic charge, if there was we would put it on the right hand side.

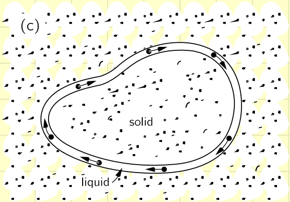
Outline

Circulation

Just like "flux" we are familiar to the concept of "circulation" from fluid mechanics. It is measure of the "circulation/rotation/curl" of that fluid flow. How can we quantify this?

Circulation

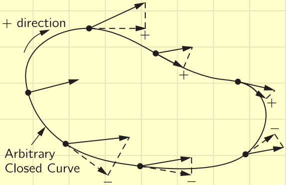
- First consider an imaginary (that does not disturb the flow) pipe.
- Imagine all the fluid except the part in the pipe is frozen
- Circulation is a measure of the fluid to move along the pipe.



Circulation

- Only the component of the velocity along the pipe will be significant for moving the fluid along the pipe.
- If the angle that the velocity vector $\vec{\mathbf{v}}$ makes with the element of the pipe at a certain point, $\mathrm{d}\vec{\mathbf{l}}$ is θ then the projection of $\vec{\mathbf{v}}$ onto $\mathrm{d}\vec{\mathbf{l}}$ will give $(v\mathrm{d}l\cos\theta = \vec{\mathbf{v}}\cdot\mathrm{d}\vec{\mathbf{l}})$ contribution to circulation in the element of the pipe.
- The flow through the whole pipe is then obtained by integrating the all contributions

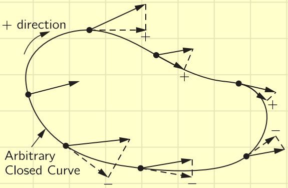
$$\int \vec{\mathbf{v}}\cdot\mathrm{d}\vec{\mathbf{l}}$$



Circulation

Going back to the basic question whether the flow rotates: For this we need to integrate the same thing for a closed loop. Then the circulation of a vector field $\vec{\mathbf{v}}$ is defined as

$$C \equiv \oint \vec{\mathbf{v}} \cdot d\vec{\mathbf{l}} \qquad (55)$$



The whole purpose of choosing a closed loop is that the result is independent of the shape of the pipe!

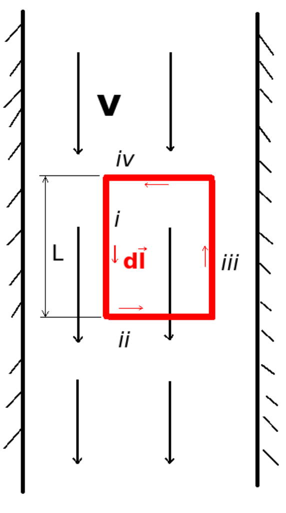
Ex:

What is the circulation in the river

- Consider a rectangular pipe
- **dl** is an element tangential to the pipe
- Evaluating the integral $C \equiv \oint \vec{\mathbf{v}} \cdot d\vec{\mathbf{l}}$ for each segment we get

$$C = vL\cos 0 + 0 + vL\cos\pi + 0 = 0$$

- I knew this result, but now found it mathematically.

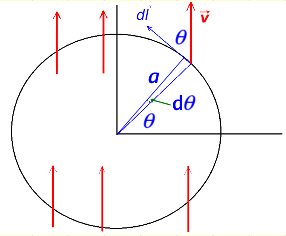


Ex:

What is the circulation in the river with uniform flow?

- Pick up a circular loop
- $\mathrm{d}l = a\mathrm{d}\theta$ and

$$C = \oint \vec{\mathbf{v}} \cdot \mathrm{d}\vec{\mathbf{l}} = \int_0^{2\pi} v\cos\theta a\mathrm{d}\theta = 0$$



- We again find $C = 0$, this result is independent of the shape of the loop.
- *The circulation through a closed loop is independent of the path, it is a property of the flow.*

Conservation

If the circulation over a closed loop is zero the integral $\int \vec{\mathbf{v}} \cdot d\vec{\mathbf{l}}$ between two points is independent of path. Such a vector field is said to be *conservative*.

In fact the circulation is a local quantity changing from point to point.

In order to have an idea of the circulation at a certain point I should keep the size of the loop as small as possible.

The practical way to understand the presence of circulation at a point is to float a cork with toothpicks pointing out radially at that point; if it starts to rotate, then you placed it at a point of nonzero circulation.

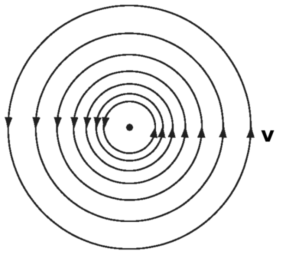
A whirlpool would be a region of large circulation.

Ex: Circulation

- Consider the flow with obvious circulation and calculate it.
- To do this I pick a circular pipe with radius a. In this case I obtain

$$C = \oint \vec{\mathbf{v}} \cdot d\vec{\mathbf{l}} = 2\pi a v$$

.

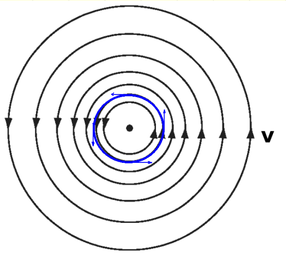


Ex: Circulation

- Consider the flow with obvious circulation and calculate it.
- To do this I pick a circular pipe with radius a. In this case I obtain

$$C = \oint \vec{\mathbf{v}} \cdot \mathrm{d}\vec{\mathbf{l}} = 2\pi a v$$

.



Outline

Helmholtz' theorem

According to Helmholtz' theorem in order to define a vector field $\vec{\mathbf{V}}$ it is sufficient to give its flux over closed surface $\Phi \equiv \oint \vec{\mathbf{V}} \cdot \mathrm{d}\vec{\mathbf{A}}$ and circulation over a closed loop $C \equiv \oint \vec{\mathbf{V}} \cdot \mathrm{d}\vec{\mathbf{l}}$.

Maxwell's Equations

- Gauss's law

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

- Faraday's law

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d}{dt} \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$$

- Gauss's law for magnetism

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

- Generalized Ampere's law

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 i_C + \mu_0 \epsilon_0 \frac{d}{dt} \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

From a long view of the history of mankind - seen from, say, ten thousand years from now - there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics.

Richard Feynman