Gauss' Law

FIZ102E: Electricity & Magnetism

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Outline

Learning Goals

- How you can determine the amount of charge within a closed surface by examining the electric field on the surface.
- What is meant by electric flux, and how to calculate it.
- How Gauss's law relates the electric flux through a closed surface to the charge enclosed by the surface.
- How to use Gauss's law to calculate the electric field due to a symmetric charge distribution.
- Where the charge is located on a charged conductor.

Maxwell's Equations

- Gauss' law

$$\boxed{\oint \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}} = \frac{Q_{\mathrm{enc}}}{\epsilon_0}}$$

- Faraday's law

$$\oint \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{l}} = -\frac{\mathrm{d}}{\mathrm{d}t} \int \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{A}}$$

- Gauss's law for magnetism

$$\oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{A}} = 0$$

- Generalized Ampere's law

$$\oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{l}} = \mu_0 i_{\mathrm{C}} + \mu_0 \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}}$$

Lorentz's Force

Force acting on a charged particle given the $\vec{\mathbf{E}}$ & $\vec{\mathbf{B}}$ fields is

$$\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})$$

where $\vec{\mathbf{v}}$ is the velocity of the particle.

Recall: Coulomb's law

- In Chapter 21 we asked the question, "Given a charge distribution, what is the electric field produced by that distribution at a point P?"
- We saw that the answer could be found by representing the distribution as an assembly of point charges, each of which produces an electric field $\vec{\mathbf{E}}$ given by

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

- The total field at a point is then the vector sum of the fields due to all the point charges.

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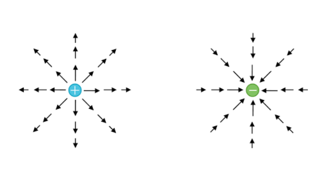

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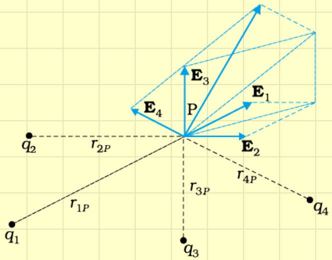

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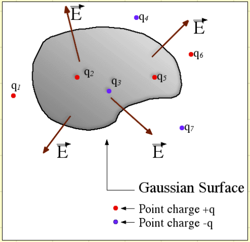
Summary: Gauss' law

- In this chapter we ask the opposite, "If the electric field pattern is known in a given region, what can we determine about the charge distribution in that region?"
- This will lead to an alternative relationship between charge distributions and electric fields:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{enc}}}{\epsilon_0}, \qquad \text{Gauss' law}$$

- Gauss' law is a relationship between the field at all the points on the surface and the total charge enclosed within the surface.



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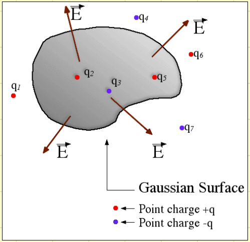

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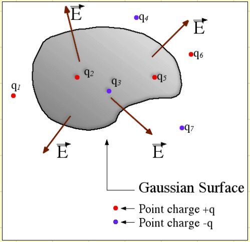

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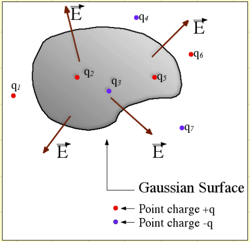
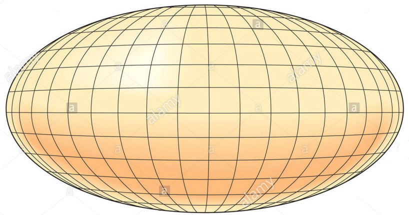


- If the electric-field pattern is known in a given region, what can we determine about the charge distribution in that region?

Closed surface

- A *closed surface* (Gaussian surface) is one enclosing a *volume*.
- A sphere is a *closed* surface.
- A cube is a *closed* surface.
- A circle is *not* a closed surface.
- A square is *not* a closed surface.
- An *open* semi-sphere
- An *closed* semi-sphere
- An open surface.



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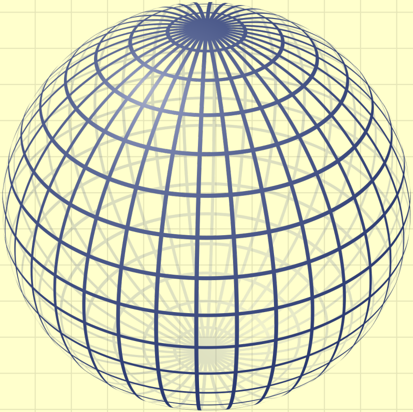

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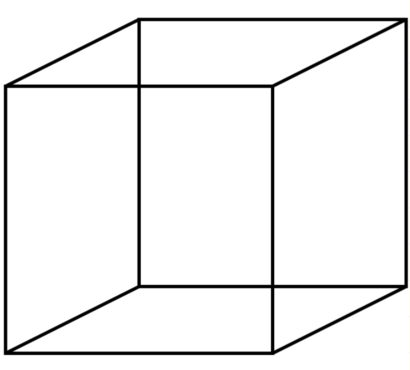

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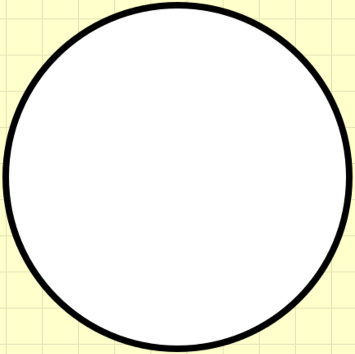

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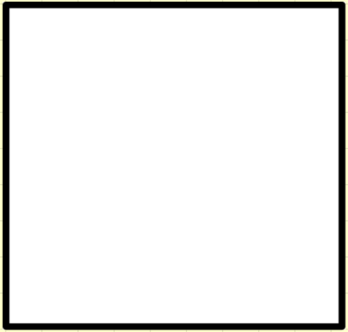

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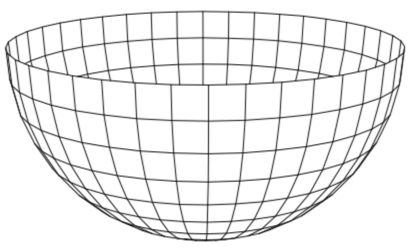

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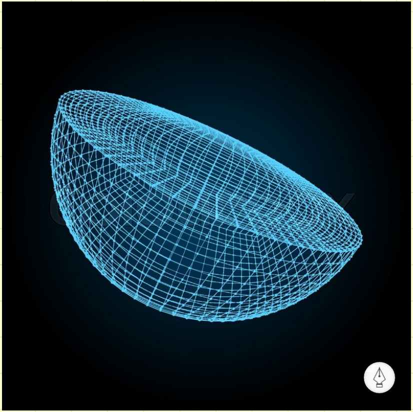

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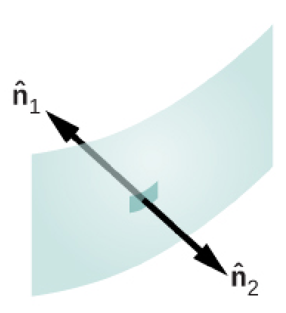
Convention

- Unit normal has two possible directions for an open surface.
- For an open surface, we can use either direction, as long as we are consistent over the entire surface.
- The outward normal is used to calculate the flux through a closed surface.



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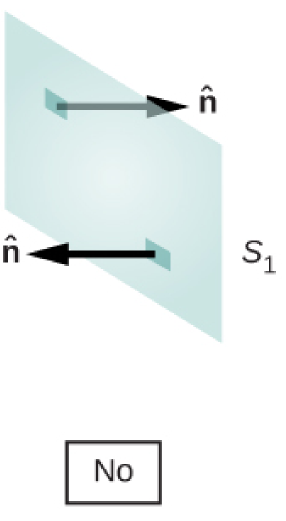

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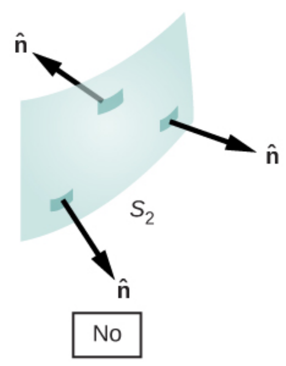

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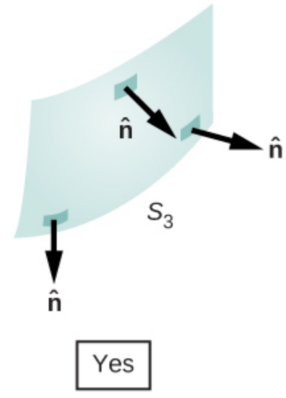

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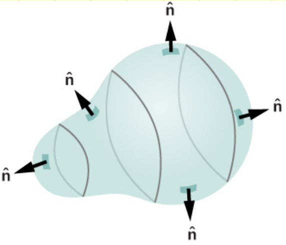
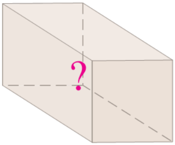


Charge and Electric Flux

- How can you determine how much (if any) electric charge lies within the box (closed surface)?
- Assume the box is made of a material that has no effect on any electric fields



(a) A box containing an unknown amount of charge

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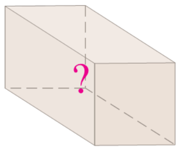
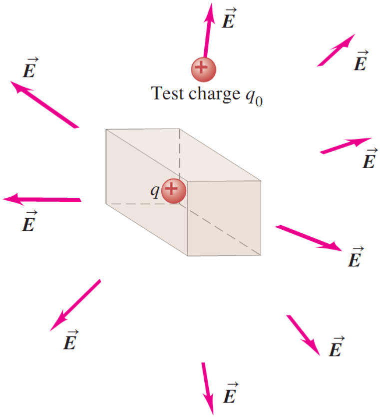


Charge and Electric Flux

- A charge distribution produces an electric field
- An electric field exerts a force on a test charge
- Move a test charge q_0 around the vicinity of the box.
- By measuring the force $\vec{\mathbf{F}}$ experienced by the test charge at different positions, make a 3D map of the $\vec{\mathbf{E}} = \vec{\mathbf{F}}/q_0$ field.

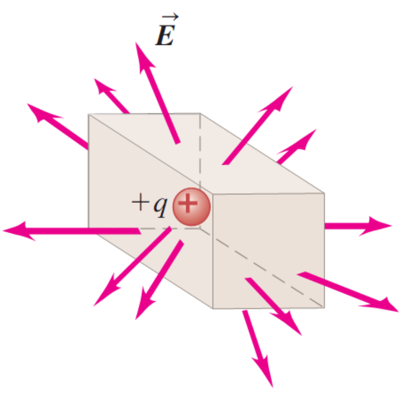
(b) Using a test charge outside the box to probe the amount of charge inside the box



Charge and Electric Flux

To determine the contents of the box, we actually need to measure $\vec{\mathbf{E}}$ on only the surface of the box.

A single positive point charge inside the box.



Charge and Electric Flux

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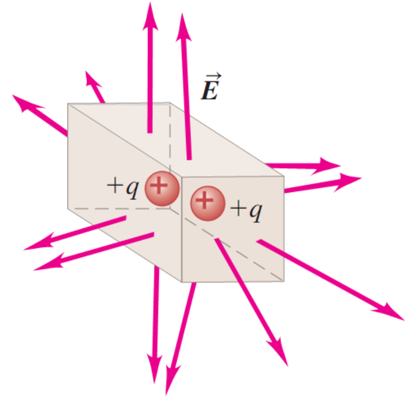
There are two positive charges.



Charge and Electric Flux

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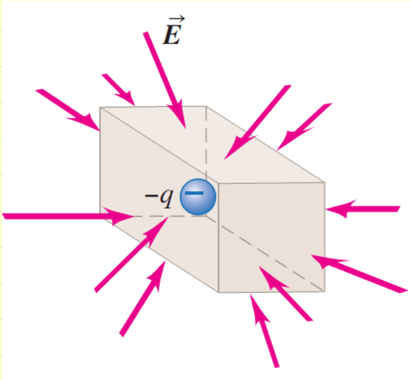
A single negative charge.



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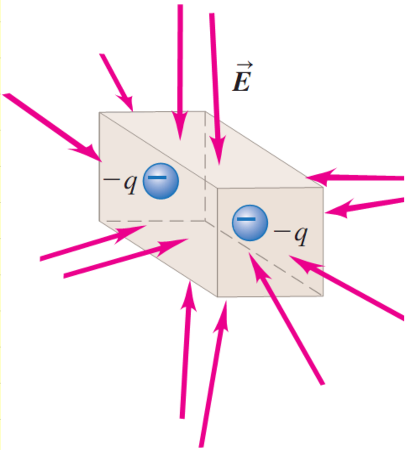
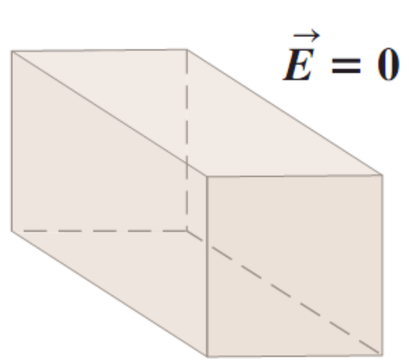
Two negative charges.



Charge and Electric Flux

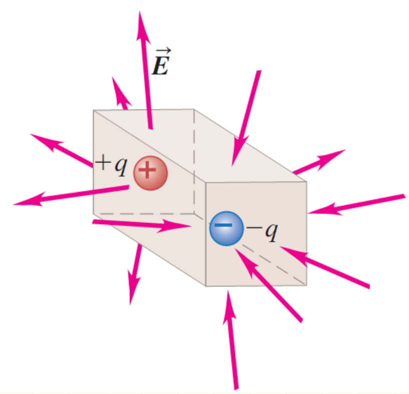
To determine the contents of the box, we actually need to measure $\vec{\mathbf{E}}$ on only the surface of the box.

No charge in the box, no flux.

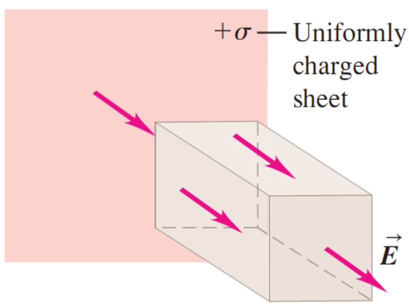


Charge and Electric Flux

To determine the contents of the box, we actually need to measure $\vec{\mathbf{E}}$ on only the surface of the box.
Zero net charge: inward flux cancels outward flux.



Charge and Electric Flux



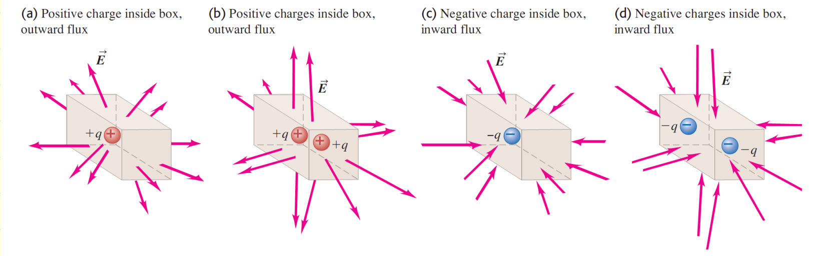
To determine the contents of the box, we actually need to measure $\vec{\mathbf{E}}$ on only the surface of the box.

No charge inside box: inward flux cancels outward flux.

There is an electric field, but it "flows into" the box on half of its surface and "flows out of" the box on the other half.

Hence no *net* electric flux into or out of the box.

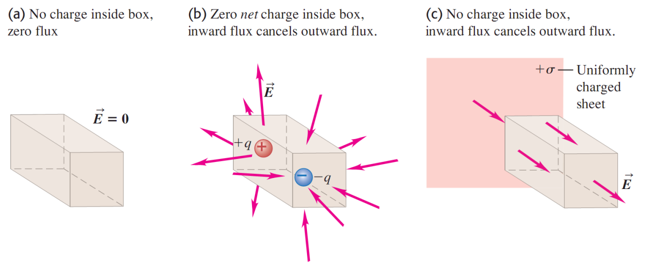
Electric Flux and Enclosed Charge



To determine the contents of the box measure $\vec{\mathbf{E}}$ on the surface of the box.

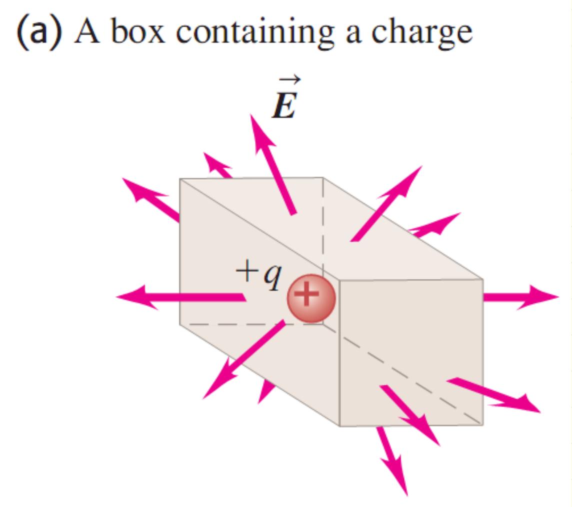
- If the enclosed charge is positive the electric field points out of the box.
- If the enclosed charge is negative the electric field points into the box.

Electric Flux and Enclosed Charge



Summary: The net electric flux through the surface of the box is directly proportional to the magnitude of the net charge enclosed by the box.

Electric Flux and Enclosed Charge



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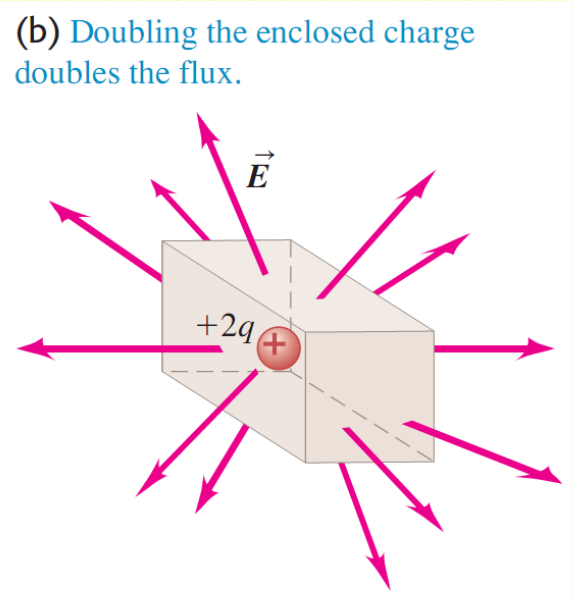

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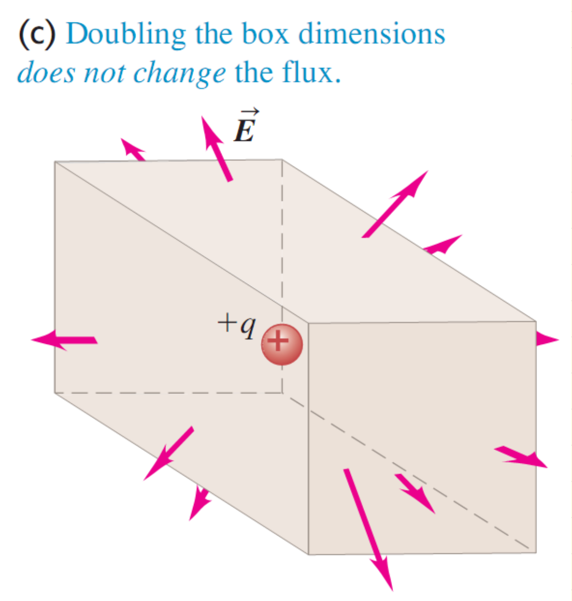


Qualitative statement of Gauss's law

1. Whether there is a net outward or inward electric flux through a closed surface depends on the sign of the enclosed charge.
2. Charges outside the surface do not give a net electric flux through the surface.
3. The net electric flux is directly proportional to the net amount of charge enclosed within the surface but is otherwise independent of the size of the closed surface.

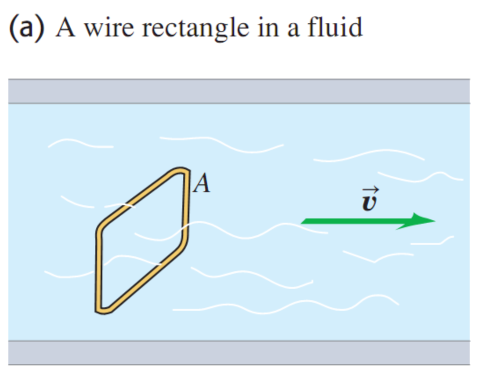
Calculating Electric Flux

- The net electric flux through a closed surface is directly proportional to the net charge inside that surface.
- To be able to make full use of this law, we need to know how to *calculate* the electric flux.
- We'll again refer to analogy between an electric field $\vec{\mathbf{E}}$ and the field of velocity vectors $\vec{\mathbf{v}}$ in a flowing fluid.

Flux: Fluid-Flow Analogy

When the area is perpendicular to the flow velocity $\vec{\mathbf{v}}$ and the flow velocity is the same at all points in the fluid, the volume flow rate $\mathrm{d}V/\mathrm{d}t$ is the area A multiplied by the flow speed v:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = vA$$

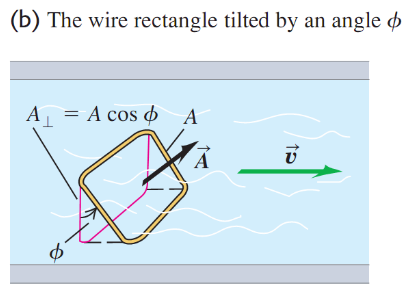


Flux: Fluid-Flow Analogy

When the rectangle is tilted at an angle ϕ so that its face is not perpendicular to $\vec{\mathbf{v}}$, the area that counts is the silhouette area that we see when we look in the direction of $\vec{\mathbf{v}}$. The projected area $A_\perp$ is equal to $A\cos\phi$:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = vA\cos\phi$$

Check: If $\phi = 90°$, $\mathrm{d}V/\mathrm{d}t = 0$; the wire rectangle is edge-on to the flow, and no fluid passes through the rectangle.

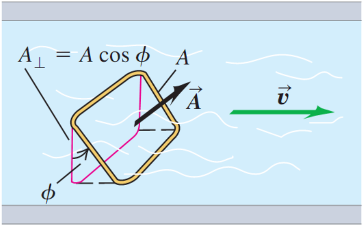


Flux: Fluid-Flow Analogy

We can express the volume flow rate more compactly by using the concept of vector area $\vec{\mathbf{A}}$, a vector quantity with magnitude A and a direction perpendicular to the plane of the area we are describing.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \vec{\mathbf{v}} \cdot \vec{\mathbf{A}}$$

(b) The wire rectangle tilted by an angle ϕ

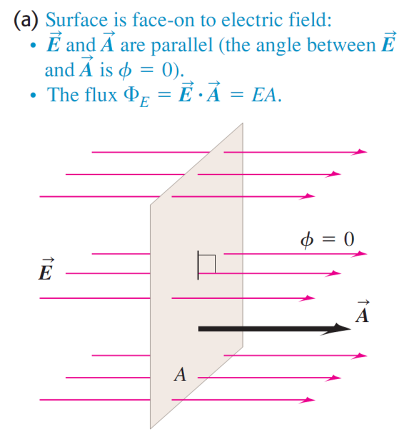


Flux of a Uniform Electric Field

- Let us replace the fluid velocity $\vec{\mathbf{v}}$ by the electric field $\vec{\mathbf{E}}$.
- The electric flux through the area is the product of the field magnitude E and the area A:

$$\Phi_E = EA$$

- The SI unit for Φ_E is $\mathrm{N \cdot m^2/C}$.



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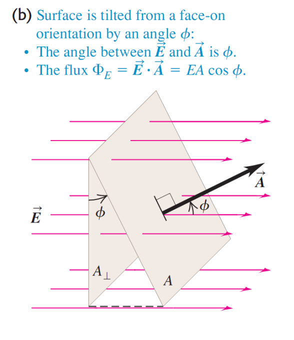


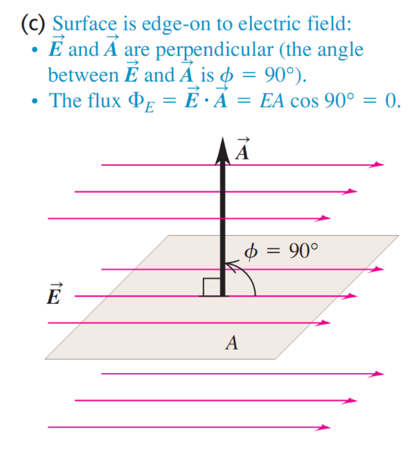
- If the area A is flat but not perpendicular to the field $\mathbf{E}$, then fewer field lines pass through it.
- In this case the area that counts is the silhouette area that we see when looking in the direction of $\vec{\mathbf{E}}$:

$$\Phi_E = EA\cos\phi = \vec{\mathbf{E}}\cdot\vec{\mathbf{A}}, \qquad \text{(electric flux for uniform } \vec{\mathbf{E}}\text{, flat surface)}$$

Flux of a Uniform Electric Field

If the area is edge-on to the field, $\vec{\mathbf{E}}$ and $\vec{\mathbf{A}}$ are perpendicular and the flux is zero:

$$\Phi_E = 0$$



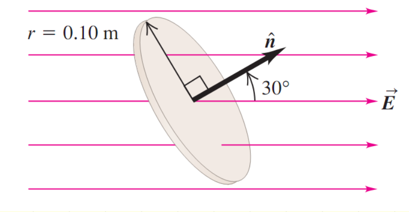
Exercise: Electric flux through a disk

A disk of radius $0.10\,\mathrm{m}$ is oriented with its normal unit vector $\hat{\mathbf{n}}$ at 30° to a uniform electric field E of magnitude $2.0 \times 10^3\,\mathrm{N/C}$.

(a) What is the electric flux through the disk?

(b) What is the flux through the disk if it is turned so that $\hat{\mathbf{n}}$ is perpendicular to $\vec{\mathbf{E}}$?

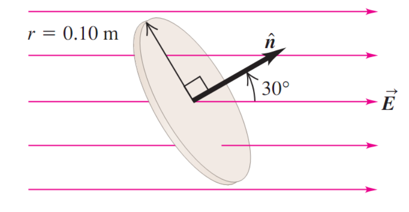
(c) What is the flux through the disk if $\hat{\mathbf{n}}$ is parallel to $\vec{\mathbf{E}}$?



Exercise: Electric flux through a disk

a) The area is

$$A = \pi r^2 = 3.14\,(0.10\,\mathrm{m})^2$$
$$= 0.0314\,\mathrm{m}^2$$

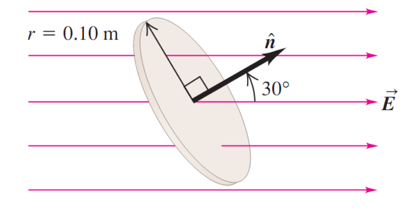


The flux is then

$$\Phi_E = EA\cos\phi$$
$$= (2.0 \times 10^3\,\mathrm{N/C})\,(0.0314\,\mathrm{m}^2)\ \cos 30°$$
$$= 54\,\mathrm{N}\cdot\mathrm{m}^2/\mathrm{C}$$

Exercise: Electric flux through a disk

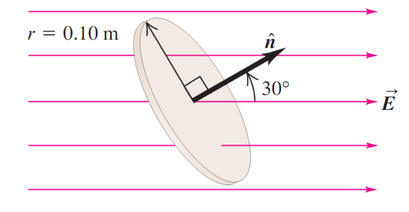
b) The normal to the disk is now perpendicular to $\vec{\mathbf{E}}$, so $\phi = 90°$, $\cos\phi = 0$, and $\Phi_E = 0$.



Exercise: Electric flux through a disk

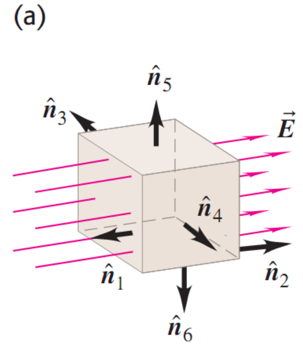
c) The normal to the disk is parallel to $\vec{\mathbf{E}}$, so $\phi = 0$ and $\cos\phi = 1$:

$$\Phi_E = (2.0 \times 10^3\,\mathrm{N/C})\,(0.0314\,\mathrm{m}^2)$$
$$= 63\,\mathrm{N \cdot m^2/C}$$



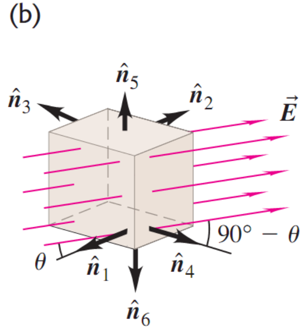
Exercise: Electric flux through a cube

An imaginary cubical surface of side L is in a region of uniform electric field $\vec{\mathbf{E}}$. Find the electric flux through each face of the cube and the total flux through the cube when
(a) it is oriented with two of its faces perpendicular to $\vec{\mathbf{E}}$



Exercise: Electric flux through a cube

An imaginary cubical surface of side L is in a region of uniform electric field $\vec{\mathbf{E}}$. Find the electric flux through each face of the cube and the total flux through the cube when
(a) it is oriented with two of its faces perpendicular to $\vec{\mathbf{E}}$
and (b) the cube is turned by an angle θ about a vertical axis.



Exercise: Electric flux through a cube

An imaginary cubical surface of side L is in a region of uniform electric field $\vec{\mathbf{E}}$. Find the electric flux through each face of the cube and the total flux through the cube when

(a) it is oriented with two of its faces perpendicular to $\vec{\mathbf{E}}$

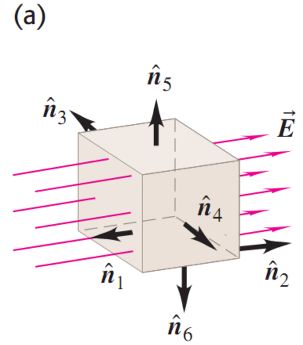
$\Phi_{E1} = EL^2 \cos 180° = -EL^2$

$\Phi_{E2} = EL^2 \cos 0 = +EL^2$

$\Phi_{E3} = \Phi_{E4} = \Phi_{E5} = \Phi_{E6} = EL^2 \cos 90° = 0$

The total flux through the cube is

$$\begin{aligned}\Phi_{\text{tot}} &= \Phi_{E1}+\Phi_{E2}+\Phi_{E3}+\Phi_{E4}+\Phi_{E5}+\Phi_{E6} \\ &= -EL^2 + EL^2 + 0 + 0 + 0 + 0 = 0\end{aligned}$$



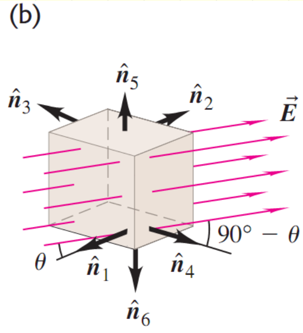
Exercise: Electric flux through a cube

and (b) the cube is turned by an angle θ about a vertical axis.

$\Phi_{E1} = EL^2 \cos(180° - \theta) = -EL^2 \cos\theta$
$\Phi_{E2} = EL^2 \cos\theta$
$\Phi_{E3} = EL^2 \cos(90° + \theta) = -EL^2 \sin\theta$
$\Phi_{E4} = EL^2 \cos(90° - \theta) = +EL^2 \sin\theta$
$\Phi_{E5} = \Phi_{E6} = EL^2 \cos 90°$

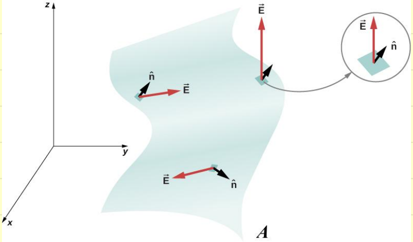
The total flux through the cube is

$$\Phi_{\text{tot}} = \Phi_{E1} + \Phi_{E2} + \Phi_{E3} + \Phi_{E4} + \Phi_{E5} + \Phi_{E6}$$
$$= 0$$



Flux of a Nonuniform Electric Field

- What happens if the $\vec{\mathbf{E}}$ isn't uniform but varies from point to point over the area A?
- Or what if A is part of a curved surface?
- Divide the surface into many small elements $\mathrm{d}A$ each of which has a unit vector $\hat{\mathbf{n}}$ perpendicular to it and a vector area $\mathrm{d}\vec{\mathbf{A}} = \hat{\mathbf{n}}\,\mathrm{d}A$.

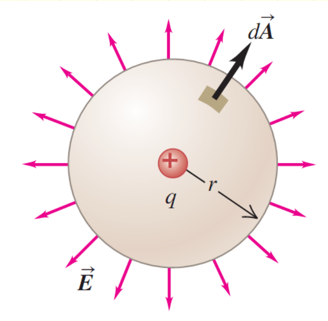


- Calculate flux through each element and integrate the results to obtain the total flux.

$$\Phi_E = \int \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}}$$

Exercise: Electric flux through a sphere

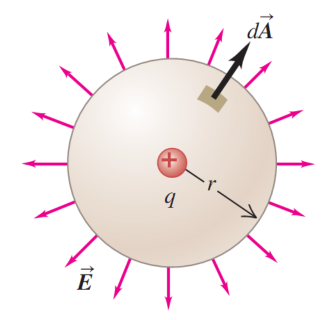
A point charge $q = +3.0\,\mu\mathrm{C}$ is surrounded by an imaginary sphere of radius $r = 0.20\,\mathrm{m}$ centered on the charge. Find the resulting electric flux through the sphere.



Exercise: Electric flux through a sphere

The surface is not flat and the electric field is not uniform, so to calculate the electric flux (our target variable) we must use the general definition

$$\Phi_E = \int \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}}$$



Exercise: Electric flux through a sphere

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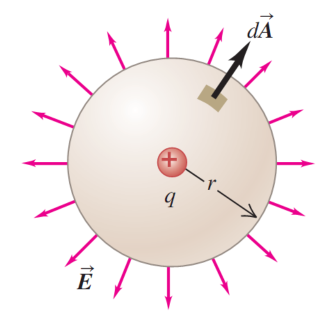
$$\Phi_E = \int \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}}$$

Here, $\vec{\mathbf{E}}$ and $\mathrm{d}\vec{\mathbf{A}}$ at all points on the surface are in the same direction ($\cos\phi = 1$). Thus

$$\Phi_E = \int \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}} = \int E \, \mathrm{d}A$$



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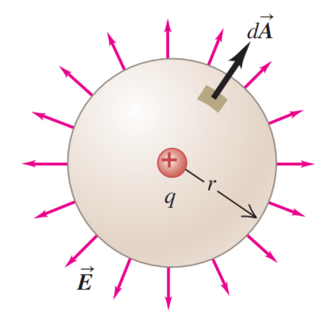

Because of the symmetry, at any point on the sphere of radius r the electric field has the same magnitude $E = \frac{q}{4\pi\epsilon_0 r^2}$ hence we can take it outside the integral

$$\Phi_E = E \int \mathrm{d}A = EA$$

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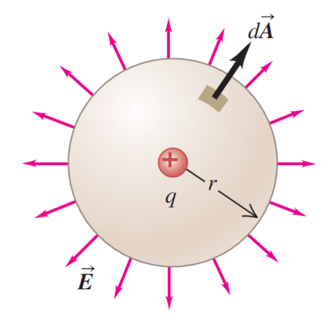


As $A = 4\pi r^2$ we obtain

$$\Phi_E = EA = \frac{q}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{q}{\epsilon_0}$$

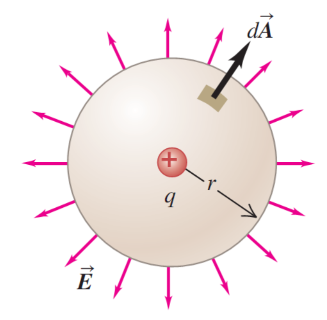
Exercise: Electric flux through a sphere

As $A = 4\pi r^2$ we obtain

$$\Phi_E = EA = \frac{q}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{q}{\epsilon_0}$$

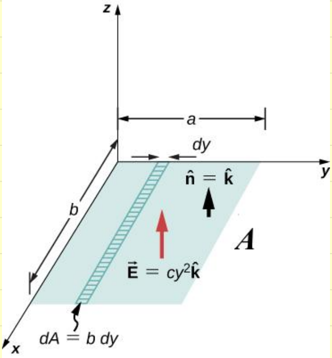
Plugging in the numbers

$$\begin{aligned}\Phi_E &= \frac{q}{\epsilon_0} \\ &= \frac{+3.0\,\mu\mathrm{C}}{8.85\times 10^{-12}\,\mathrm{C^2/N\cdot m^2}} \\ &= 3.4\times 10^5\,\mathrm{N\cdot m^2/C}\end{aligned}$$



Exercise

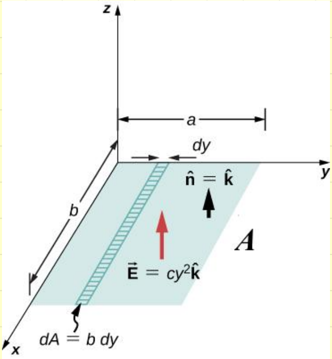
What is the total flux of the electric field through the rectangular surface shown in the figure?



Exercise

We will apply $\Phi_E = \int \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}}$ where $\vec{\mathbf{E}} = cy^2\hat{\mathbf{k}}$ and $\mathrm{d}\vec{\mathbf{A}} = b\mathrm{d}y\hat{\mathbf{k}}$ and so

$$\Phi_E = \int_0^a \underbrace{cy^2}_{E}\,\underbrace{b\,\mathrm{d}y}_{\mathrm{d}A}\,\underbrace{\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}}_{=1}$$



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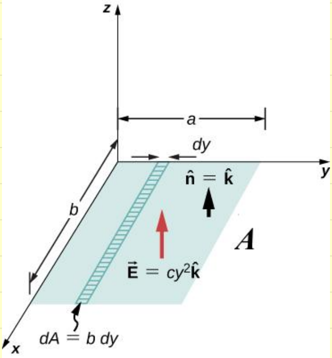
$$\Phi_E = \int_0^a \underbrace{cy^2}_{E} \underbrace{b\,\mathrm{d}y}_{\mathrm{d}A} \underbrace{\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}}_{=1}$$

$$= cb \int_0^a y^2\,\mathrm{d}y$$



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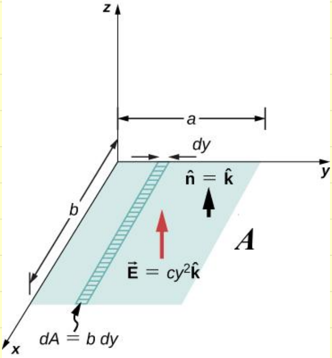
$$\Phi_E = \int_0^a \underbrace{cy^2}_{E} \underbrace{b\,\mathrm{d}y}_{\mathrm{d}A} \underbrace{\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}}_{=1}$$

$$= \frac{1}{3}a^3bc$$



Carl Friedrich Gauss (1777-1855)

- one of the greatest mathematicians
- helped develop several branches of mathematics
- calculated the orbit of the first asteroid to be discovered
- also made state-of-the-art investigations of the earths magnetism
- CGS unit of magnetic field named after him

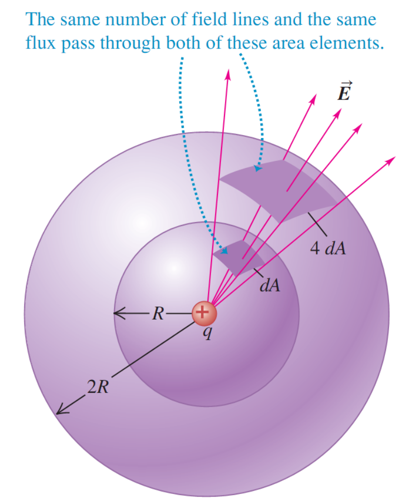
Gauss's Law

$\vec{\mathbf{E}}$ is the <u>total</u> field at the position of the surface area element $\mathrm{d}\vec{\mathbf{A}}$.

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

The total electric flux Φ_E through a closed surface is equal to the total (net) electric charge inside the surface, divided by ϵ_0.

Point Charge Inside a Spherical Surface



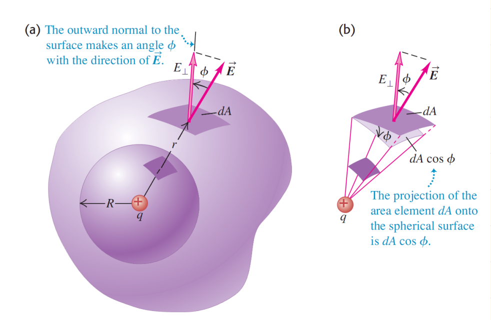
- E-field magnitude at radius R: $E(R) = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}$
- Radial direction is normal to spherical surface
- Surface area $A = 4\pi R^2$
- Flux

$$\Phi_E = EA = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}4\pi R^2$$

$$\Rightarrow \Phi_E = \frac{q}{\epsilon_0}$$

Flux is independent of the radius of the sphere.

Point Charge Inside a Nonspherical Surface



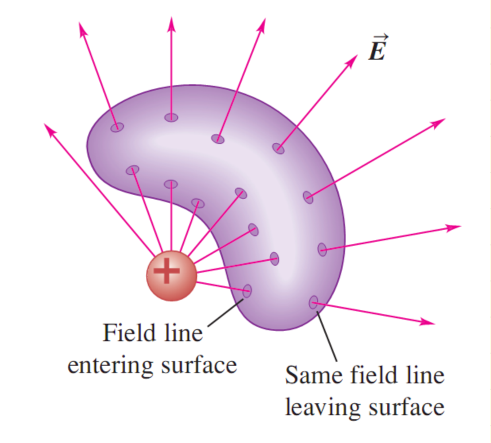
E-field not in the direction of the normal to the surface:
If $\vec{\mathbf{E}}$ makes an angle with the normal, then

$$\mathrm{d}\Phi_E = E\mathrm{d}A\cos\phi$$

For a closed surface enclosing the point charge q

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}} = \frac{q}{\epsilon_0}$$

Point Charge Inside a Nonspherical Surface



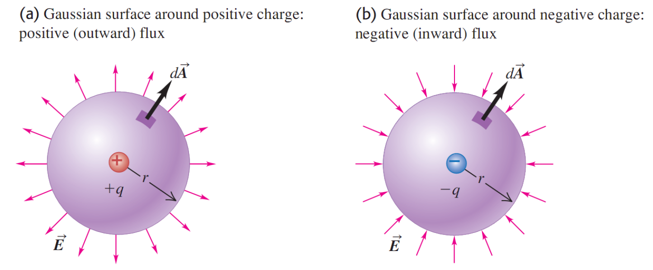
For a closed surface enclosing no charge

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}} = 0$$

General Form of Gausss Law

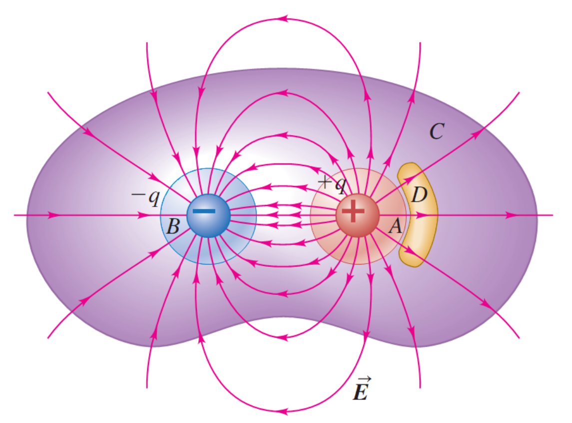
$$\Phi_E = \oint E\cos\phi\,\mathrm{d}A = \oint E_\perp \mathrm{d}A = \oint \vec{\mathbf{E}}\cdot \mathrm{d}\vec{\mathbf{A}} = \frac{Q_{\mathrm{enc}}}{\epsilon_0}$$

For a negative charge, the direction of the field is reversed.



The flux in (b) is negative, i.e. $\Phi_E = -q/\epsilon_0 < 0$.

Electric flux and enclosed charge



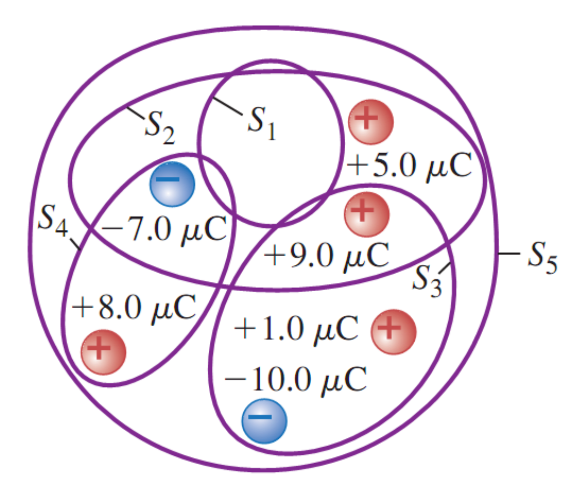
$\Phi_E^A = ?$

$\Phi_E^B = ?$

$\Phi_E^C = ?$

$\Phi_E^D = ?$

Electric flux and enclosed charge



$\Phi_E^{S_1} = ?$

$\Phi_E^{S_2} = ?$

$\Phi_E^{S_3} = ?$

$\Phi_E^{S_4} = ?$

$\Phi_E^{S_5} = ?$

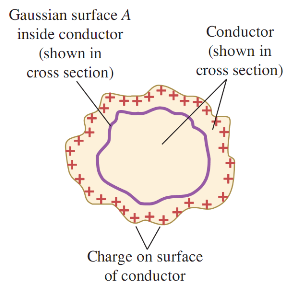
Applications of Gauss' Law

Gauss' law can be used in two ways.

- If we know the charge distribution, and if it has enough symmetry to let us evaluate the integral in Gauss' law, we can find the field.
- If we know the field, we can use Gauss' law to find the charge distribution, such as charges on conducting surfaces.

Charged conductor in equilibrium

When *excess* charge is placed on a solid conductor and is at rest, it resides entirely on the surface, not in the interior of the material.



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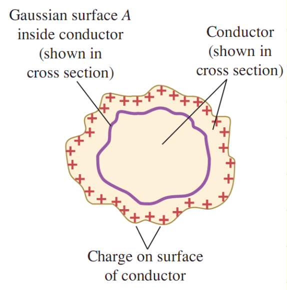
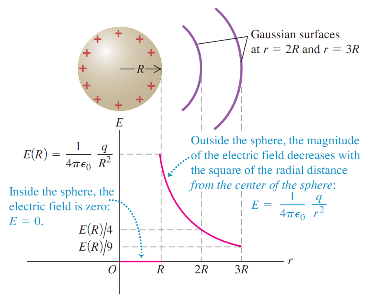


Proof:

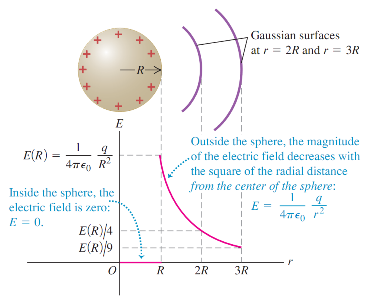
- In an electrostatic situation (with all charges at rest) $\vec{\mathbf{E}}$ at every point in the interior of a conducting material is zero.
- If $\vec{\mathbf{E}}$ were not zero, the excess charges would move.
- Consider a Gaussian surface inside the conductor, such as surface A. Because $\vec{\mathbf{E}} = \vec{\mathbf{0}}$ everywhere on this surface, Gauss' law requires that the net charge inside the surface is zero.
- Thus, there can be no excess charge at any point within a solid conductor; any excess charge must reside on the conductor's surface.

Conducting sphere with charge $q > 0$



- conductor $\Rightarrow$ charge on the surface
- symmetry $\Rightarrow$ $\vec{\mathbf{E}} = E(r)\hat{\mathbf{r}}$
- symmetry $\Rightarrow$ choose a spherical Gaussian surface
- $Q_{\text{enc}} = q$ for $r > R$ and zero otherwise
- Flux $\Phi_E = E(r)\,4\pi r^2$

Conducting sphere with charge $q > 0$



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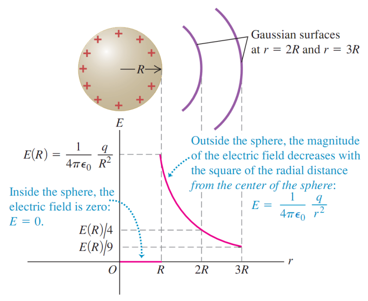

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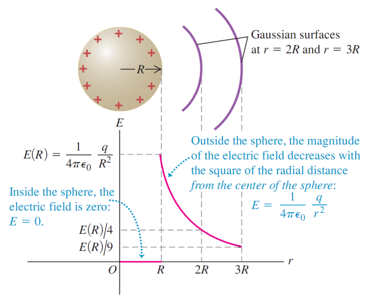

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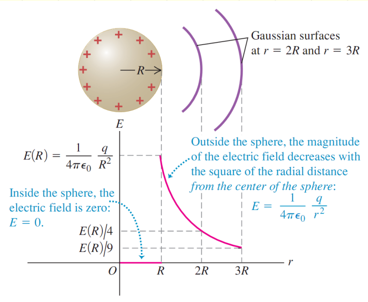

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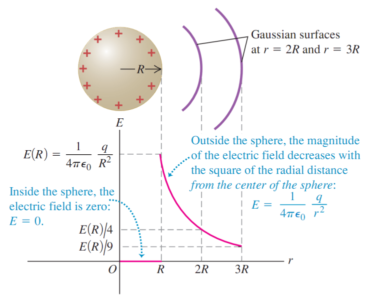

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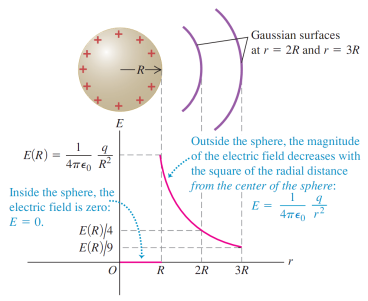
- Flux $\Phi_E = E(r)\, 4\pi r^2$

$$\Phi_E = Q_{\text{enc}}/\epsilon_0$$

$$E(r) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}, & r > R \\ 0, & r < R \end{cases}$$

Conducting sphere with charge $q > 0$



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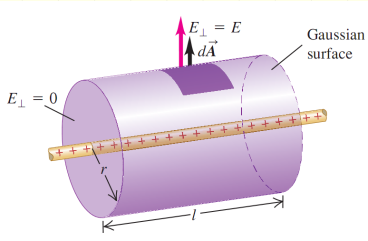
- Flux $\Phi_E = E(r)\,4\pi r^2$

$$\Phi_E = Q_{\text{enc}}/\epsilon_0$$

$$E(r) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}, & r > R \\ 0, & r < R \end{cases}$$

For $r > R$ electric field same as a point charge q located at the center.

an infinitely long, charged wire



- symmetry $\Rightarrow \vec{\mathbf{E}} = E(r)\hat{\mathbf{r}}$
- symmetry $\Rightarrow$ choose a cylindrical Gaussian surface
- $Q_{\mathrm{enc}} = \lambda l$ with linear charge density λ
- Adjacent side of the cylinder: $2\pi r l$
- Flux $\Phi_E = \oint \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}} = 2\pi r l E(r) + 0 + 0$ (no flux from the caps)

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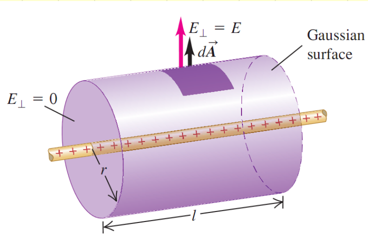

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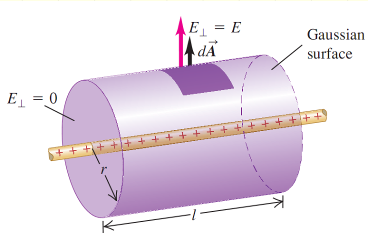

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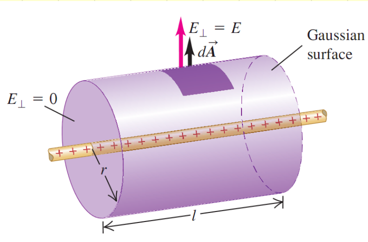

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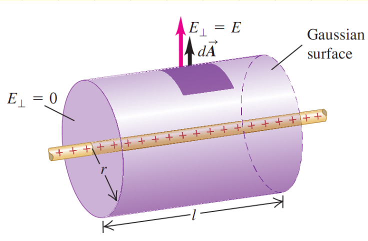

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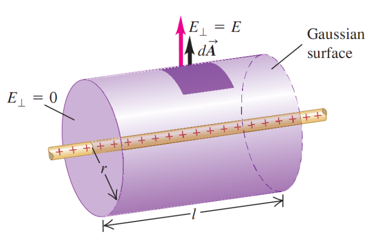

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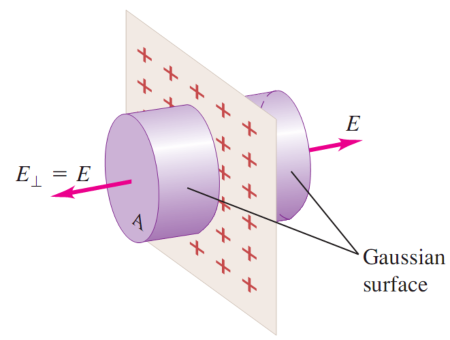
$\Phi_E = Q_{\text{enc}}/\epsilon_0$

$$\vec{\mathbf{E}}(r) = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}\hat{\mathbf{r}}$$

Exercise: Check units of the E-field.
Recall: We found the same result last week by integrating the field of a line of charge.

infinite sheet of charge

Choose coordinates such that the plane is at $z = 0$.



- surface charge density σ
- symmetry $\Rightarrow$
  $\vec{\mathbf{E}} = E(|z|)\hat{\mathbf{k}}$ for $z > 0$,
  $\vec{\mathbf{E}} = E(|z|)(-\hat{\mathbf{k}})$, $z < 0$.
- choose a cylindrical Gaussian surface
- $Q_{\text{enc}} = \sigma A$
- Flux $\Phi_E = \oint \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}} = 2AE(|z|)$ (no flux from the adjacent side)

infinite sheet of charge

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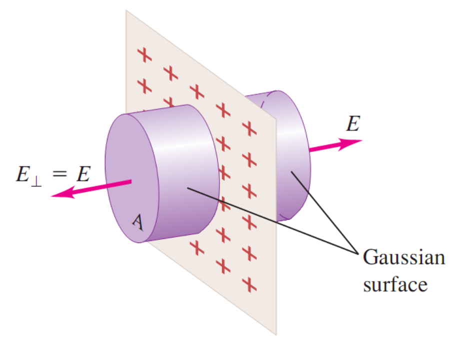

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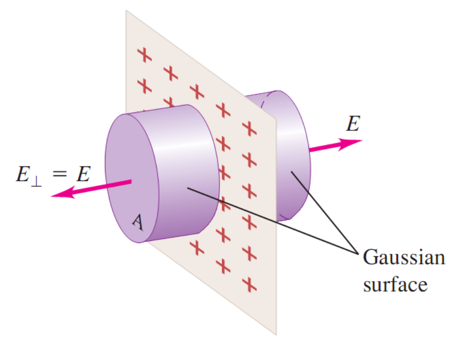

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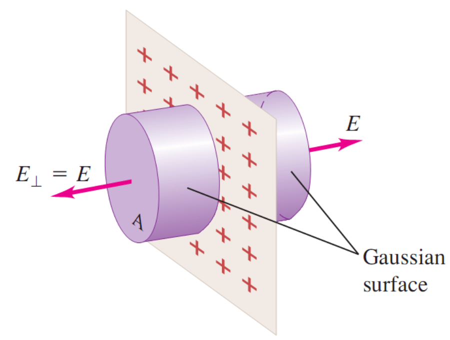

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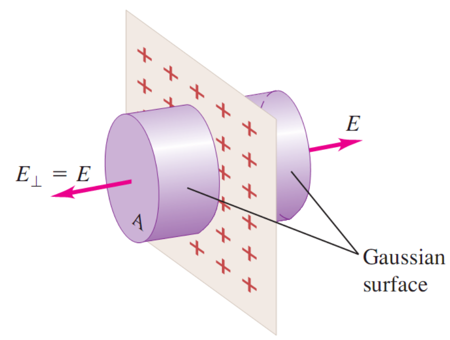

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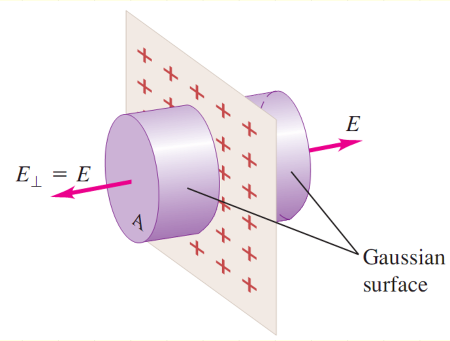

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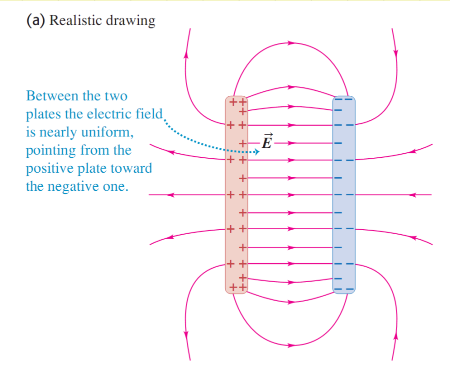
$$\oint \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}} = Q_{\mathrm{enc}}/\epsilon_0 \rightarrow 2EA = \sigma A/\epsilon_0 \rightarrow$$

$$E = \frac{\sigma}{2\epsilon_0} \qquad \vec{\mathbf{E}} = \frac{\sigma}{2\epsilon_0}\frac{z}{|z|}\hat{\mathbf{k}}$$

The field magnitude is independent of z.

Note: Discontinuity in E-field crossing a charged surface

oppositely charged parallel conducting plates



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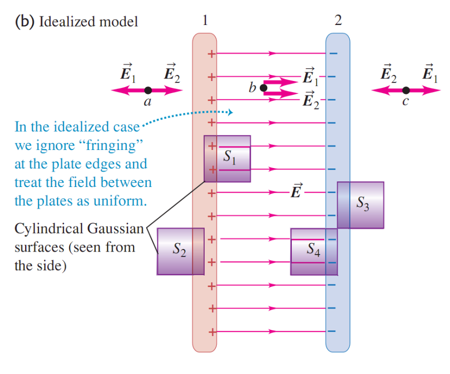


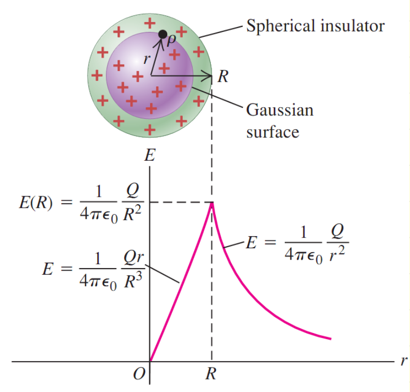
- idealize as two infinite sheets of charge
- use superposition principle or Gauss' law
- E-field in between

$$E = \sigma/\epsilon_0$$

- E-field outside

$$E = 0$$

uniformly charged sphere



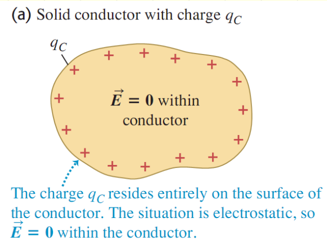
- uniform charge density ρ
- $\rho = Q/(4\pi R^3/3)$
- symmetry $\Rightarrow \vec{\mathbf{E}} = E(r)\hat{\mathbf{r}}$
- symmetry $\Rightarrow$ choose a spherical Gaussian surface
- $Q_{\rm enc} = Q$ for $r > R$
- $Q_{\rm enc} = 4\pi r^3\rho/3$ for $r < R$
- Flux $\Phi_E = 4\pi r^2 E(r)$

For $r < R$, E-field increases linearly with r

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{4\pi r^3\rho/3}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}, \quad r < R\,.$$

For $r > R$, E is the same as a point charge Q located at the center.

Charges on conductors

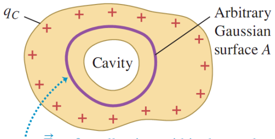


- Apply Gauss' law for any closed surface inside the conductor
- Since $\vec{\mathbf{E}} = \vec{\mathbf{0}}$ inside the conductor, any volume inside a conductor contains zero net charge.
- $\Rightarrow$ all the excess charge on the conductor must be on its surface

What if there is a *cavity* inside the conductor?
$\Rightarrow$ Again, no charge on the cavity surface.

Charges on conductors

(b) The same conductor with an internal cavity



Because $\vec{E} = \mathbf{0}$ at all points within the conductor, the electric field at all points on the Gaussian surface must be zero.

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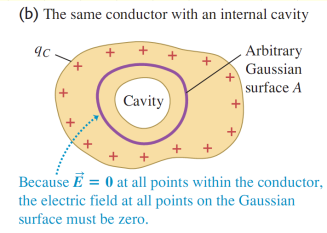

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What if there is a *cavity* inside the conductor?
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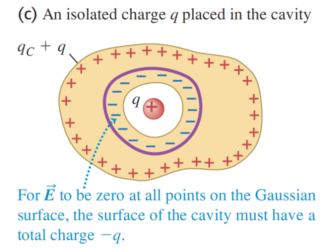
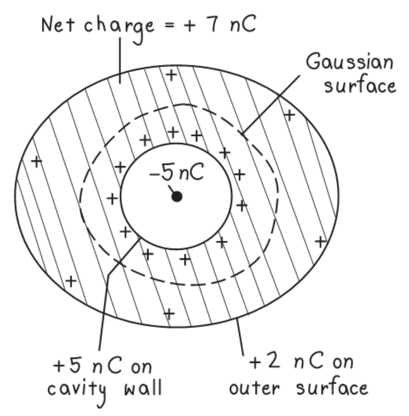
Charge inside a cavity in conductor

Consider charge q show in the figure. Assume charge q_C on the conductor.



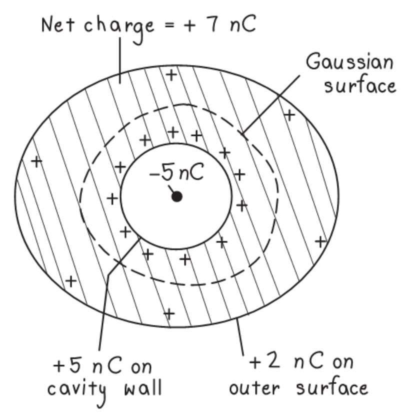
- Apply Gauss' law for any closed surface enclosing the cavity.
- Since $\vec{\mathbf{E}} = \vec{\mathbf{0}}$ inside the conductor, cavity surface must have charge $-q$.
- $\Rightarrow$ charge $q + q_C$ must be on the outer surface of the conductor.

Example



- **Q:** How much charge is on the inner and outer surface of the conductor?
- Consider the Gaussian surface shown with dashed lines.
- Net charge inside must be zero.
- $\Rightarrow$ charge on the inner surface is $+5\,\mathrm{nC}$.
- $\Rightarrow$ charge on the outer surface is $+7\,\mathrm{nC} - 5\,\mathrm{nC} = 2\,\mathrm{nC}$.

Example



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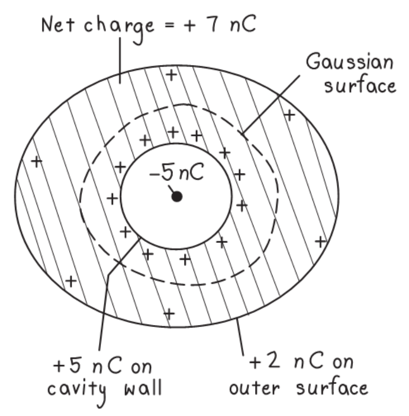

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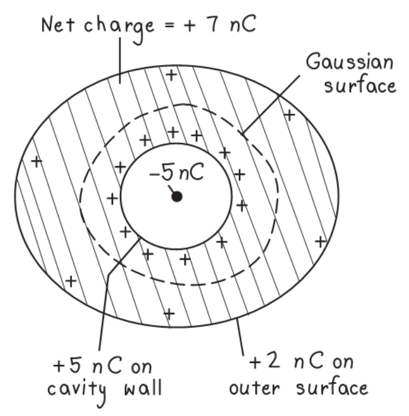

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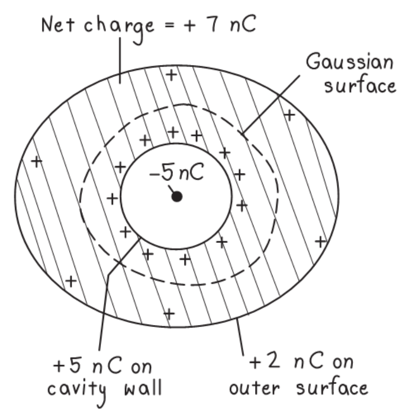

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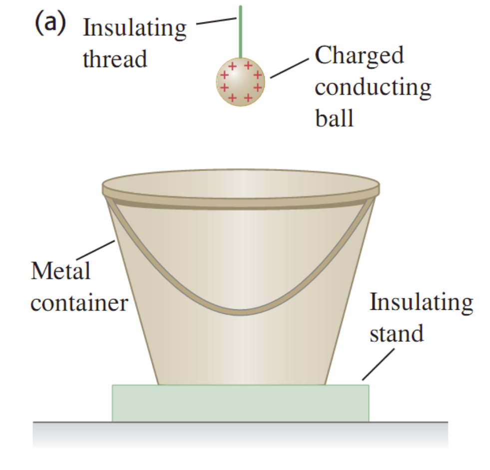
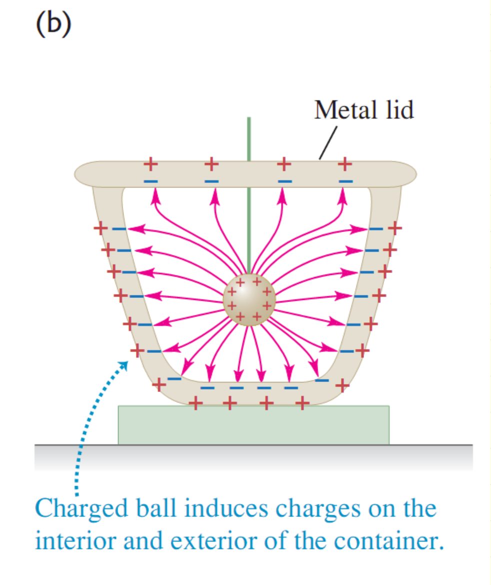
Testing Gauss' Law

A charged metal sphere is lowered into a conducting container.



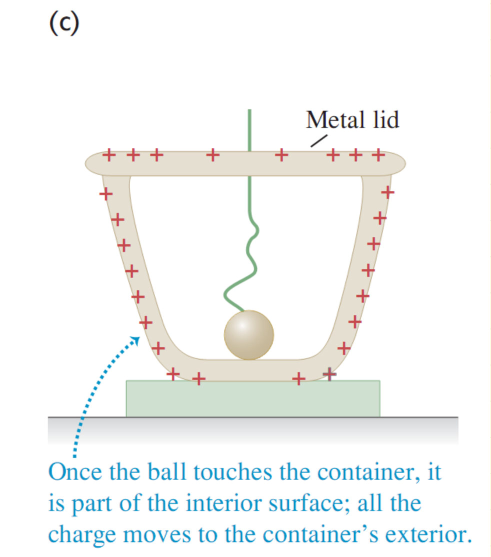
Testing Gauss' Law

The charged sphere is enclosed in the conducting container.

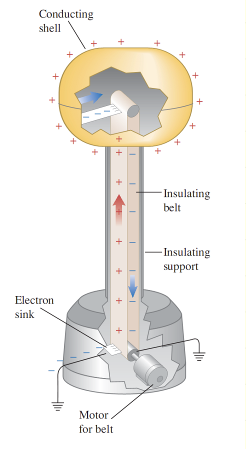


Testing Gauss' Law

The charged metal sphere touches the conducting container.

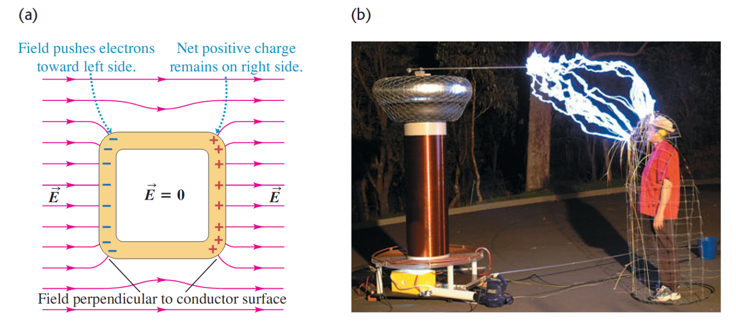


Van de Graaff electrostatic generator



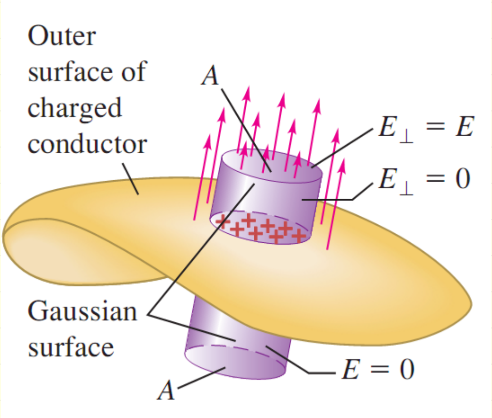
- belt produces charge build up
- by Gauss' law charge moves to the outside
- large E-fields around the shell
- used as an accelerator of charged particles and physics demos

Electrostatic shielding



- redistribution of the free electrons in the conductor
- redistribution causes an additional E-field such that the total field at every point inside the box is zero
- also alters the shapes of the field lines near the box
- such a setup is often called a *Faraday cage*.

Field at the Surface of a Conductor



- Assume surface charge density σ.
- Apply Gauss' law:

$$\Phi_E = E_\perp A = EA$$

and

$$Q_{\text{enc}} = \sigma A.$$

$$\Rightarrow E = \frac{\sigma}{\epsilon_0}$$