Electric potential

FIZ102E: Electricity & Magnetism

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Outline

Learning Goals

- How to calculate the electric potential energy of a collection of charges.
- The meaning and significance of electric potential.
- How to calculate the electric potential that a collection of charges produces at a point in space.
- How to use equipotential surfaces to visualize how the electric potential varies in space.
- How to use electric potential to calculate the electric field.

Maxwell's Equations and the Lorentz Force

Gauss' law

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

Gauss' law for magnetism

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

Faraday's law

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d}{dt} \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$$

Generalized Ampere's law

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 i_{\mathrm{C}} + \mu_0 \epsilon_0 \frac{d}{dt} \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

Lorentz force on a particle with charge q and velocity $\vec{\mathbf{v}}$ moving in $\vec{\mathbf{E}}$ & $\vec{\mathbf{B}}$ fields

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right)$$

Introduction

- When a charged particle moves in an electric field, the field exerts a force that can do work on the particle.
- This work can be expressed in terms of electric potential energy.
- Just as gravitational potential energy depends on the height of a mass above the earth's surface, electric potential energy depends on the position of the charged particle in the electric field.
- We'll use a new concept called electric potential, or simply potential to describe electric potential energy.
- In circuits, a difference in potential from one point to another is often called voltage.

Electric potential energy

Review

- In mechanics work, potential energy, and conservation of energy were extremely useful.
- these concepts are just as useful for understanding and analyzing electrical interactions.
- When a force $\vec{\mathbf{F}}$ acts on a particle that moves from point a to point b, the work $W_{a\to b}$ done by the force is given by a line integral:

$$W_{a\to b} = \int_a^b \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{l}} = \int_a^b F\cos\phi\,\mathrm{d}l$$

where $\mathrm{d}\vec{\mathbf{l}}$ is an infinitesimal displacement along the particles path and ϕ is the angle between $\vec{\mathbf{F}}$ and $\mathrm{d}\vec{\mathbf{l}}$ at each point along the path.

Review

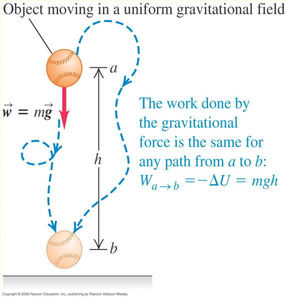
- If $\vec{\mathbf{F}}$ is conservative, the work done by $\vec{\mathbf{F}}$ can always be expressed in terms of a *potential energy* U.
- When the particle moves from a point where U is U_a to a point where it is U_b, the change in potential energy is $\Delta U = U_b - U_a$ and

$$W_{a \to b} = U_a - U_b = -(U_b - U_a) = -\Delta U$$

- When $W_{a \to b} > 0$, $U_a > U_b$, $\Delta U < 0$, and U *decreases.*

Review

- The baseball falls from a high point (a) to a lower point (b) under the influence of the earth's gravity; the force of gravity does positive work, and the gravitational potential energy decreases
- When a tossed ball is moving upward, the gravitational force does negative work during the ascent, and the potential energy increases.



Review

- The work-energy theorem says that the change in kinetic energy $\Delta K = K_b - K_a$ during a displacement equals the total work done on the particle.
- If only conservative forces do work, then $W_{a \to b} = U_a - U_b$ gives the total work, and $K_b - K_a = -(U_b - U_a)$.
- We usually write this as

$$K_a + U_a = K_b + U_b$$

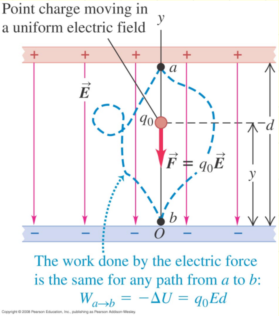
- The total mechanical energy (kinetic plus potential) is conserved under these circumstances.

Electric Potential Energy in a Uniform Field

- A pair of large, charged, parallel metal plates sets up a uniform, downward E.
- E exerts a downward force with magnitude $F = q_0 E$ on a positive test charge q_0.
- As the q_0 moves downward a distance d from a to b, F on q_0 is constant and independent of its location.
- The work done by E is

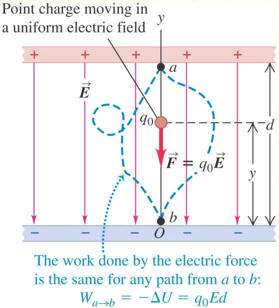
$$W_{a\to b} = Fd = q_0 Ed$$

- This work is $+$, since F is in the same direction as the net displacement of q_0.



Electric Potential Energy in a Uniform Field

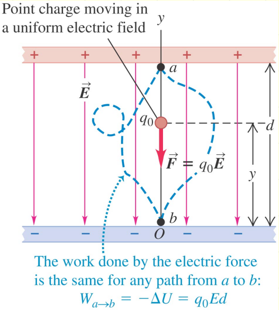
- The y-component of the electric force, $F_y = -q_0 E$, is constant, and there is no x- or z-component.
- This is exactly analogous to the gravitational force on a mass m near the earth's surface; for this force, there is a constant y-component $F_y = -mg$ and the x- and z-components are zero.
- Because of this analogy, we can conclude that the force exerted on q_0 by the uniform electric field is conservative, just as is the gravitational force.



Electric Potential Energy in a Uniform Field

- This means that the work $W_{a\to b}$ done by the field is independent of the path the particle takes from a to b.
- We can represent this work with a potential-energy function U.
- The potential energy for the gravitational force $F_y = -mg$ was $U = mgy$; hence the potential energy for the electric force $F_y = -q_0 E$ is

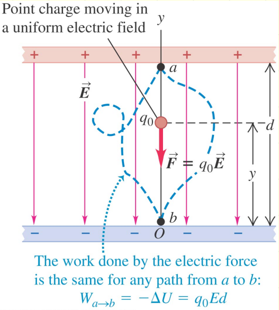
$$U = q_0 E y$$



Electric Potential Energy in a Uniform Field

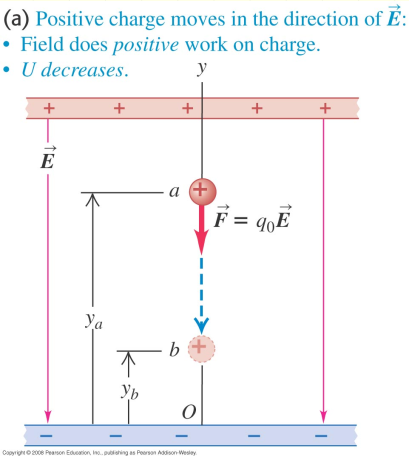
- When the test charge moves from height y_a to height y_b, the work done on the charge by the field is given by

$$W_{a\to b} = -\Delta U = qE(y_a - y_b)$$



Electric Potential Energy in a Uniform Field

- When $y_a > y_b$, the $+$ test charge q_0 moves downward, in the same direction as $\vec{\mathbf{E}}$;
- The displacement is in the same direction as the force $\vec{\mathbf{F}} = q_0\vec{\mathbf{E}}$, so the field does positive work and U decreases.



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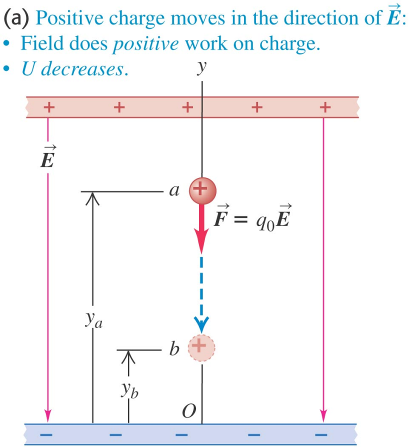
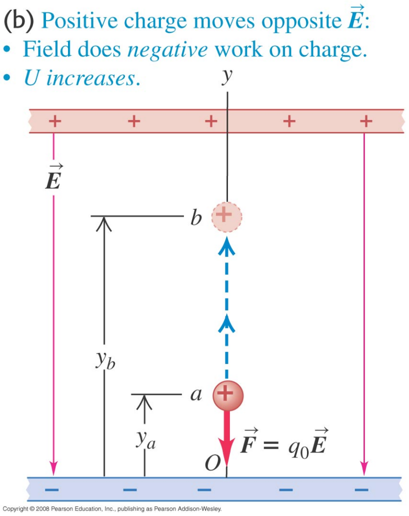


Electric Potential Energy in a Uniform Field

- When $y_a < y_b$, the positive test charge q_0 moves upward, in the opposite direction to $\vec{\mathbf{E}}$.
- the displacement is opposite the force, the field does negative work, and U increases.



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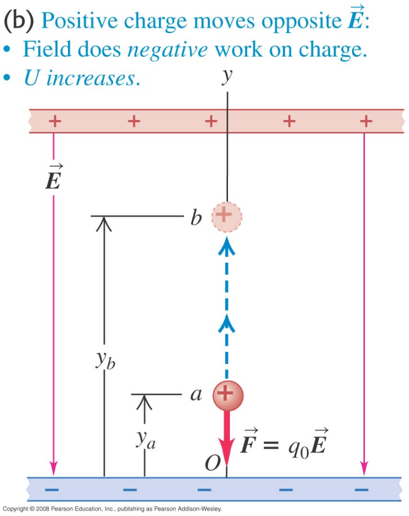
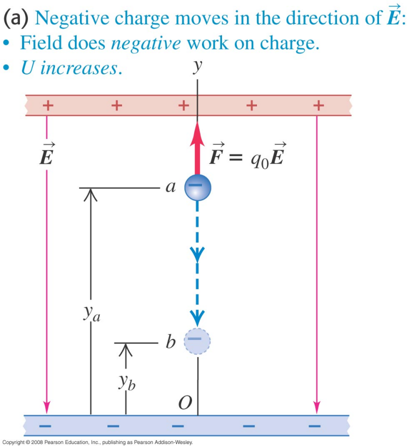


Electric Potential Energy in a Uniform Field

If the test charge $q_0 < 0$

- U increases when it moves with the field and
- U decreases when it moves against the field.



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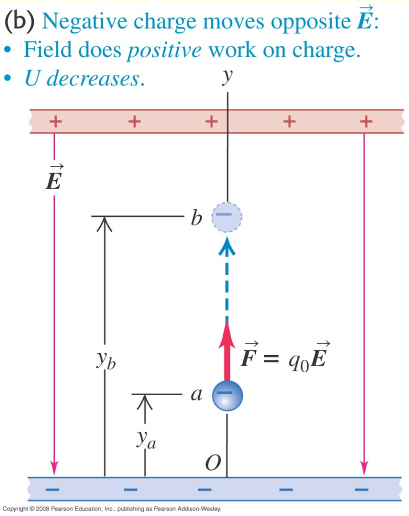
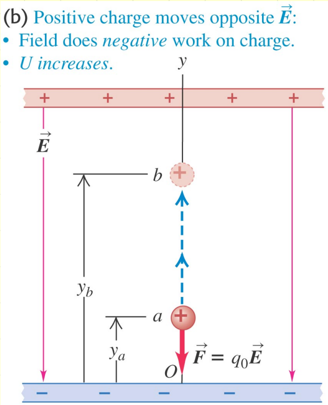


Electric Potential Energy in a Uniform Field

Whether the test charge is $+$ or $-$, the following general rules apply:

- U *increases* if q_0 moves in the direction *opposite* the electrical force $\vec{\mathbf{F}} = q_0\vec{\mathbf{E}}$
- U *decreases* if q_0 moves in the *same* direction as $\vec{\mathbf{F}} = q_0\vec{\mathbf{E}}$



$q_0 > 0$

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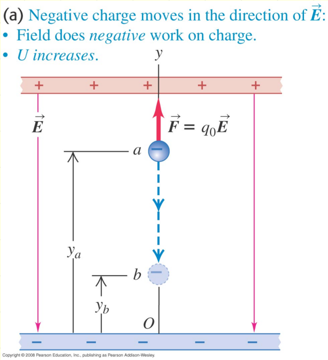

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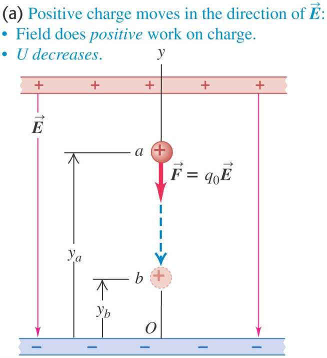

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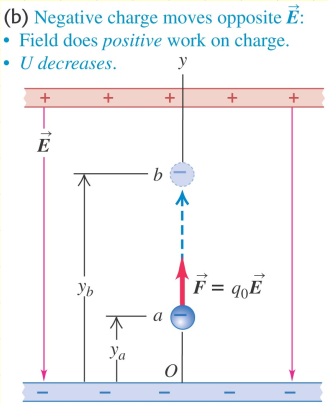

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Electric Potential Energy in a Uniform Field

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This is the same behavior as for gravitational potential energy which increases if a mass m moves upward (opposite the direction of the gravitational force) and decreases if m moves downward (in the same direction as the gravitational force).

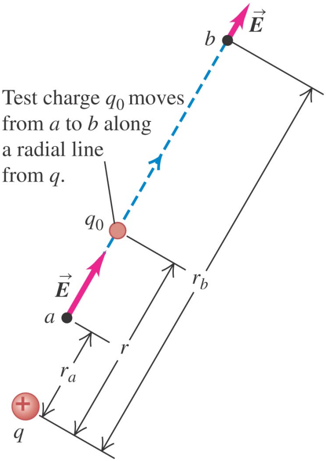
Electric Potential Energy of Two Point Charges

- The idea of electric potential energy isn't restricted to the special case of a uniform $\vec{\mathbf{E}}$.
- We can apply this concept to a point charge in *any* $\vec{\mathbf{E}}$ caused by a *static* charge distribution.
- Recall: we can represent any charge distribution as a collection of point charges.
- Therefore it's useful to calculate the work done on a test charge q_0 moving in the electric field caused by a single, stationary point charge q.

Electric Potential Energy of Two Point Charges

- First consider a displacement along the radial line.
- The force on q_0 is given by Coulomb's law, and its radial component is

$$F_r = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2}$$

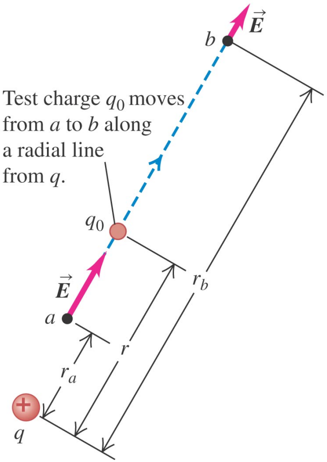


Electric Potential Energy of Two Point Charges

- The force is *not* constant during the displacement, and we must integrate to calculate the work $W_{a\to b}$ done on q_0 by this force as q_0 moves from a to b:

$$W_{a\to b} = \int_{r_a}^{r_b} F_r \, \mathrm{d}r = \frac{qq_0}{4\pi\epsilon_0}\int_{r_a}^{r_b}\frac{\mathrm{d}r}{r^2}$$
$$= \frac{qq_0}{4\pi\epsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$$

- The work done by the electric force for this path depends on only the endpoints.

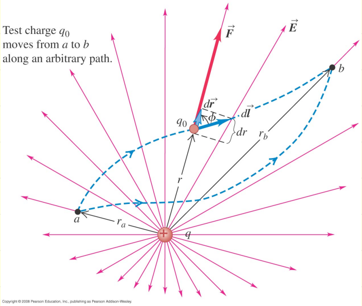


Electric Potential Energy of Two Point Charges

- Now consider a more general displacement in which a and b do not lie on the same radial line.
- From $W_{a\to b} = \int_a^b \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{l}}$ the work done on q_0 during this displacement is given by

$$W_{a\to b} = \int_{r_a}^{r_b} F \cos\phi \, \mathrm{d}l$$

$$= \frac{q q_0}{4\pi\epsilon_0} \int_{r_a}^{r_b} \frac{\cos\phi \, \mathrm{d}l}{r^2}$$

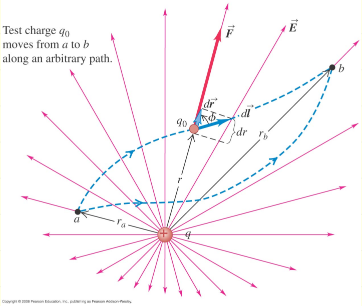


Electric Potential Energy of Two Point Charges

- But Fig. shows that $\cos\phi\,\mathrm{d}l = \mathrm{d}r$.
- The work done during a small displacement $\mathrm{d}\vec{\mathbf{l}}$ depends only $\mathrm{d}r$
- Thus

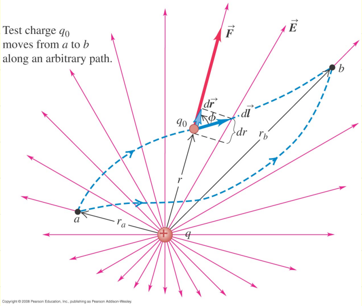
$$W_{a\to b} = \frac{qq_0}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

is valid even for this more general displacement.
- The work done on q_0 by $\vec{\mathbf{E}}$ produced by q depends only on the endpoints r_a and r_b, not on the details of the path.



Electric Potential Energy of Two Point Charges

- Also, if q_0 returns to its starting point a by a different path, the total work done in the round-trip displacement is zero.
- These are the needed characteristics for a conservative force.
- Thus the force on q_0 is a conservative force.

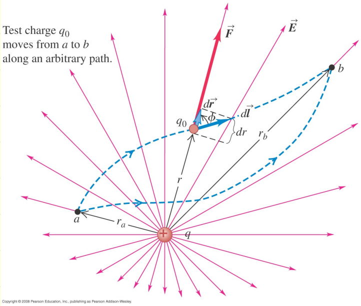


Electric Potential Energy of Two Point Charges

- We see that $W_{a\to b} = -\Delta U$ and

$$W_{a\to b} = \frac{qq_0}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

  are consistent if we define the potential energy to be $U_a = qq_0/(4\pi\epsilon_0 r_a)$ when q_0 is a distance r_a to q, and to be $U_b = qq_0/(4\pi\epsilon_0 r_b)$ when q_0 is a distance r_b to q.



Electric Potential Energy of Two Point Charges

- Thus

$$\boxed{U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r}}$$

  is valid no matter what the signs of the charges q and q_0.

- The potential energy is positive if the charges q and q_0 have the same sign.

- and negative if they have opposite signs

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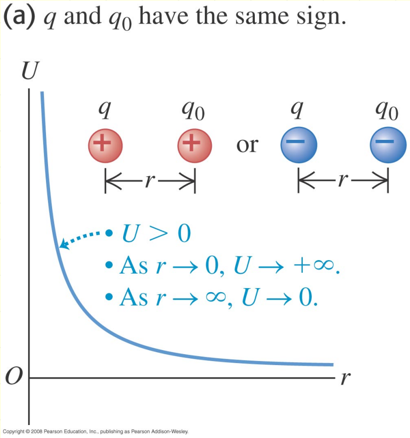

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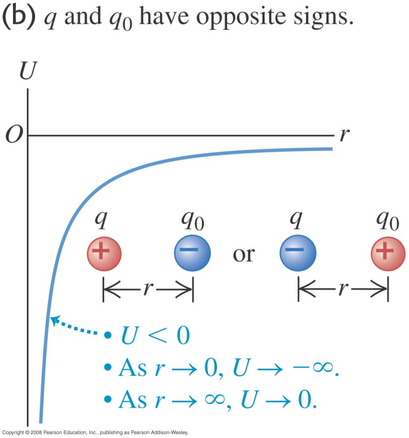

Electric Potential Energy of Two Point Charges

- Potential energy is always defined relative to some reference point where $U = 0$.
- In $U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r}$, U is zero when q and q_0 are infinitely far apart $(r \to \infty)$.
- Therefore U represents the work that would be done on the test charge q_0 by the field of q if q_0 moved from an initial distance r to infinity.

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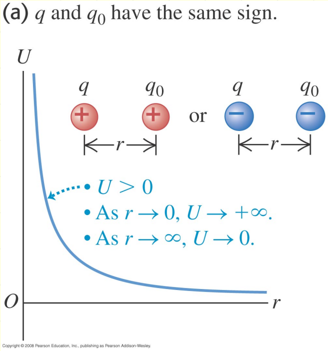
Electric Potential Energy of Two Point Charges

- If q and q_0 have the same sign, the interaction is repulsive, $W_{a\to b} > 0$ and $U > 0$ at any finite separation.
- If q and q_0 have the opposite signs, the interaction is attractive, $W_{a\to b} < 0$ and $U < 0$.



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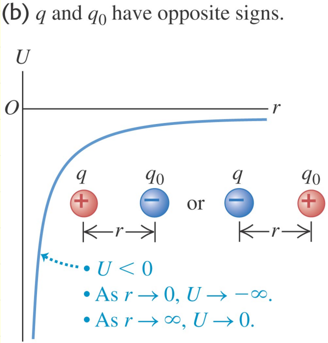


Electric Potential Energy of Two Point Charges

- The potential energy U given by

$$U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r}$$

  is a *shared* property of the two charges.
- If the distance between q and q_0 is changed from r_a to r_b, the change in potential energy is the same whether q is held fixed and q_0 is moved or q_0 is held fixed and q is moved.
- For this reason, we never say "the electric potential energy of a point charge."
- Likewise, if a mass m is at a height h above the earth's surface, the gravitational potential energy is a shared property of the mass m and the earth.

Electric Potential Energy of Two Point Charges

- The equation

$$U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r}$$

  also holds if the charge q_0 is outside a spherically symmetric charge distribution with total charge q; the distance r is from q_0 to the center of the distribution.
- This is because Gauss' law tells us that the electric field outside such a distribution is the same as if all of its charge q were concentrated at its center.

Example: Conservation of energy with electric forces

A positron (the electron's antiparticle) has mass $9.11 \times 10^{-31}\,\mathrm{kg}$ and charge $q_0 = +e = +1.60 \times 10^{-19}\,\mathrm{C}$. Suppose a positron moves in the vicinity of an α (alpha) particle, which has charge $q = +2e = 3.20 \times 10^{-19}\,\mathrm{C}$ and mass $6.64 \times 10^{-27}\,\mathrm{kg}$. The α particle's mass is more than 7000 times that of the positron, so we assume that the α particle remains at rest. When the positron is $1.00 \times 10^{-10}\,\mathrm{m}$ from the α particle, it is moving directly away from the α particle at $3.00 \times 10^{6}\,\mathrm{m/s}$.

(a) What is the positron's speed when the particles are $2.00 \times 10^{-10}\,\mathrm{m}$ apart?

(b) What is the positron's speed when it is very far from the α particle?

(c) Suppose the initial conditions are the same but the moving particle is an electron (with the same mass as the positron but charge $q_0 = -e$). Describe the subsequent motion.

Example: Conservation of energy with electric forces

IDENTIFY and SET UP

- The electric force between a positron (or an electron) and an α particle is conservative, so the total mechanical energy (kinetic plus potential) is conserved. $U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r}$ gives the potential energy U at any separation r.
- We are given the positron speed $3.00 \times 10^6\,\mathrm{m/s}$ when the separation between the particles is $r_a = 1.00 \times 10^{-10}\,\mathrm{m}$.
- In parts (a) and (b) we use $K_a + U_a = K_b + U_b$ and $U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r}$ to find the speed for $r = r_b = 2.00 \times 10^{-10}\,\mathrm{m}$ and $r = r_c \to \infty$, respectively.
- In part (c) we replace the positron with an electron and reconsider the problem.

Example: Conservation of energy with electric forces

Question (a): What is the positron's speed when the particles are $2.00 \times 10^{-10}\,\mathrm{m}$ apart?

Solution (a):

- Both particles have positive charge, so the positron speeds up as it moves away from the α particle.
- From the energy-conservation equation $K_a + U_a = K_b + U_b$ $K_b = \frac{1}{2}mv_b^2 = K_a + U_a - U_b$.
- Here $K_a = \frac{1}{2}mv_a^2 = \frac{1}{2}(9.11 \times 10^{-31}\,\mathrm{kg})(3.00 \times 10^{6}\,\mathrm{m/s})^2 = 4.10 \times 10^{-18}\,\mathrm{J}$.
- $U_a = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r_a} = (9 \times 10^{9}\,\mathrm{N \cdot m^2/C^2})\frac{(3.20\times10^{-19}\,\mathrm{C})(1.60\times10^{-19}\,\mathrm{C})}{1.00\times10^{-10}\,\mathrm{m}} = 4.61 \times 10^{-18}\,\mathrm{J}$
- $U_b = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r_b} = 3.30 \times 10^{-18}\,\mathrm{J}$
- Hence $K_b = K_a + U_a - U_b = 4.10 \times 10^{-18}\,\mathrm{J} + 4.61 \times 10^{-18}\,\mathrm{J} - 3.30 \times 10^{-18}\,\mathrm{J} = 6.41 \times 10^{-18}\,\mathrm{J}$.
- From this follows $v_b = \sqrt{\frac{2K_b}{m}} = \sqrt{\frac{2(6.41\times10^{-18}\,\mathrm{J})}{(9.11\times10^{-31}\,\mathrm{kg})}} = 3.8 \times 10^{6}\,\mathrm{m/s}$

Example: Conservation of energy with electric forces

Question (b): What is the positron's speed when it is very far from the α particle?

Solution (b):

- When the positron and α particle are very far apart so that $r = r_c \to \infty$, the final potential energy U_c approaches zero.
- Again from energy conservation, the final kinetic energy is $K_c = K_a + U_a - U_c = 4.10 \times 10^{-18}\,\mathrm{J} + 4.61 \times 10^{-18}\,\mathrm{J} - 0 = 8.71 \times 10^{-18}\,\mathrm{J}$
- and speed of the positron is $v_c = \sqrt{\frac{2K_c}{m}} = \sqrt{\frac{2(8.71\times10^{-18}\,\mathrm{J})}{(9.11\times10^{-31}\,\mathrm{kg})}} = 4.4 \times 10^{6}\,\mathrm{m/s}$

Example: Conservation of energy with electric forces

Question (c): Suppose the initial conditions are the same but the moving particle is an electron (with the same mass as the positron but charge $q_0 = -e$). Describe the subsequent motion.

Solution (c):

- The electron and α particle have opposite charges, so the force is attractive and the electron slows down as it moves away.
- Changing the moving particle's sign from $+e$ to $-e$ means that the initial potential energy is now $U_a = -4.61 \times 10^{-18}\,\mathrm{J}$, which makes the total mechanical energy negative:
  $K_a + U_a = 4.10 \times 10^{-18}\,\mathrm{J} - 4.61 \times 10^{-18}\,\mathrm{J} = -0.51 \times 10^{-18}\,\mathrm{J}$
- The total mechanical energy would have to be positive for the electron to move infinitely far away from the α particle.
- Like a rock thrown upward at low speed from the earths surface, it will reach a maximum separation $r = r_d$ from the α particle before reversing direction.
- At this point its speed and its kinetic energy K_d are zero
  $U_d = K_a + U_a - \cancelto{0}{K_d} = -0.51 \times 10^{-18}\,\mathrm{J}$

Example: Conservation of energy with electric forces

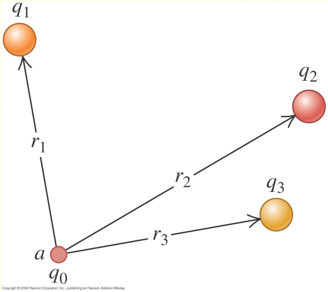
Question (c): Suppose the initial conditions are the same but the moving particle is an electron (with the same mass as the positron but charge $q_0 = -e$). Describe the subsequent motion.

Solution (c):

- $U_d = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r_d} = -0.51 \times 10^{-18}\,\mathrm{J} \to r_d = 9.0 \times 10^{-10}\,\mathrm{m}$.
- For $r_b = 2.00 \times 10^{-10}\,\mathrm{m}$ we have $U_b = 3.30 \times 10^{-18}\,\mathrm{J}$ so the electron kinetic energy and speed at this point are $K_b = K_a + U_a - U_b = (4.10 \times 10^{-18}\,\mathrm{J}) + (-4.61 \times 10^{-18}\,\mathrm{J}) - (-2.30 \times 10^{-18}\,\mathrm{J}) = 1.79 \times 10^{-18}\,\mathrm{J}$
- $v_b = \sqrt{\frac{2K_b}{m}} = \sqrt{\frac{2(1.79\times10^{-18}\,\mathrm{J})}{(9.11\times10^{-31}\,\mathrm{kg})}} = 2.0 \times 10^{6}\,\mathrm{m/s}$

Electric Potential Energy with Several Point Charges

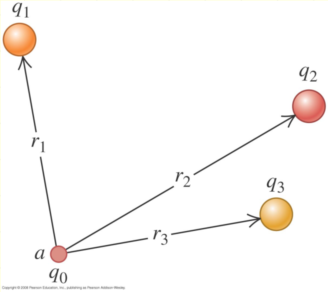
- Suppose the electric field $\vec{\mathbf{E}}$ in which charge q_0 moves is caused by several point charges $q_1, q_2, q_3, \cdots$ at distances $r_1, r_2, r_3, \cdots$ from q_0
- The total electric field at each point is the vector sum of the fields due to the individual charges, and
- the total work done on q_0 during any displacement is the sum of the contributions from the individual charges.



Electric Potential Energy with Several Point Charges

- The potential energy associated with the test charge q_0 at point a is the algebraic sum:

$$U = \frac{q_0}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots\right)$$
$$= \frac{q_0}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$$



Electric Potential Energy with Several Point Charges

- When q_0 is at a different point b, the potential energy is given by the same expression, but r_1, r_2, $\cdots$ are the distances from q_1, q_2, $\cdots$ to point b.
- The work done on charge q_0 when it moves from a to b along any path is equal to the difference $U_a - U_b$ between the potential energies when q_0 is at a and at b.

$$U = \frac{q_0}{4\pi\epsilon_0} \sum_i \frac{q_i}{r_i}$$

Electric Potential Energy with Several Point Charges

- We can represent any charge distribution as a collection of point charges.
- The equation
$$U = \frac{q_0}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$$
shows that we can always find a potential-energy function for any static electric field.
- It follows that **for every electric field due to a static charge distribution, the force exerted by that field is conservative**.

Electric Potential Energy with Several Point Charges

- Equations $U = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r}$ and $U = \frac{q_0}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$ define U to be zero when distances r_1, r_2, $\cdots$ are infinite.
- As with any potential-energy function, the point where $U = 0$ is arbitrary; we can always add a constant to make U equal zero at any point we choose.
- In electrostatics problems it's usually simplest to choose this point to be at infinity.
- When we analyze electric circuits in other choices will be more convenient.

Electric Potential Energy with Several Point Charges

- Equations $U = \frac{q_0}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$ gives the potential energy associated with the presence of the test charge q_0 in the E field produced by q_1, q_2, q_3, $\cdots$.
- But there is also potential energy involved in assembling these charges.
- If we start with charges q_1, q_2, q_3, $\cdots$ all separated from each other by infinite distances and then bring them together so that the distance between q_i and q_j is r_{ij}, the total potential energy U is the sum of the potential energies of interaction for each pair of charges:

$$U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}$$

Electric Potential Energy with Several Point Charges

- The potential energies of interaction for each pair of charges:

$$U = \frac{1}{4\pi\epsilon_0} \sum_{i<j} \frac{q_i q_j}{r_{ij}}$$

- This sum extends over all pairs of charges; we don't let $i = j$
- We include only terms with $i < j$ to make sure that we count each pair only once.

Electric Potential Energy with Several Point Charges

- For 2 particles (1 term)

$$U = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}}\right)$$

- For 3 particles (3 terms)

$$U = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

- For 4 particles (6 terms)

$$U = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}}\right)$$

- For N particles $N(N-1)/2$ terms

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- For N particles $N(N-1)/2$ terms

Interpreting Electric Potential Energy

- There are two viewpoints on electric potential energy.
- We have defined it in terms of the work done by the electric field on a charged particle moving in the field.
- When a particle moves from point a to point b, the work done on it by the electric field is $W_{a \to b} = U_a - U_b$.

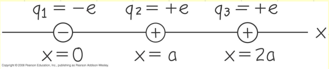
Interpreting Electric Potential Energy

- An alternative but equivalent viewpoint is to consider how much work we would have to do to "raise" a particle from a point b where the potential energy is U_b to a point a where it has a greater value U_a.
- To move the particle slowly (so as not to give it any kinetic energy), we need to exert an additional external force $\vec{\mathbf{F}}_{\text{ext}}$ that is equal and opposite to the electric-field force and does positive work.
- The potential-energy difference $U_a - U_b$ is then defined as the work that must be done by an external force to move the particle slowly from b to a against the electric force.
- Because $\vec{\mathbf{F}}_{\text{ext}}$ is the negative of the electric-field force and the displacement is in the opposite direction, this definition of the potential difference $U_a - U_b$ is equivalent to that given above.

Example: A system of point charges

Question:

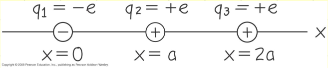
- Two point charges are at fixed positions on the x-axis, $q_1 = -e$ at $x = 0$ and $q_2 = +e$ at $x = a$. (a) Find the work that must be done by an external force to bring a third point charge $q_3 = +e$ from infinity to $x = 2a$. (b) Find the total potential energy of the system of three charges.



Example: A system of point charges

Solution a:

- We need to find the work W that must be done on q_3 by an external force $\vec{\mathbf{F}}_{\text{ext}}$ to bring q_3 in from infinity to $x = 2a$. We do this by using $U = \frac{q_0}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$ to find the potential energy associated with q_3 in the presence of q_1 and q_2:



$$W = U = \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$$

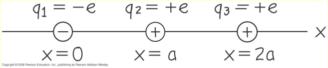
$$= \frac{+e}{4\pi\epsilon_0}\left(\frac{-e}{2a} + \frac{+e}{a}\right)$$

$$= \frac{e^2}{8\pi\epsilon_0 a}$$

Example: A system of point charges

Solution b:

- In part (b) we use $U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}$, the expression for the potential energy of a collection of point charges, to find the total potential energy of the system.



- This becomes

$$U = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$
$$= \frac{1}{4\pi\epsilon_0}\left(\frac{(-e)(e)}{a} + \frac{(-e)(e)}{2a} + \frac{(e)(e)}{a}\right)$$
$$= -\frac{e^2}{8\pi\epsilon_0 a}$$

Exercise: Energy required to assemble a sphere of uniform charge

Question:

- Consider a sphere of radius a and total charge Q distributed uniformly over its volume.
- What is the potential energy associated with this assembly of charges?

Exercise: Energy required to assemble a sphere of uniform charge

Solution:

- The density of charge is $\rho = Q/\frac{4}{3}\pi a^3$
- The charge of a sphere with radius $r < a$ is $q = Q(r/a)^3$
- The charge of a spherical shell with radius r thickness $\mathrm{d}r$ is $\mathrm{d}q = \rho 4\pi r^2 \,\mathrm{d}r$
- Consider the potential energy of a sphere of radius $r < a$ and the surrounding spherical shell of thichness $\mathrm{d}r$:

$$\mathrm{d}U = \frac{1}{4\pi\epsilon_0}\frac{q\,\mathrm{d}q}{r}$$

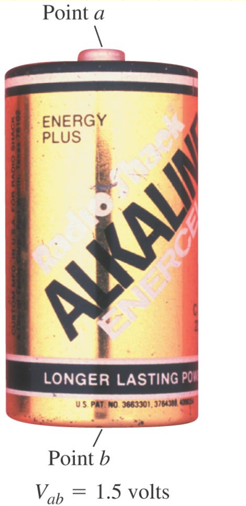
- To find the total potential energy associated with this assembly of charges add up the potential energy of each sphere+shell pair

$$U = \frac{1}{4\pi\epsilon_0}\int \frac{q\,\mathrm{d}q}{r} = \frac{3}{5}\frac{1}{4\pi\epsilon_0}\frac{Q^2}{a}$$

Electric potential

Electric potential

- *Potential* is potential energy per unit charge.
- The SI unit is volt: $1\,\mathrm{V} = 1\,\mathrm{J/C}$
- Work done by the electric force: $W_{a\to b} = -\Delta U = q_0(V_a - V_b)$
- $V_{ab} = V_a - V_b$ the potential of a wrt to b
- Alternatively, V_{ab} equals the work that must be done to move a *unit* charge slowly from b to a against the electric force.
- measured by a *voltmeter*.
- V is a scalar field.



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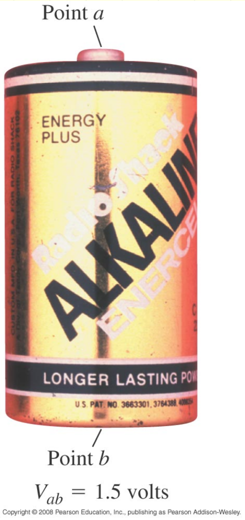

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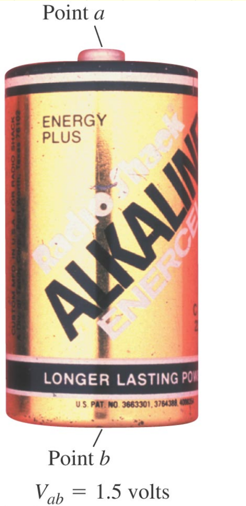

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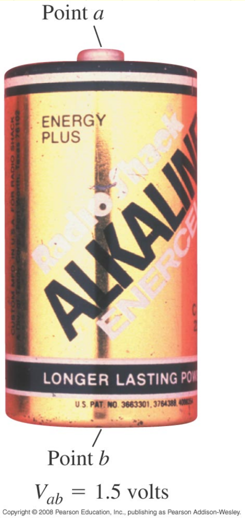

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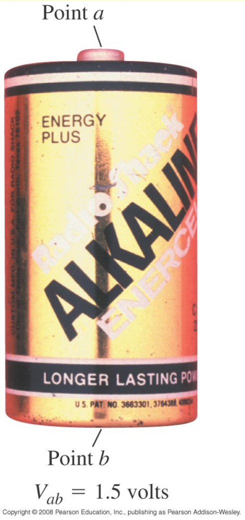

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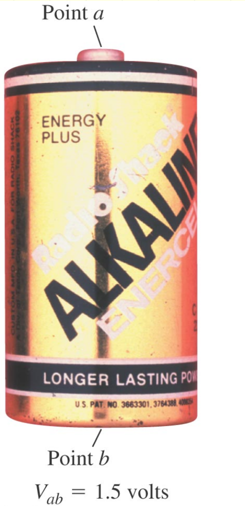

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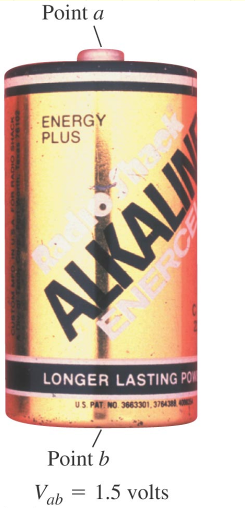


Calculating Electric Potential

Potential due to a single point charge

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

where r is the distance from charge q.

Potential due to a collection of charges

$$V = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$$

where r_i is the distance from charge q_i.

For a continuous distribution of charge

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\mathrm{d}q}{r}$$

where r is the distance from charge $\mathrm{d}q$.

Finding Electric Potential from Electric Field

- the work done by the electric force as the test charge moves from a to b

$$W_{a\to b} = \int_a^b \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{l}} = \int_a^b q_0\vec{\mathbf{E}}\cdot \mathrm{d}\vec{\mathbf{l}}$$

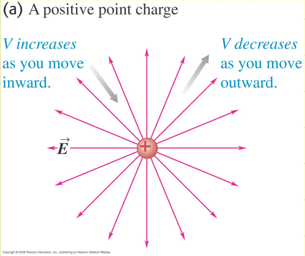
- Recall $W_{a\to b} = -\Delta U = -(U_b - U_a)$
- The potential difference is $-\Delta U/q_0 = -\Delta V = -(V_b - V_a)$

$$\boxed{V_b - V_a = -\int_a^b \vec{\mathbf{E}}\cdot \mathrm{d}\vec{\mathbf{l}}}$$

- The value of ΔV is independent of the path taken from a to b, just as the value of $W_{a\to b}$ is independent of the path.

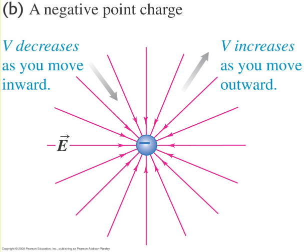
Finding Electric Potential from Electric Field

- Consider a positive point charge.
- The electric field is directed away from the charge, and $V = q/4\pi\epsilon_0 r$ is positive at any finite distance from the charge.
- If you move away from the charge, in the direction of $\vec{\mathbf{E}}$, you move toward lower values of V;
- if you move toward the charge, in the direction opposite $\vec{\mathbf{E}}$, you move toward greater values of V.



Finding Electric Potential from Electric Field

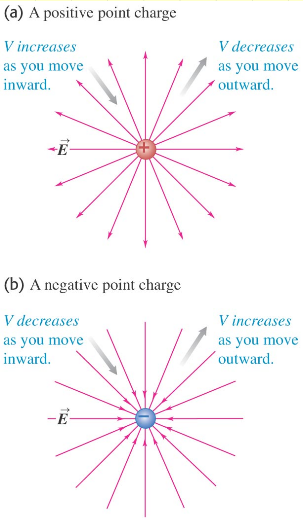
- Consider a negative point charge.
- The electric field is directed toward the charge, and $V = q/4\pi\epsilon_0 r$ is negative at any finite distance from the charge.
- If you move toward the charge, in the direction of $\vec{\mathbf{E}}$, you move toward lower (more negative) values of V;
- Moving away from the charge, in the direction opposite $\vec{\mathbf{E}}$, moves you toward increasing (less negative) values of V.



Finding Electric Potential from Electric Field

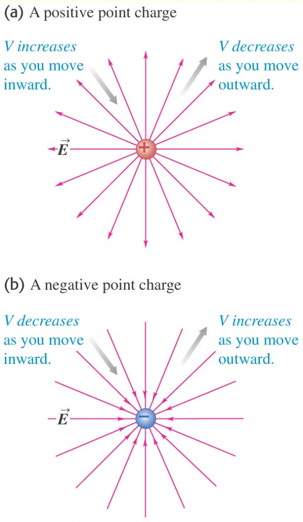
The general rule, valid for *any* electric field, is:

- Moving *with* the direction of $\vec{\mathbf{E}}$ means moving in the direction of *decreasings* V.
- Moving *against* the direction of $\vec{\mathbf{E}}$ means moving in the direction of *increasing* V.



Finding Electric Potential from Electric Field

- A positive test charge q_0 experiences an electric force in the direction of $\vec{\mathbf{E}}$, toward lower values of V;
- a negative test charge experiences a force opposite $\vec{\mathbf{E}}$, toward higher values of V.
- Thus a positive charge tends to "fall" from a high-potential region to a lower-potential region.
- The opposite is true for a negative charge.



Potential difference

The equation

$$V_b - V_a = -\int_a^b \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

is interpreted as follows:

To move a unit charge slowly against the electric force, we must apply an external force per unit charge equal to $-\vec{\mathbf{E}}$, equal and opposite to the electric force per unit charge $\vec{\mathbf{E}}$. It says that $V_b - V_a = V_{ba}$, the potential of b with respect to a, equals the work done per unit charge by this external force to move a unit charge from a to b.

Unit of electric field-again

The equation

$$V_b - V_a = -\int_a^b \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

show that the unit of electric field is

$$1\,\mathrm{V/m} = 1\,\mathrm{N/C}$$

In practice, the volt per meter is the usual unit of electric-field magnitude.

Electron Volts

- When a particle with charge q moves from a point where the potential is V_a to a point where it is V_b, the change in the potential energy is
  $U_b - U_a = q(V_b - V_a) = qV_{ba}$
- If charge q equals the magnitude e of the electron charge, $1.602 \times 10^{-19}\,\mathrm{C}$, and the potential difference is $V_{ba} = 1\,\mathrm{V} = 1\,\mathrm{J/C}$, the change in energy is
  $U_b - U_a = (1.602 \times 10^{-19}\,\mathrm{C})(1\,\mathrm{J/C}) = 1.602 \times 10^{-19}\,\mathrm{J}$
- This quantity of energy is defined to be 1 electron volt:
  $1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}$
- Useful in many calculations with atomic and nuclear systems.
- The multiples meV, keV, MeV, GeV, and TeV are often used.

Electron Volts

- the electron volt is a unit of energy, not a unit of potential or potential difference!
- When a particle with charge e moves through a potential difference of $1\,\text{volt}$, the change in potential energy is $1\,\text{eV}$.
- Although we defined the electron volt in terms of potential energy, we can use it for any form of energy, such as the kinetic energy of a moving particle.
- Ex: The Large Hadron Collider near Geneva, Switzerland, is designed to accelerate protons to a kinetic energy of $7\,\text{TeV}$ (i.e. $7 \times 10^{12}\,\text{eV}$).

Example:

Question

A proton (charge $+e = 1.602 \times 10^{-19}\,\mathrm{C}$) moves a distance $d = 0.50\,\mathrm{m}$ in a straight line between points a and b in a linear accelerator. The electric field is uniform along this line, with magnitude $E = 1.5 \times 10^{7}\,\mathrm{V/m} = 1.5 \times 10^{7}\,\mathrm{N/C}$ in the direction from a to b. Determine

(a) the force on the proton;

(b) the work done on it by the field;

(c) the potential difference $V_a - V_b$.

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(b) the work done on it by the field;

(c) the potential difference $V_a - V_b$.

Answer (a):

The force on the proton is in the same direction as the E field

$F = qE = (1.602 \times 10^{-19}\,\mathrm{C})(1.5 \times 10^{7}\,\mathrm{N/C}) = 2.4 \times 10^{-12}\,\mathrm{N}$

Example:

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(a) the force on the proton;

(b) the work done on it by the field;

(c) the potential difference $V_a - V_b$.

Answer (b):

The force is constant and in the same direction as the displacement, so the work done on the proton is

$W_{a \to b} = Fd = (2.4 \times 10^{-12}\,\mathrm{N})(0.50\,\mathrm{m}) = 1.2 \times 10^{-12}\,\mathrm{J} = 7.5\,\mathrm{MeV}$

Example:

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(a) the force on the proton;

(b) the work done on it by the field;

(c) the potential difference $V_a - V_b$.

Answer (c):

the potential difference is the work per unit charge

$V_a - V_b = \frac{W_{a\to b}}{q} = \frac{1.2\times 10^{-12}\,\mathrm{J}}{1.602\times 10^{-19}\,\mathrm{C}} = 7.5 \times 10^{6}\,\mathrm{V}$

This could also be found by $V_a - V_b = \int_a^b E\,\mathrm{d}l = E\int_a^b \mathrm{d}l = Ed = (1.5 \times 10^{7}\,\mathrm{N/C})(0.50\,\mathrm{m}) = 7.5 \times 10^{6}\,\mathrm{eV}$

Key concepts-summary

- The potential difference V_{ab} between point a and point b, equal to the difference $V_a - V_b$ of the potentials V at the two points, is the amount of work the electric force does on a unit charge as it moves from a to b.
- If you move in the direction of the electric field, V decreases (V_{ab} is positive);
- if you move opposite to the direction of the electric field, V increases (V_{ab} is negative).

Potential due to two point charges

What is V_a, V_b and V_c?

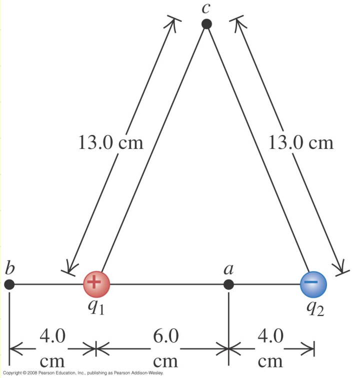
$$V_a = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2}$$

$$V_a = (9.0\times 10^9\,\mathrm{Nm^2/C^2})\times$$

$$\left(\frac{12\,\mathrm{nC}}{6.0\,\mathrm{cm}} + \frac{-12\,\mathrm{nC}}{4.0\,\mathrm{cm}}\right)$$

$$= 1800\,\mathrm{Nm/C} + (-2700\,\mathrm{Nm/C})$$

$$= -900\,\mathrm{V}$$



Similarly, $V_b = 1930\,\mathrm{V}$ and $V_c = 0$.

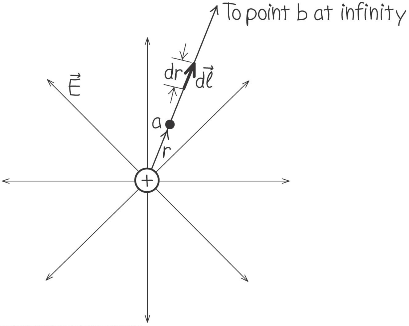
Evaluate: Does $V_b > V_a$ make sense? How about $V_c = 0$?

Finding potential by integration

Choose the potential to be zero at an infinite distance from the charge q

$$V_a - 0 = \int_a^{\infty} \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

The most convenient path is a radial line as shown



$$\int_a^{\infty} \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \mathrm{d}r = -\frac{q}{4\pi\epsilon_0 r}\bigg|_a^{\infty}$$

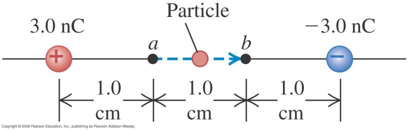
$$\Rightarrow V_a = \frac{q}{4\pi\epsilon_0 a}$$

Evaluate: Check sign for $(\pm)$ve charge.

Moving through a potential difference

Mass $m = 5.0 \times 10^{-9}\,\mathrm{kg} = 5.0\,\mu\mathrm{g}$ with charge $q_0 = 2.0\,\mathrm{nC}$ starts from rest and moves in a straight line from point a to point b.

What is its speed v at point b?



Use energy conservation $K_a + U_a = K_b + U_b$ with $K_a = 0$ and $K_b = mv^2/2$

$$0 + q_0 V_a = \frac{1}{2} m v^2 + q_0 V_b \Rightarrow v = \sqrt{\frac{2q_0(V_a - V_b)}{m}}$$

Calculating $V = q/4\pi\epsilon_0 r$ gives $V_a = 1350\,\mathrm{V}$ and $V_b = -1350\,\mathrm{V}$.

$$v = \sqrt{\frac{2(2.0 \times 10^{-9}\,\mathrm{C})(2700\,\mathrm{V})}{5.0 \times 10^{-9}\,\mathrm{kg}}} = 46\,\mathrm{m/s}$$

Calculating electric potential

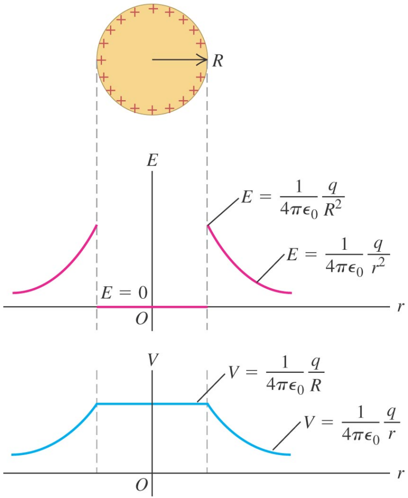
A charged conducting sphere

has radius R and total charge q.

- Take $V = 0$ at infinity
- the potential at a point outside the sphere is the same as that due to a point charge q at the center

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

- $V_{\text{surface}} = q/4\pi\epsilon_0 R$
- Inside $\vec{\mathbf{E}} = \vec{\mathbf{0}}$
- Inside $V = V_{\text{surface}}$



Ionization and Corona Discharge

- The max. V to which a conductor in air can be raised is limited since air molecules become ionized.
- Air becomes conductor at $E_{\mathrm{m}} \sim 3 \times 10^6\,\mathrm{V/m}$ (dielectric strength of air).
- Max. V a spherical conductor can be raised: $V_{\mathrm{m}} = RE_{\mathrm{m}}$
- Van de Graaff generators use spherical terminals with very large radii

Corona discharge: Even small V applied to sharp points with a very small radius of curvature, produce sufficiently high E to ionize the surrounding air.

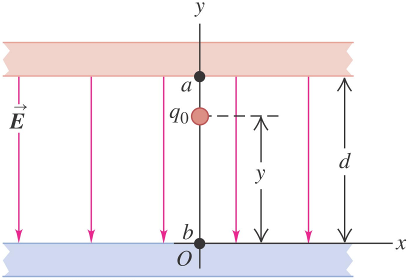
Oppositely charged parallel plates

- The electric potential energy U for a test charge q_0 at y:
- $U = q_0 E y$
- The potential is

$$V(y) = \frac{U(y)}{q_0} = Ey$$

- Using $V_a - V_b = Ed$:

$$E = \frac{V_{ab}}{d}$$



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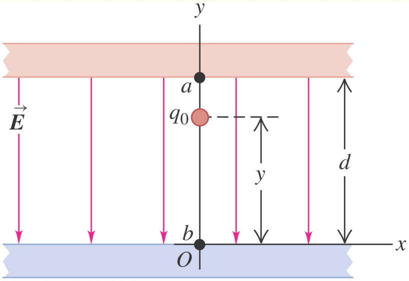

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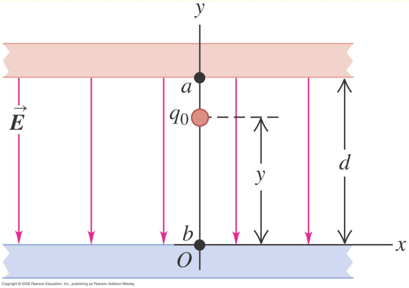

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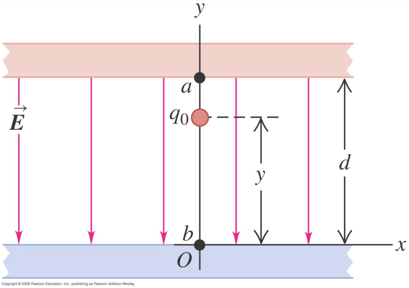


- Using $V_a - V_b = Ed$:

$$E = \frac{V_{ab}}{d}$$

Ex: Potential of a uniformly charged sphere

Question:

Consider an non-conducting sphere with radius a and total charge Q. The charge is distributed uniformly in the volume. Assuming the potential at infinity to be zero

(a) What is the potential at a radial distance r outside the sphere?

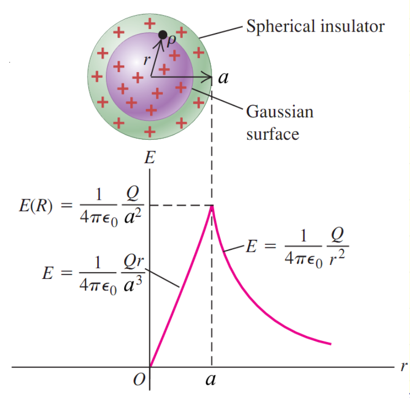
(b) What is the potential on the surface of the sphere?

(c) What is the potential at a radial distance r inside the sphere?

Ex: Potential of a uniformly charged sphere

Recall that the electric field is found by the Gauss' law and it is

$$E = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 a^3} r & \text{for } r < a \\ \dfrac{Q}{4\pi\epsilon_0 r^2} & \text{for } r > a \end{cases}$$



Ex: Potential of a uniformly charged sphere

Answer (a):

- Outside the sphere ($r > a$) the the electric field is $E = \frac{Q}{4\pi\epsilon_0 r^2}$
- We use $V_f - V_i = -\int_i^f \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{l}}$ assuming $f \to \infty$ and i is at any radial distance $r > a$.

$$\cancelto{0}{V_\infty} - V(r) = -\int_r^\infty E(r')\,\mathrm{d}r' = -\int_r^\infty \frac{Q}{4\pi\epsilon_0 r'^2}\,\mathrm{d}r'$$

$$= -\frac{Q}{4\pi\epsilon_0}\,\frac{-1}{r}\bigg|_r^\infty = \frac{Q}{4\pi\epsilon_0}\left(0 - \frac{1}{r}\right) = -\frac{Q}{4\pi\epsilon_0 r}$$

- This gives

$$V(r) = \frac{Q}{4\pi\epsilon_0 r}, \qquad \text{for } r > a$$

same as the potential of a point charge as expected.

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Ex: Potential of a uniformly charged sphere

Answer (b):

- Given that the potential outside is

$$V(r) = \frac{Q}{4\pi\epsilon_0 r}$$

- ...the potential at the surface must be

$$V(a) = \frac{Q}{4\pi\epsilon_0 a}$$

Ex: Potential of a uniformly charged sphere

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Ex: Potential of a uniformly charged sphere

Answer (c):

- The electric field inside is $E = \frac{Q}{4\pi\epsilon_0 a^3} r$
- The potential difference between a point in the sphere and the potential at the surface ($V(a) = \frac{Q}{4\pi\epsilon_0 a}$) is

$$V(a) - V(r) = -\int_r^a E(r')\,\mathrm{d}r' = -\frac{Q}{4\pi\epsilon_0 a^3}\int_r^a r'\,\mathrm{d}r'$$
$$= -\frac{Q}{4\pi\epsilon_0 a^3}\left.\frac{r'^2}{2}\right|_r^a = -\frac{Q}{8\pi\epsilon_0 a^3}(a^2 - r^2)$$

- Using $V(a) = \frac{Q}{4\pi\epsilon_0 a}$ from (b)

$$V(r) = \frac{Q}{4\pi\epsilon_0 a} + \frac{Q}{8\pi\epsilon_0 a}\left(1 - \frac{r^2}{a^2}\right) = \frac{Q}{8\pi\epsilon_0 a}\left(3 - \frac{r^2}{a^2}\right), \quad \text{for } r < a$$

Ex: Potential of a uniformly charged sphere

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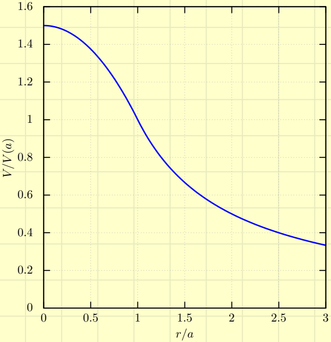
Ex: Potential of a uniformly charged sphere

- We have found

$$V = \begin{cases} \frac{Q}{8\pi\epsilon_0 a}\left(3 - \frac{r^2}{a^2}\right) & \text{for } r < a \\ \frac{Q}{4\pi\epsilon_0 r} & \text{for } r > a \end{cases}$$



- The potential at the center is then

$$V_c = \frac{3Q}{8\pi\epsilon_0 a}$$

which is not very trivial from the beginning.

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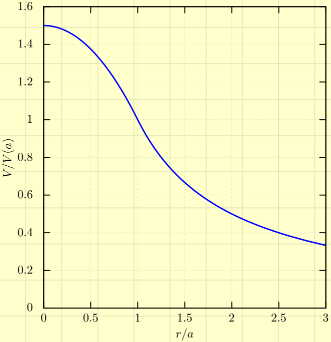

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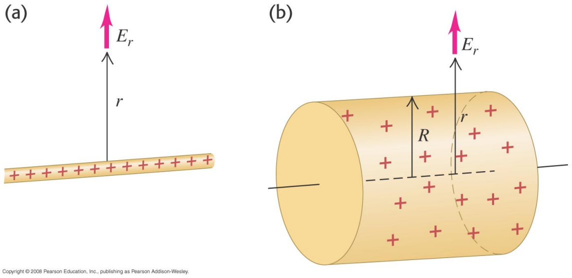
An infinite line charge or charged conducting cylinder



$$V_a - V_b = -\int \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

$$V_a - V_b = -\int_a^b E_r dr$$

For inf. line of charge and $r > r_0$

- $E_r = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}$
- Take $V_b = 0$ at $r = r_0$ (different from the usual reference at ∞!)
- $V - 0 = \frac{\lambda}{2\pi\epsilon_0}\int_a^b \frac{dr}{r} = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r_0}{r}$

For the cylinder $r > R$

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- Take $V_b = 0$ at $r = R$
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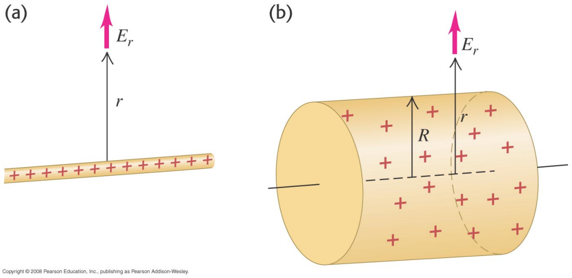
- $\vec{\mathbf{E}} = \vec{\mathbf{0}}$ inside $\Rightarrow V = 0$ inside.

An infinite line charge or charged conducting cylinder



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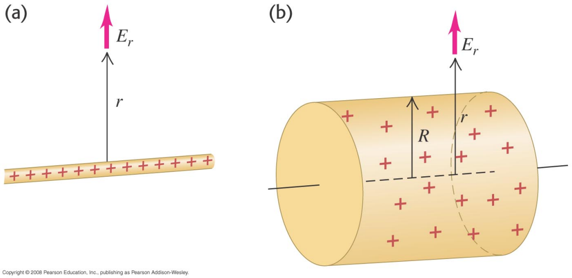

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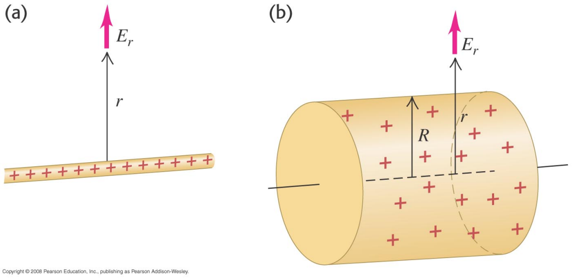
An infinite line charge or charged conducting cylinder



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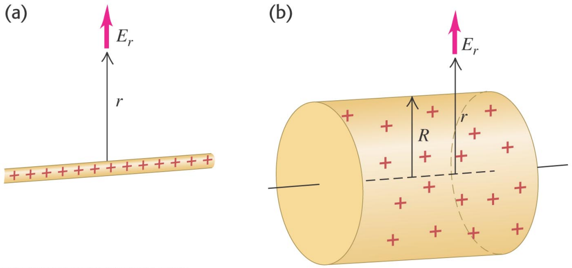
- $\vec{\mathbf{E}} = \vec{\mathbf{0}}$ inside $\Rightarrow V = 0$ inside.

An infinite line charge or charged conducting cylinder



$$V_a - V_b = -\int \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

$$V_a - V_b = -\int_a^b E_r dr$$

For inf. line of charge and $r > r_0$

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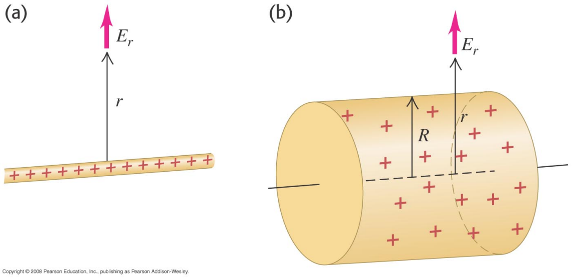

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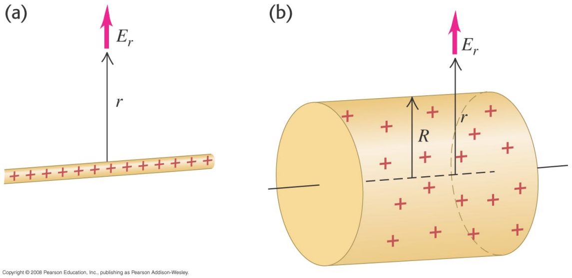

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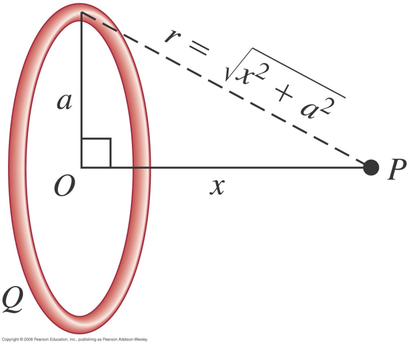
A ring of charge

Divide the ring into infinitesimal segments.

- distance of dq to P: $r = \sqrt{x^2 + a^2}$
- potential due to dq: $\mathrm{d}V = \mathrm{d}q/(4\pi\epsilon_0 r)$
- sum all the potentials: $V = \frac{1}{4\pi\epsilon_0}\frac{1}{\sqrt{a^2+x^2}}\int \mathrm{d}q$
  Note that the distance r is independent of the position of dq on the ring.
- so that we get: $V = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{a^2+x^2}}$



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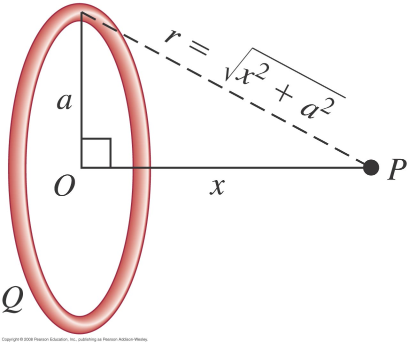

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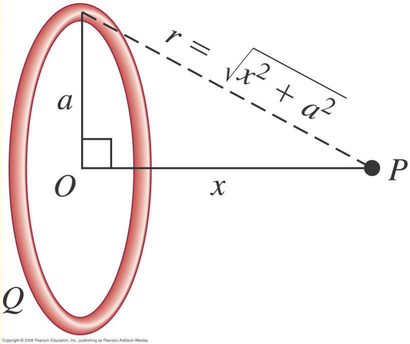

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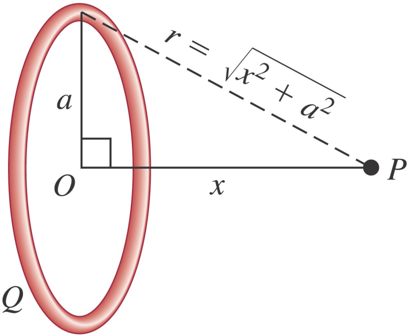
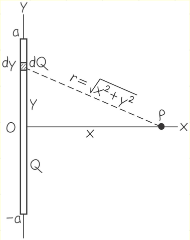


A line of charge

Problem:

Positive electric charge Q is distributed uniformly along a line of length $2a$ lying along the y-axis between $y = -a$ and $y = +a$. Find the electric potential at a point P on the x-axis at a distance x from the origin.

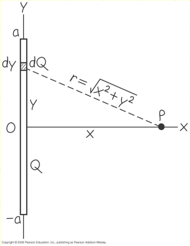


A line of charge

Solution:

- The element of charge $\mathrm{d}Q$ corresponding to an element of length $\mathrm{d}y$ on the rod is $\mathrm{d}Q = (Q/2a)\,\mathrm{d}y$.
- The distance from $\mathrm{d}Q$ to P is $\sqrt{x^2+y^2}$, so the contribution $\mathrm{d}V$ that the charge element makes to the potential at P is

$$\mathrm{d}V = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\frac{\mathrm{d}y}{\sqrt{x^2+y^2}}$$



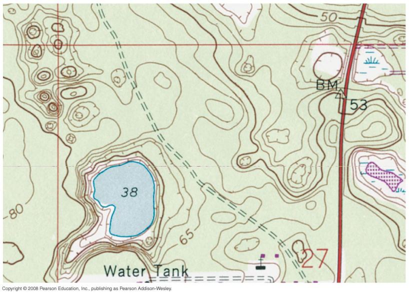
- Integrating this from $y=-a$ to $+a$ we get

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\ln\left(\frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a}\right)$$

Equipotential surfaces

Equipotential surfaces

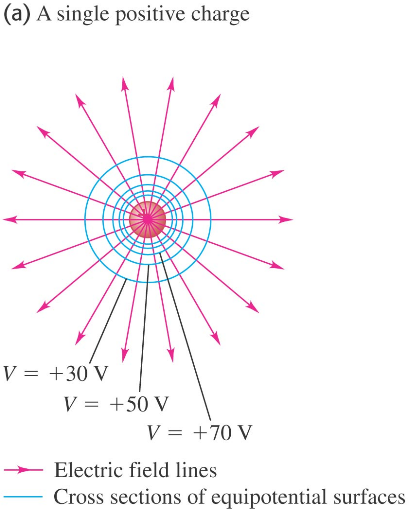
- By analogy to contour lines on a topographic map, an equipotential surface is a 3-D surface on which V is the same at every point.
- the electric potential energy $q_0 V$ of a charge q_0 remains constant when moved on an equipotential surface.
- No point can be at two different potentials, so equipotential surfaces for different potentials can never touch or intersect.



Contour lines on a topographic map are curves of constant elevation and hence of constant gravitational potential energy.

Equipotential Surfaces and Field Lines

- Field lines and equipotential surfaces are always mutually perpendicular. Why?
- In the figures on the right, equipotentials are drawn so that there are equal potential differences between adjacent surfaces. In regions where the magnitude of $\vec{\mathbf{E}}$ is large. Why?
- $\vec{\mathbf{E}}$ need not be constant over an equipotential surface



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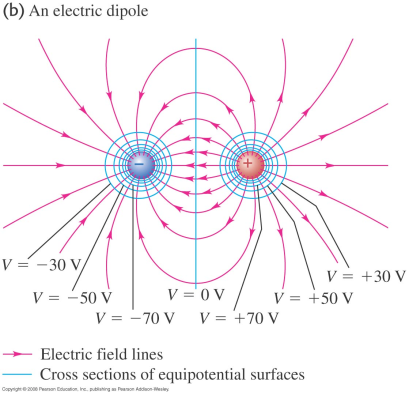

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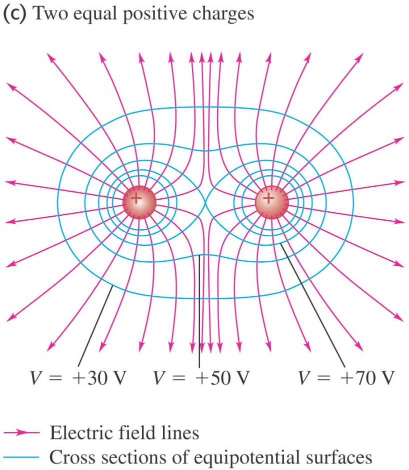
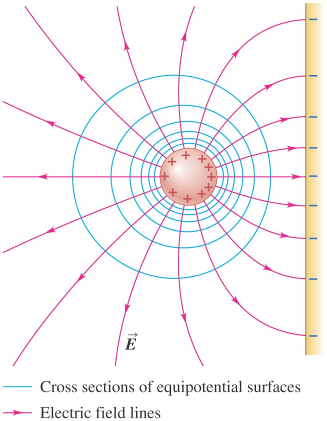


Equipotentials and Conductors

When all charges are at rest,

- the surface of a conductor is always an equipotential surface,
- the electric field just outside a conductor must be perpendicular to the surface at every point,
- the entire solid volume of a conductor is at the same potential.



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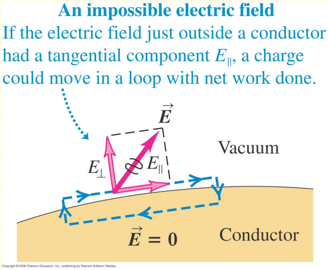
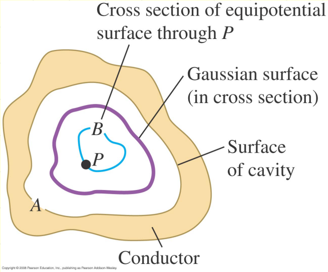


Equipotentials and Conductors

If a conductor contains a cavity and if no charge is present inside the cavity,

- then there can be no net charge anywhere on the surface of the cavity. Every point in the cavity is at the same potential. The electric field inside the cavity is zero everywhere.



Potential gradient

Potential gradient

- Electric field and electric potential are closely related.

$$V_a - V_b = \int_a^b \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

  If we know $\vec{\mathbf{E}}$ at various points, we can use this equation to calculate potential differences.
- if we know $V(\vec{\mathbf{r}})$, can we use it to determine $\vec{\mathbf{E}}(\vec{\mathbf{r}})$?
- Yes! The components of $\vec{\mathbf{E}}$ are related to the partial derivatives of V with respect to x, y, and z.
- Using $V_a - V_b = \int_b^a \mathrm{d}V = -\int_a^b \mathrm{d}V$ we can write

$$-\mathrm{d}V = \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

  This remind us that moving in the direction of $\vec{\mathbf{E}}$, the electric potential V decreases most rapidly.

Potential gradient

In Cartesian coordinates

$$-\mathrm{d}V = \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{l}} = (E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}) \cdot (\mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}y\hat{\mathbf{j}} + \mathrm{d}z\hat{\mathbf{k}})$$

$$-\mathrm{d}V = E_x\mathrm{d}x + E_y\mathrm{d}y + E_z\mathrm{d}z \quad \Rightarrow \quad E_x = -\left.\frac{\partial V}{\partial x}\right|_{y,z}$$

and similarly for the y- and z-components.

If $\vec{\mathbf{E}}$ has a radial component E_r with respect to a point or an axis and r is the distance from the point or axis, the relationship is

$$E_r = -\frac{\partial V}{\partial r}$$

Electric field of a point charge from electric potential

Given $V(r) = q/(4\pi\epsilon_0 r)$ we can calculate

$$E_r = -\frac{\partial V}{\partial r} = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2} \quad \Rightarrow \quad \vec{\mathbf{E}} = E_r \hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$

Since $r = \sqrt{x^2 + y^2 + z^2}$

$$E_x = -\left.\frac{\partial V}{\partial x}\right|_{y,z} = \frac{q}{4\pi\epsilon_0}\frac{x}{r^3}$$

and similarly calculating $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$ give

$$\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0}\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r^3} = \frac{q}{4\pi\epsilon_0}\frac{\vec{\mathbf{r}}}{r^3} \qquad \checkmark$$