Math 332E (Measure Theory) Quiz II October 26, 2016

Name and Last Name:______________________ Student Number:______________________

1. (a) (20 points) Write the definition of a σ-algebra on a set X.

Solution: A σ-algebra on X is a collection of sets $\mathcal{A} \subseteq 2^X$ such that

(i) For every A and B in $\mathcal{A}$ the sets $A \cup B$, $A \cap B$, $A \setminus B$ and $B \setminus A$ are also in $\mathcal{A}$.

(ii) For every countable family $\{E_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$, the union $\bigcup_{n\in\mathbb{N}} E_n$ is also in $\mathcal{A}$.

(b) (20 points) Write the definition of an upper measure on a set X.

Solution: An upper measure ν is a partial function $\nu\colon 2^X \to [0,\infty) \cup \{\infty\}$ which satisfies:

(i) $\nu(\emptyset) = 0$,

(ii) $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$,

(iii) $\nu\left(\bigcup_{n=0}^{\infty} E_n\right) \leq \sum_{n=0}^{\infty} \nu(E_n)$ for every countable family of sets $\{E_n\}_{n\in\mathbb{N}}$

2. Assume $n \geq 2$ and let $X = \{1,\ldots,n\}$. Let us define a set function $\eta\colon 2^X \to [0,\infty)$ by letting

$$\eta(U) = \frac{\sum_{x\in U} x^2}{|U|}$$

where $|U|$ is the number of elements in U. In other words, we take the average of the squares of the numbers in U.

(a) (30 points) Show that η is not additive by giving a counter-example. In other words, find two disjoint subsets U and V in X such that $\eta(U \cup V) \neq \eta(U) + \eta(V)$.

Solution: We have

$$\eta(\{1\}) = \frac{1^2}{1} = 1 \quad \text{and} \quad \eta(\{2\}) = \frac{2^2}{1} = 4$$

but

$$\eta(\{1,2\}) = \frac{1+4^2}{2} = \frac{9}{2} \neq 1 + 4$$

(b) (30 points) Show that η is sub-additive. In other words, verify that for every pair of subsets U and V in X we have $\eta(U \cup V) \leq \eta(U) + \eta(V)$.

Solution: We have

$$\frac{\sum_{x\in U\cup V} x^2}{|U\cup V|} \leq \frac{\sum_{x\in U} x^2}{|U\cup V|} + \frac{\sum_{x\in V} x^2}{|U\cup V|} \leq \frac{\sum_{x\in U} x^2}{|U|} + \frac{\sum_{x\in V} x^2}{|V|}$$

In other words

$$\mu(U \cup V) \leq \mu(U) + \mu(V)$$