Math 332E (Measure Theory) Midterm November 2, 2016

Name and Last Name:________________________ Student Number:________________________

1. (30 points) Write the definitions of the followings:
   (a) An algebra on a set X.
   (b) A σ-algebra on a set X.
   (c) A measure on a σ-algebra on X.
   (d) An outer measure on a set X.
   (e) The Borel σ-algebra on a topological space X.
   (f) A complete measure space.

2. (10 points) Assume (x_n) is an increasing sequence of real numbers $x_n \leq x_{n+1}$ such that $\lim_{n\to\infty} x_n = a$ for some real number a. Consider the collection of intervals

$$E_n = [x_n, x_{n+1})$$

Calculate $\mu(\bigcup_{n=0}^{\infty} E_n)$. [Hint: Verify that $E_n \cap E_m = \emptyset$ whenever $n \neq m$.]

Solution: Assume $n \neq m$. Without loss of generality, we can assume $n < m$. Assume on the contrary that there is an element $u \in [x_n, x_{n+1}) \cap [x_m, x_{m+1})$ Then

$$x_n \leq u < x_{n+1} \leq x_m \leq u < x_{m+1}$$

which can not be true because we are saying $u < u$. So, we get $E_n \cap E_m = \emptyset$ when $n \neq m$. Then by using the σ-additivity of the standard measure we get

$$\mu(\bigcup_{n=0}^{\infty} E_n) = \sum_{n=0}^{\infty} \mu([x_n, x_{n+1})) = \sum_{n=0}^{\infty}(x_{n+1} - x_n)$$

This is a telescopic series

$$\lim_{N\to\infty} \sum_{n=0}^{N}(x_{n+1} - x_n) = -x_0 + x_1 - x_1 + x_2 + \cdots - x_N + x_{N+1} = \lim_{n\to\infty} x_{N+1} - x_0 = a - x_0$$

3. (10 points) Assume $(X, \mathcal{A})$ is a measurable space such that one-point subsets $\{x\}$ are all in $\mathcal{A}$.
   (a) Show that every countable subset $U \subseteq X$ is in $\mathcal{A}$.

Solution: A set U is countable if there is a one-to-one and onto function $f\colon \mathbb{N} \to U$. So, if $U \subseteq X$ is countable we can write it as a sequence of elements

$$U = \{u_0, u_1, \ldots, u_n, \ldots\}$$

Then

$$U = \bigcup_{n=0}^{\infty} \{u_n\}$$

Since $\mathcal{A}$ is a σ-algebra and 1-point sets $\{u_n\}$ are in $\mathcal{A}$ we see that $U \in \mathcal{A}$ because it is closed under taking countable unions.

(b) Let μ be an outer measure on $\mathcal{A}$ such that $\mu(\{x\}) = 0$ for every $x \in X$. Show that $\mu(U) = 0$ for every countable subset $U \subseteq X$.

> Solution: Since $U = \bigcup_{n=0}^{\infty} \{u_n\}$ and μ is an outer measure we get
>
> $$0 \leq \mu(U) \leq \sum_{n=0}^{\infty} \mu(\{u_n\}) = 0$$
>
> So, $\mu(U) = 0$.

4. (20 points) Recall that for any subset $X \subseteq \mathbb{R}$ and real number $\alpha \in \mathbb{R}$, we defined new subsets $\alpha X = \{\alpha x : x \in X\}$ and $\alpha + X = \{\alpha + x : x \in X\}$. Now, let $0 < \lambda < \frac{1}{3}$ be a fixed real number. Let $C_0 = [0, 1]$ and let us define recursively

$$C_{n+1} = \lambda C_n \cup ((1 - 2\lambda) + \lambda C_n)$$

(a) Show that λC_n and $(1 - 2\lambda) + \lambda C_n$ are disjoint subsets. [Hint: Sketch a picture.]

> Solution: We can see that $C_n \subseteq [0, 1]$ for every $n \geq 0$. Then $\lambda C_n \subseteq [0, \lambda]$ and
>
> $$(1 - 2\lambda) + \lambda C_n \subseteq [1 - 2\lambda, 1 - \lambda]$$
>
> Since $\lambda < \frac{1}{3}$ we have $\lambda < 1 - 2\lambda$. So, λC_n and $(1 - 2\lambda) + \lambda C_n$ are disjoint.

(b) Calculate the measure of C_n for every $n \geq 0$.

> Solution: Since the pieces of C_{n+1} are disjoint
>
> $$\mu(C_{n+1}) = \mu(\lambda C_n) + \mu((1 - 2\lambda) + \lambda C_n) = 2\lambda \cdot \mu(C_n)$$
>
> Since $\mu(C_0) = 1$, we get that $\mu(C_n) = (2\lambda)^n$.

(c) Show that $C_{n+1} \subset C_n$. [Hint: Sketch a picture.]

> Solution: We do this by induction on n. We see that $C_1 = [0, \lambda] \cup [1 - 2\lambda, 1 - \lambda] \subset [0, 1] = C_0$. Now, assume $C_{n+1} \subseteq C_n$ and we would like to show that $C_{n+2} \subseteq C_{n+1}$. Since $C_{n+1} \subseteq C_n$ we get
>
> $$\lambda C_{n+1} \subseteq \lambda C_n \quad \text{and} \quad (1 - 2\lambda) + \lambda C_{n+1} \subseteq (1 - 2\lambda) + \lambda C_n$$
>
> Then
>
> $$(\lambda C_{n+1}) \cup ((1 - 2\lambda) + \lambda C_{n+1}) = C_{n+2} \subseteq C_{n+1} = (\lambda C_n) \cup ((1 - 2\lambda) + \lambda C_n)$$

(d) Using the continuity of the ordinary measure on $\mathbb{R}$, calculate the measure of $C = \bigcap_{n=0}^{\infty} C_n$.

> Solution: We saw above $C_n \supseteq C_{n+1}$. Then
>
> $$\mu(C) = \mu\left(\bigcap_{n=0}^{\infty} C_n\right) = \lim_{n\to\infty} \mu(C_n) = \lim_{n\to\infty} (2\lambda)^n = 0$$
>
> since $2\lambda < 1$.

5. (30 points) Consider $\mathbb{R}^2$ together with the σ-algebra generated by bounded convex sets $\mathcal{C}$. Now, for a bounded convex set $\Omega \subseteq \mathbb{R}^2$ define

$$\eta(\Omega) = diam(\Omega) = \sup\left\{\sqrt{(a-c)^2 + (b-d)^2}\,:\,(a,b),(c,d) \in \Omega\right\}$$

and then let

$$\eta^*(A) = \inf\left\{\sum_{n=0}^{\infty} \eta(\Omega_n)\,:\, A \subseteq \bigcup_{n=0}^{\infty} \Omega_n\right\}$$

(a) Calculate η-measure of the interior of the rectangle determined by the points $(0,0)$, $(2\sqrt{3},0)$, $(0,2)$ and $(2\sqrt{3},2)$.

Solution: The measure $\eta^*(Rectangle)$ is the length of the diagonal of this rectangle which is 4.

(b) Show that η^* is monotone, i.e. $\eta^*(A) \leq \eta^*(B)$ whenever $A \subseteq B$.

Solution: Whenever $A \subseteq B$, we also have an inclusion of the form

$$\left\{\sum_{n=0}^{\infty} \eta(\Omega_n)\,:\, B \subseteq \bigcup_{n=0}^{\infty} \Omega_n\right\} \subseteq \left\{\sum_{n=0}^{\infty} \eta(\Omega_n)\,:\, A \subseteq \bigcup_{n=0}^{\infty} \Omega_n\right\}$$

This is because if $B \subseteq \bigcup_n \Omega_n$ then we also have $A \subseteq \bigcup_n \Omega_n$. Then

$$\eta^*(B) = \inf\left\{\sum_{n=0}^{\infty} \eta(\Omega_n)\,:\, B \subseteq \bigcup_{n=0}^{\infty} \Omega_n\right\} \geq \inf\left\{\sum_{n=0}^{\infty} \eta(\Omega_n)\,:\, A \subseteq \bigcup_{n=0}^{\infty} \Omega_n\right\} = \eta^*(A)$$

(c) Show that $\eta^*(\emptyset) = 0$.

Solution: Any 1-point set $\{x\}$ is convex and $\eta(\{x\}) = 0$. Then

$$0 \leq \eta^*(\emptyset) \leq \eta(\{x\}) = 0$$

since $\emptyset \subseteq \{x\}$ and η^* is monotone by the previous part.

(d) Show that η^* is σ-subadditive, i.e. for any countable family of set $\{E_n\}_{n\in\mathbb{N}}$ we have

$$\eta^*\left(\bigcup_{n=0}^{\infty} E_n\right) \leq \sum_{n=0}^{\infty} \eta^*(E_n)$$

Solution: Fix an $\epsilon > 0$. Then for every $n \geq 0$, there is a countable cover $\bigcup_{r=0}^{\infty} F_{n,r} \supseteq E_n$ by convex sets such that

$$\eta(E_n) \leq \sum_{r=0}^{\infty} \eta(F_{n,r}) < \eta(E_n) + \frac{\epsilon}{2^n}$$

Since we have

$$\bigcup_{n=0}^{\infty}\bigcup_{r=0}^{\infty} F_{n,r} \supseteq \bigcup_{n=0}^{\infty} E_n$$

we get

$$\eta^*\left(\bigcup_{n=0}^{\infty} E_n\right) \leq \eta\left(\bigcup_{n=0}^{\infty}\bigcup_{r=0}^{\infty} F_{n,r}\right) \leq \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \eta(F_{n,r}) \leq \sum_{n=0}^{\infty} \eta^*(E_n) + \frac{\epsilon}{2^n} = \epsilon + \sum_{n=0}^{\infty} \eta^*(E_n)$$

using the fact that $diam(A \cup B) \leq diam(A) + diam(B)$ for all convex sets A and B. Since $\epsilon > 0$ was arbitrary, we get that

$$\eta^* \left(\bigcup_{n=0}^{\infty} E_n \right) \leq \sum_{n=0}^{\infty} \eta^*(E_n)$$