Math 332E (Measure Theory) Quiz III November 30, 2016

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1. Assume $X$ is an arbitrary set. Recall that for a set $U \subset X$ we defined *the characteristic function* of $U$ as

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

Now, for two functions $f, g\colon X \to \mathbb{R}$ let us define

$$(f \cdot g)(x) = f(x)g(x) \quad \text{and} \quad (f \vee g)(x) = \max\{f(x), g(x)\}$$

Assume $U$ and $V$ are two subsets of $X$. Show that

(a) $\chi_U \cdot \chi_V = \chi_{U \cap V}$

**Solution:** We see that $(\chi_U \cdot \chi_V)(x) = 1$ if and only if $\chi_U(x) = \chi_V(x) = 1$ which works when, and only when, $x \in U \cap V$.

(b) $\chi_U \vee \chi_V = \chi_{U \cup V}$

**Solution:** We see that $(\chi_U \vee \chi_V)(x) = 1$ if and only if $\chi_U(x) = 1$ or $\chi_V(x) = 1$ which works when, and only when, $x \in U \cup V$.

2. Assume $(X, \mathcal{A}, \mu)$ is a measure space. Let $f, g\colon X \to \mathbb{R}$ be two measurable functions.

(a) Show that the subset $\{x \in X |\ f(x) < g(x)\}$ is measurable, i.e. it is in $\mathcal{A}$.
[Hint: $f - g$ is a measurable function.]

**Solution:** The difference $f - g$ measurable since both $f$ and $g$ are measurable. Then we have

$$\{x \in X |\ f(x) < g(x)\} = \{x \in X |\ (f - g)(x) < 0\}$$

which is a measurable set by definition.

(b) If $\{x \in X |\ f(x) < g(x)\}$ has measure 0, show that $\int_X f d\mu \geq \int_X g d\mu$.

**Solution:** Let $N = \{x \in X |\ f(x) < g(x)\}$ and $P = \{x \in X |\ f(x) \geq g(x)\}$. Our hypothesis says $\mu(N) = 0$ and we have

$$\int_X (f - g) d\mu = \int_P (f - g) d\mu + \int_N (f - g) d\mu$$

Now, let

$$N_k = \{x \in X |\ -k \leq f(x) - g(x) < 0\}$$

and we have $N = \bigcup_{k=1}^{\infty} N_k$. Since each $N_k$ is measurable we also get $0 \leq \mu(N_k) \leq \mu(N) = 0$ for every $k \geq 1$ and

$$0 \leq \int_N (g - f) d\mu = \sup_k \int_{N_k} (g - f) d\mu \leq k\mu(N_k) = 0$$

Then

$$\int_X (f - g) d\mu = \int_P (f - g) d\mu$$

and by the monotonicity of the integrals, we get that $\int_P (f - g) d\mu \geq 0$.

3. Recall that $U \subseteq \mathbb{R}$ is open when for every $x \in U$ there is a $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq U$. Now, let us consider $\mathbb{R}$ as a measurable space by considering $\mathbb{R}$ together with the Borel $\sigma$-algebra $\mathcal{B}$ generated by open subsets of $X$. Assume $\mu$ is the standard Lebesgue measure on $(\mathbb{R}, \mathcal{B})$.

   (a) Show that $\mu(U) > 0$ for all non-empty open sets $U \subseteq \mathbb{R}$.

   **Solution:** Take any $x \in U$. Then there is a $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq U$. Since $\mu$ is a measure, it is a monotone set function. Then

   $$\mu(U) \geq \mu((x-\epsilon, x+\epsilon)) = 2\epsilon > 0$$

   (b) Assume $U \subseteq \mathbb{R}$ is open and let $f\colon \mathbb{R} \to [0, \infty)$ be a measurable function such that

   $$\int_U f d\mu = 0$$

   Show that the set $\{x \in U | \, f(x) > 0\}$ has measure 0.

   **Solution:** We will use proof by contradiction. Assume that the set $\{x \in U | \, f(x) > 0\}$ has positive measure. Then there is a $\epsilon > 0$ such that the set $\{x \in U | \, f(x) > \epsilon\}$ has positive measure. Then

   $$\int_U f d\mu \geq \int_U \epsilon d\mu = \epsilon\mu(U) > 0$$

   which is a contradiction.

   (c) If $f\colon \mathbb{R} \to [0, \infty)$ is continuous and satisfies the condition $\int_U f d\mu = 0$ for an open subset $U \subseteq \mathbb{R}$, show that $f(x) = 0$ for all $x \in U$.

   **Solution:** Since $f$ is continuous and $U$ is open, the set

   $$\{x \in U | \, f(x) > 0\} = U \cap f^{-1}(0, \infty)$$

   is open. By part (b) the set must have 0 measure. But part (a) says all non-empty open sets have positive measure. So, the set must be empty.