# Math 332E (Measure Theory) Quiz IV

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1. (a) Consider the set $\mathbb{N}$ together with the set function

$$\alpha(U) = \sum_{x \in U} x$$

Show that $\alpha$ is a measure on $\mathbb{N}$.

Solution: Since every element of $\mathbb{N}$ is positive, it is clear that $\alpha(U) \geq 0$ for every $U \subseteq \mathbb{N}$. We need to show that $\alpha$ is countably additive. So, assume $\{U_n\}_{n \in \mathbb{N}}$ is a countable disjoint family of subsets of $\mathbb{N}$, i.e. $U_n \cap U_m = \emptyset$ whenever $n \neq m$. By removing empty sets from the family, we can assume $U_n \neq \emptyset$ for every $n$. But then, we either have a finite family $\{U_n\}_{n=0}^{N}$ or an infinite family $\{U_n\}_{n=0}^{\infty}$ with $|U_n| > 0$ for every $n \geq 0$. It is easy to see that $\alpha$ is finitely additive since

$$\alpha\left(\bigcup_{n=0}^{N} U_n\right) = \sum_{x \in \bigcup_{n=0}^{N} U_n} x = \sum_{n=0}^{N} \sum_{x \in U_n} x = \sum_{n=0}^{N} \alpha(U_n)$$

In case the family is infinite we have $\infty = \alpha\left(\bigcup_{n=0}^{\infty} U_n\right)$ and

$$\sum_{n=0}^{\infty} \alpha(U_n) = \lim_{n \to \infty} \sum_{n=0}^{N} \alpha(U_n) \geq \lim_{n \to \infty} \sum_{n=0}^{N} 1 = \infty$$

(b) Now, we consider the cartesian product $\mathbb{N} \times \mathbb{N}$ together with the product measure $\alpha \otimes \alpha$. Calculate the product measure $(\alpha \otimes \alpha)(A)$ of the set

$$A = \{(1,2), (2,2), (1,3), (2,3), (3,3)\}$$

Solution: We have

$$\begin{aligned}
(\alpha \otimes \alpha)(A) &= \sum_{(a,b) \in A} (\alpha \otimes \alpha)(\{(a,b)\}) \\
&= \sum_{(a,b) \in A} (\alpha \otimes \alpha)(\{a\} \times \{b\}) \\
&= \sum_{(a,b) \in A} \alpha(\{a\})\alpha(\{b\}) \\
&= \sum_{(a,b) \in A} a \cdot b \\
&= 1 \cdot 2 + 2 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 + 3 \cdot 3 = 24
\end{aligned}$$

2. Let $\Omega = \{(x, y) \in \mathbb{R}^2 | \, x \in [-1, 1], y \in [-x, x] \cap \mathbb{Q}\}$ and calculate

$$\int_{\Omega} |x + y| d(\mu \otimes \mu)$$

Solution: We know that

$$\Omega_x = \{(a, b) \in \Omega | \, a = x\} = [-x, x] \cap \mathbb{Q}$$

and therefore $\mu(\Omega_x) = 0$ since $\Omega_x$ is countable for every $x \in [-1, 1]$. Then

$$0 \leq \int_{\Omega} |x + y| d(\mu \otimes \mu) = \int_{[-1,1]} \int_{\Omega_x} |x + y| d\mu(y) d\mu(x)$$

On the other hand $|x + y| \leq 2$ for every $(x, y) \in \Omega$ and

$$0 \leq \int_{\Omega} |x + y| d(\mu \otimes \mu) \leq \int_{[-1,1]} \int_{\Omega_x} 2 d\mu(y) d\mu(x) = 2 \int_{[-1,1]} 2\mu(\Omega_x) d\mu(x) = 0$$

i.e. $\int_{\Omega} |x + y| d(\mu \otimes \mu) = 0$.