Name and Last Name:
Student Number:

Question:	1	2	3	4	5	6	Total
Points:	40	20	30	35	35	40	200
Score:							

1. (40 points) Give a definition of the following terms:
   (a) $\mathcal{A}$ is a σ-algebra on a set X.
   (b) μ is a measure on a measurable space $(X, \mathcal{A})$.
   (c) μ is an outer measure on a measurable space $(X, \mathcal{A})$.
   (d) $f\colon X \to Y$ is a measurable function from a measurable space $(X, \mathcal{A})$ to another measurable space $(Y, \mathcal{B})$.

2. (20 points) Assume X is a set. Recall that for every $Y \subseteq X$ we defined

$$\chi_Y(a) = \begin{cases} 1 & \text{if } a \in Y \\ 0 & \text{if } a \notin Y \end{cases}$$

(a) Show that $\chi_Y \cdot \chi_Z = \chi_{Y \cap Z}$ for every $Y, Z \subseteq X$.

Solution: We can see that $(\chi_Y \cdot \chi_Z)(x) = 1$ only when both $\chi_Y(x) = 1$ and $\chi_Z(x) = 1$. This means $(\chi_Y \cdot \chi_Z)(x) = (\chi_{Y \cap Z}(x)$ for every $x \in X$.

(b) Show that $\max(\chi_Y, \chi_Z) = \chi_{Y \cup Z}$ for every $Y, Z \subseteq X$.

Solution: We see that $\max(\chi_Y, \chi_Z)(x) = 0$ only when $\chi_Y(x) = 0$ and $\chi_Z(x) = 0$. This means $\max(\chi_Y, \chi_Z)(x) = \chi_{Y \cup Z}(x)$ for every $x \in X$.

3. (30 points) Assume $(X, \mathcal{A})$ is a σ-algebra and let $Y \subseteq X$ be an arbitrary subset. Show that the family of sets

$$\mathcal{B} := \{U \cap Y | \, U \in \mathcal{A}\}$$

is a σ-algebra on Y.

Solution:

i) Since $\emptyset \in \mathcal{A}$ we see that $\emptyset \cap Y = \emptyset \in \mathcal{B}$. Also, since $X \in \mathcal{A}$ we also have $X \cap Y = Y \in \mathcal{B}$.

ii) If $U, V \in \mathcal{B}$ then there are $U', V' \in \mathcal{A}$ such that

$$U = U' \cap Y \quad \text{and} \quad V' = V \cap Y$$

Then $U' \setminus V' \in \mathcal{A}$ since $\mathcal{A}$ is an algebra. But then

$$\begin{aligned} U \setminus V =& U \cap V^c = (U' \cap Y) \cap (V' \cap Y)^c \\ =& (U' \cap Y) \cap ((V')^c \cup Y^c) = (U' \cap (V')^c \cap Y) \cup (U' \cap Y \cap Y^c) \\ =& (U' \setminus V') \cap Y \end{aligned}$$

is in $\mathcal{B}$ since $U' \setminus V'$ is in $\mathcal{A}$.

iii) Assume we have a countable family of subsets $\{U_i\}_{i \in \mathbb{N}}$ in $\mathcal{B}$. Then by definition there is another family of subsets $\{U_i'\}_{i \in \mathbb{N}}$ in $\mathcal{A}$ such that $U_i = U_i' \cap Y$. Since $\mathcal{A}$ is a σ-algebra we get that $\bigcup_{i \in \mathbb{N}} U_i' \in \mathcal{A}$. Then we also see that

$$\bigcup_{i \in \mathbb{N}} U_i = \bigcup_{i \in \mathbb{N}} U_i' \cap Y = Y \cap \bigcup_{i \in \mathbb{N}} U_i'$$

is also in $\mathcal{B}$.

4. (35 points) Let $(X, \mathcal{A})$ be a measurable space. Let $\eta\colon \mathcal{A} \to [0, \infty)$ be a set function such that

i) $\eta(\emptyset) = 0$

ii) $\eta(X) < \infty$

iii) for all $U \subseteq V$ in $\mathcal{A}$ we have $\eta(U) \leq \eta(V) \leq \eta(U) + \eta(V \setminus U)$

Let us define

$$\eta^*(Z) = \inf \left\{ \sum_i \eta(U_i) : \bigcup_{i=0}^{\infty} U_i \supseteq Z \right\}$$

for every $Z \subseteq X$.

(a) Show that $\eta(U) = \eta^*(U)$ for every $U \in \mathcal{A}$.

Solution: One can see that $\eta(A \cup B) \leq \eta(A) + \eta(B)$ whenever $A \cap B = \emptyset$ by setting $V = A \cup B$ and $U = A$. In case $A \cap B \neq \emptyset$ the inequality still holds since $B \subseteq (A \cup B) \setminus A$ and η is monotone. This proves η is finitely subadditive. Then given any cover $U \subseteq \bigcup_i U_i$ we get

$$\eta(U) \leq \sum_i \eta(U_i)$$

which means $\eta(U) \leq \eta^*(U)$ On the other hand U as a singleton family is a cover of U since $U \in \mathcal{A}$. Thus $\eta^*(U) \leq \eta(U)$ which proves $\eta^*(U) = \eta(U)$.

(b) Show that η^* is an outer measure on X.

Solution: We know that $\eta(\emptyset) = 0$. The fact that η is monotone is also given. We must show that η is countably subadditive. Take $E \subseteq X$ and take a countable cover $E \subseteq \bigcup_n E_n$. Fix an $\epsilon > 0$. Then for every $n \geq 0$, there is a countable cover $\bigcup_{r=0}^{\infty} F_{n,r} \supseteq E_n$ in $\mathcal{A}$ such that

$$\eta^*(E_n) \leq \sum_{r=0}^{\infty} \eta(F_{n,r}) < \eta^*(E_n) + \frac{\epsilon}{2^n}$$

Since we have

$$\bigcup_{n=0}^{\infty} \bigcup_{r=0}^{\infty} F_{n,r} \supseteq \bigcup_{n=0}^{\infty} E_n$$

we get

$$\eta^* \left(\bigcup_{n=0}^{\infty} E_n \right) \leq \eta \left(\bigcup_{n=0}^{\infty} \bigcup_{r=0}^{\infty} F_{n,r} \right) \leq \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \eta(F_{n,r}) \leq \sum_{n=0}^{\infty} \eta^*(E_n) + \frac{\epsilon}{2^n} = \epsilon + \sum_{n=0}^{\infty} \eta^*(E_n)$$

using the fact that η is monotone. Since $\epsilon > 0$ was arbitrary, we get that

$$\eta^* \left(\bigcup_{n=0}^{\infty} E_n \right) \leq \sum_{n=0}^{\infty} \eta^*(E_n)$$

5. (35 points) Let us define

$$f_n(x) = \begin{cases} x - \frac{1}{n} & \text{if } 0 \leq x \leq 1 - \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$.

Solution: The pointwise limit is

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) We would like to calculate the Lebesgue integral $\int_{\mathbb{R}} f d\mu$. However, as you might recall, it is not easy to say

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n d\mu$$

State which theorem we can use to safely say that the equality above holds. Explain why we can use that specific theorem.

Solution: We can use both *Dominated Convergence Theorem* (DCT) and *Monotone Convergence Theorem* (MCT). We can use MCT because $f_n(x) \leq f_{n+1}(x)$ for every $x \in \mathbb{R}$. We can use DCT because $f_n(x) \leq \chi_{[0,1)}$ and $\chi_{[0,1)}$ is a measurable and integrable function. One can replace $\chi_{[0,1)}$ with $f(x)$ or any other measurable integrable function $g(x)$ as long as $f_n(x) \leq g(x)$ for every $x \in \mathbb{R}$.

(c) Calculate $\int_{\mathbb{R}} f d\mu$.

Solution: We can see that $\int_{\mathbb{R}} f d\mu = \int_{[0,1)} x d\mu$. Since $f(x) = x$ is a continuous function, its Lebesgue integral and Riemann integral are the same. Then

$$\int_{\mathbb{R}} f d\mu = \int_{[0,1)} x d\mu = \int_0^1 x dx$$

Then by the Fundamental Theorem of Calculus we get

$$\int_0^1 x dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

6. (40 points) Consider $X = \{1, 2, \ldots, 100\}$ together with the largest σ-algebra 2^X on X. Define

$$\eta(U) = |U| \quad \text{and} \quad \mu(U) = \begin{cases} 1 & \text{if } 1 \in U \\ 0 & \text{otherwise} \end{cases}$$

Let us also define $f(n) = (-1)^n$.

(a) Calculate $\int_X f d\eta$

Solution: First, split X into two pieces

$$E = \{2n |\, n = 1, 2, \ldots, 50\} \quad \text{and} \quad O = \{2n - 1 |\, n = 1, 2, \ldots, 50\}$$

where E contains the even numbers in X while O contains the odd numbers in X. Then we can see that

$$\eta(E) = |E| = |O| = \eta(O) = 50$$

Moreover,

$$\begin{aligned}\int_X f d\eta = \int_E f d\eta + \int_O f d\eta = \int_E (-1) d\eta + \int_O (+1) d\eta \\ =(-1)\eta(E) + (+1)\eta(O) = -50 + 50 = 0\end{aligned}$$

(b) Calculate $\int_X f d\mu$

Solution: Again, we split X into two pieces. This time

$$P = \{1\} \quad \text{and} \quad Q = \{2, 3, \ldots, 100\}$$

Then

$$\begin{aligned}\int_X f d\mu =& \int_P f d\mu + \int_Q f d\mu \\ =& \int_P f d\mu + \int_{Q\cap E} f d\mu + \int_{Q\cap O} f d\mu \\ =&(-1)\mu(\{1\}) + (+1)\mu(Q \cap E) + (-1)\mu(Q \cap E) \\ =& -1 + 0 + 0 = -1\end{aligned}$$

(c) Define $g(n, m) = f(n + m)$ and calculate $\int_{X\times X} g \, d(\eta \otimes \mu)$.

Solution: We have

$$\int_{X\times X} g\, d(\eta\otimes\mu) = \int_{X\times X} (-1)^{n+m} d(\eta\otimes\mu)$$

By Fubini's Theorem we can calculate the integral over the product measure as an iterated double integral

$$\int_X \int_X (-1)^{n+m} d\eta(n) d\mu(m) = \int_X \int_X (-1)^{n+m} d\mu(m) d\eta(n)$$

However, since $(-1)^{n+m} = (-1)^n(-1)^m$ we get

$$\begin{aligned}\int_X \int_X (-1)^{n+m} d\eta(n) d\mu(m) &= \int_X (-1)^m \int_X (-1)^n d\eta(n) d\mu(m)\\ &= \int_X (-1)^m \cdot 0\, d\mu(m) = 0\end{aligned}$$

from part (a).