1. The alternative set $A = \{a_1, a_2, \ldots, a_m\}$ must be composed of mutually exclusive and independent elements a_i.

2. If the above is not possible, then an additional alternative must be defined as a combination of the dependent alternatives and the choice must be limited to only one of the dependent alternatives of their combination.

The structure of indexed objective function is similar to those of Kemeny and Snell [K27] and Levenglick [L16]. In this approach, Bernardo considers the allocation of scarce resources which set up the constrained linear assignment model.

This approach has the same properties as Kemeny's function, and is natural, decisive, anonymous and homogeneous. The resource restriction affects the function such that it is not monotonic and Paretian, because the axiomatic structure of the objective function is to find the maximization of the highest index value from the agreement between the voter's rank and the consensus rankings.

2.13 COOK AND SEIFORD'S ORDINAL INTERSECTION METHOD

Cook and Seiford [C39] present the ordinal intersection method for selecting possible fundable sets of research and development (R & D) projects. The problem is to place priorities on choosing courses of actions within the total resources available from a finite set of alternatives: $A = \{a_1, a_2, \ldots, a_m\}$, by members of the committee: $N = \{1, 2, \ldots, n\}$, with a set of multiple criteria: $S = \{s_1, s_2, \ldots, s_\ell\}$.

The Ordinal Intersection Method:

Each committee member gives the ranking of alternatives for each criterion. Cook and Seiford's social choice function (see Section 2.9) is used to determine a compromise or consensus ranking of alternatives for each criterion that best agree with all the committee's rankings. Then the ordinal intersection method is used for finding fundable projects within the total available resources.

Once the committee's order-ranking consensus of alternatives for each criterion is obtained, and the research budget is known, the alternatives can be divided into the fundable and non-fundable subsets according to each criterion.

The ordinal intersection method proceeds as follows:

Step 1. We take the intersection of all ℓ fundable sets (i.e., $g_1 = \bigcap_{i=1}^{\ell} f_i$). This set, be noted by g_1, (g_1 may be empty) is the subset of projects which are contained in all fundable sets f_i, $i = 1, 2, \ldots, \ell$. Now, if this g_1 set contains enough projects to absorb the entire budget, we stop. If not, we go to the next step.

Step 2. We select some set of $\ell-1$ of the f_i and take their intersection, $\bigcap_{i=1}^{\ell-1} f_i$. There are a set of $\ell(= {}_{\ell}C_{\ell-1})$ intersections.

If the committee has assigned cardinal weights w_i to the criteria, then the ordering of taking these intersections should be determined by the ordering of the corresponding sum of weights, $\sum_{i=1}^{\ell-1} w_i$. That is, that combination whose weights sum to the largest value is chosen as the first, and that of the second largest value the second, and so on. If the criteria are ranked ordinally only, then in distinguishing among the set of ℓ intersections, the sum of ranks could be used as a selection criterion. For example, consider the case of 5 criteria, where the ordinal ranking of the categories is (3, 1, 2, 5, 4). If sets of four criteria are to be selected, the first choice would be criteria 1, 2, 3, 5 whose sum of ranks 2 + 3 + 1 + 4 = 10. This sum is the minimal among all sums corresponding to four criteria. The next choice of four would be criteria 1, 2, 3, 4 whose sum of ranks is 2 + 3 + 1 + 5 = 11, and the last one would be 1, 2, 4, 5 whose sum of ranks is 2 + 3 + 5 + 4 = 14.

We may denote the new subset of projects by $g_2, g_3, \ldots, g_\ell$; each is $g_i = \bigcap_{j=1}^{\ell-1} f_j$, and combination of f_j is according to the ordering of the corresponding sum of weights $\sum_{i=1}^{\ell-1} w_j$, or according to the sum of ranks when the criteria are ranked ordinally only. If the set of combined projects in g_1 and g_2, that is, $g = g_1 \cup g_2$, absorbs the budget, then stop. Otherwise, consider the next set of combined projects, that is $g = \bigcup_{i=1}^{3} g_i$. If the new set of projects absorbs the budget, stop; otherwise proceed in the fashion described above.

At some stage g, the union of all intersections to date will contain enough projects to absorb the available budget. If at the end of the union of $g = \bigcup_{i=1}^{\ell-1} g_i$, where the union of all intersections to date does not contain enough projects to absorb the available budget, we must go to Step 3.

Step 3. The procedure described in Step 2 will continue to select the elements of ℓ-2 out of f_i and take their intersections, $\bigcap_{j=1}^{\ell-2} f_j$. The procedure will stop when the union of all intersections contains enough project to absorb the available budget.

Example: (Cook and Seiford [C39])

The total budget is assumed to be \$240,000 which is to be allocated over a set of 10 projects. Their estimated costs are:

C_1 = 50,000; C_2 = 60,000; C_3 = 10,000; C_4 = 70,000; C_5 = 30,000;
C_6 = 30,000; C_7 = 20,000; C_8 = 40,000; C_9 = 50,000; C_{10} = 40,000.

There are six experts who give the rank-order of projects for each criterion in order to evaluate the projects. The optimum consensus rankings are derived by Cook and Seiford's function (see Section 2.9). The results for the 5 criteria are (note that these are project lists, not rank lists):

Criterion No. 1: 10, 4, 9, 3, 5, 7 | 2, 6, 1, 8

Criterion No. 2: 3, 4, 10, 8, 7, 2 | 1, 5, 6, 9

Criterion No. 3: 1, 3, 5, 7, 2, 6, 10 | 9, 8, 4

Criterion No. 4: 10, 6, 4, 3, 2, 5 | 1, 9, 7, 8

Criterion No. 5: 1, 4, 9, 10, 3 | 2, 5, 6, 8, 7

The vertical line (|) indicates the cut-off point, given the available budget. For example, under criterion No. 1, the first six projects (projects 10, 4, 9, 3, 5, 7) which occupy the top six ranks slots, cost \$220,000; if we include project 2, then the total cost exceeds the available resource. Thus, we have the fundable sets : f_1 = {10, 4, 9, 3, 5,7}, f_2 = {3, 4, 10, 8, 7, 2}, f_3 = {1, 3, 5,7, 2, 6, 10}, f_4 = {10, 6, 4, 3, 2, 5}, f_5 = {1, 4, 9, 10, 3}.

If we assume that the five criteria have been ranked in the natural ordering, i.e., criterion No. 1 is preferred to No. 2 which is preferred to No. 3, etc., then

Step 1. The intersection of the five criteria:

$$\bigcap_{i=1}^{5} f_i = \{3, 10\} = g_1.$$

The total budget consumed by these two projects is $C_3 + C_{10}$ = \$50,000. Since some budget remains, we go to the next step.

Step 2. The intersection of four criteria (note that the sequence of process is according to the relative of criteria weights):

$f_1 \cap f_2 \cap f_3 \cap f_4 = \{3, 10\}$ (the same as g_1)

$f_1 \cap f_2 \cap f_3 \cap f_5 = \{3, 10\}$ (the same as g_1)

$f_1 \cap f_2 \cap f_4 \cap f_5 = \{3, 10, 4\} = g_2$

Then $g_1 \cup g_2 = \{3, 10, 4\} = g$. Project 4 is added; the total budget consumed to date is $C_3 + C_{10} + C_4 = \$120,000$.

$f_1 \cap f_3 \cap f_4 \cap f_5 = \{3, 10\}$ (The same as g_1)

$f_2 \cap f_3 \cap f_4 \cap f_5 = \{3, 10)$ (The same as g_1)

Since some budget remains, we go to next step.

Step 3. The intersection of three criteria (the sequence of process is according to the relative of criteria weights):

$f_1 \cap f_2 \cap f_3 = \{3, 7, 10\} = g_3$

Then, $g_1 \cup g_2 \cup g_3 = \{3, 10, 4, 7\} = g$

Project 7 is added; the total budget consumed to date is $C_3 + C_{10} + C_4 + C_7 =$ \$140,000.

$f_1 \cap f_2 \cap f_4 = \{3, 4, 10\}$ (The same as g_2)

$f_1 \cap f_2 \cap f_5 = \{3, 4, 10\}$ (The same as g_2)

$f_1 \cap f_3 \cap f_4 = \{3, 5, 10\} = g_4$.

Then $g_1 \cup g_2 \cap g_3 \cup g_4 = \{3, 10, 4, 7, 5\} = g$. Project 5 is added; the total budget consumed to date is $C_3 + C_{10} + C_4 + C_7 + C_5 = \$170,000$.

$f_1 \cap f_3 \cap f_5 = \{3, 10\}$ (The same as g_1)

$f_2 \cap f_3 \cap f_4 = \{2, 3, 10\} = g_5$

Then $\bigcup_{i=1}^{5} g_i = \{3, 10, 4, 7, 5, 2\} = g$. Project 2 is added; the total budget consumed to date is $C_3 + C_{10} + C_4 + C_7 + C_5 + C_2 = \$230{,}000$. Since only \$10,000 remains and no project is left which would be completed with this much money, we are finished. The best set of projects is, $g = \{2, 3, 4, 5, 7, 10\}$.

Note that g is not contained as a subset of any of the f_i, $i = 1, 2, 3, 4, 5$, yet each element of g is in the fundable sets corresponding to at least three criteria.

Example: Selection of a Set of Scientific Experiments for NASA's Space Shuttle (Adams, et al., [A1], Bernardo [B19]).

The problem is presented in Section 2.12. Cook and Seiford's function (see Section 2.9) has been applied to find the consensus rank-ordering of projects for each criterion. The results are given as follows:

Criterion s_1: $a_6 \;\; a_3 \;\; | \;\; a_2 \;\; a_5 \;\; a_4$

Criterion s_2: $a_3 \;\; a_6 \;\; | \;\; a_2 \;\; a_4 \;\; a_5$

or

$a_3 \;\; a_6 \;\; | \;\; a_4 \;\; a_2 \;\; a_5$

Criterion s_3: $a_3 \;\; a_6 \;\; | \;\; a_5 \;\; a_2 \;\; a_4$

or

$a_3 \;\; a_5 \;\; | \;\; a_6 \;\; a_2 \;\; a_4$

The verticalline (|) indicates the cut-off point, given the crew time-daily requirement (man-hours/day) of 20. For example, under criterion s_1, the sum of the first two alternatives (a_6 and a_3) which the total is 18.5 (= 10.25 + 8.25); however if we include a_2, then the total time exceeds the restricted value. Therefore, the fundable sets are:

$f_1 = \{a_6, a_3\}$

$f_2 = \{a_3, a_6\}$

$f_3 = \{a_3, a_6\}$

or

$f_3' = \{a_3, a_5\}$

Assume that three criteria have equal importance. The ordinal intersection method proceeds as follows:

<u>Case</u> <u>A:</u> Consider f_1, f_2, f_3

Step 1 Intersection of three criteria.

$$\bigcap_{i=1}^{3} f_i = \{a_3, a_6\} = g$$

The total times consumed by these two alternatives is $a_3 + a_6 = 10.25 + 8.25 = 18.5$. Some time remains, but we do not have any alternative which has daily time requirement less than 1.5 (= 20 - 18.5). Thus, the best set of alternatives is $g = \{a_3, a_6\}$.

Case B: Consider f_1, f_2, f_3'

<u>Step</u> <u>1</u> Intersection of three criteria

$$\bigcap_{i=1}^{3} f_i = \{a_3\} = g_1$$

The time consumed by alternative a_3 is 10.25. Since some time remains, we go to the next step.

<u>Step</u> <u>2:</u> Intersection of two criteria.

$$f_1 \cap f_2 = \{a_3, a_6\} = g_2$$

The alternative a_6 is added; the total time consumed to date is $a_3 + a_6 = 18.5$, and now $g = g_1 \cup g_2 = \{a_3, a_6\}$.

$f_1 \cap f_3' = \{a_3\}$ (the same as g_1)

$f_2 \cap f_3' = \{a_3\}$ (the same as g_1)

Therefore, nothing more needs to be added; the best set of alternatives being $g = \{a_3, a_6\}$.

Note:

When the candidate projects and the evaluation criteria are both very large this method would become cumbersome. We also assume all of the criteria must be mutually independent.

The consensus rank-ordering of projects for each criterion used in this section was obtained by Cook and Seiford's function (see Section 2.9), however, any other social choice functions presented in Sections 2.3 through 2.11 can be employed.

The problems considered in this section differ from those for Sections 2.3 through 2.11 in two points: (i) multiple criteria are explicitly presented in evaluating each alternative, (ii) the selection is constrained by one scarce resource.

3. SOCIAL WELFARE FUNCTION

3.1 Introduction

In a society, a decision often affects groups of people instead of isolated individuals. The problem of decision making is this: How can many individuals' preferences be combined to yield a collective choice? Various procedures have been proposed to accomplish this feat, all of which differ from each other in many respects. For instance, the simple majority rule is widely used in two-candidate situations and is judged to be reasonable and equitable for making decisions. On the other hand, when the simple majority rule is applied in multi-candidate situations, intransitivity among candidates may occur. For example, the simple majorities could be intransitive in the situation when x beats y, y beats z, and z beats x. This outcome is a cyclical ranking and is called the paradox of voting. The paradox was known and developed by Marquis de Condorcet in the eighteenth century, and it is referred to as the Condorcet effect. Another case, in section 1.2.1.2 (a) of paradox of voting, for example 1 of Condorcet, the result depends on the method of voting being employed. Any of the three candidates could be elected: candidate a by a plurality method, candidate b by the second ballot of the majority representation system, and candidate c by the Condorcet principle. This is clearly an