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A HYBRID METHOD FOR SECOND-ORDER WAVE RADIATION BY AN OSCILLATING AXISYMMETRIC CYLINDER

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Abstract

A numerical-analytical hybrid method is presented for the three dimensional analysis of forced oscillatory motion of a vertical axisymmetric cylinder in finite water depth. The present study extends the direct method based on Weber's integral theorem to include more general, axisymmetric, cylindrical forms. A classical perturbation procedure is employed to solve the nonlinear problem through the second-order. The fluid domain around the cylinder is separated into an interior and an exterior region. The advantage of axisymmetry is taken into consideration in the interior region by making use of Rankine ring-source distribution within the boundary element formulation. In the exterior region, nonhomogeneous second-order free surface condition is satisfied by means of a modified form of Weber's integral theorem. Eigenfunction expansion is used for the exterior homogeneous solution. The complete solution is then obtained by matching the interior and the exterior solutions on the common boundary by satisfying the continuity of pressure and radial velocities. A computational example is given for the verification of the formulation for the heave motion of a vertical truncated circular cylinder.

1. Introduction

Various approaches to the solution of the problem of the second-order radiation/diffraction of waves, by a vertical axis cylinder, can still be seen in the literature. Using the classical perturbation method, these studies aim to provide benchmark computations and understand the effect of higher-order harmonics beyond the linear theory. These attempts may be classified into two categories: (i) an indirect approach, pioneered by Lighthill (1979) and Molin (1979), which uses Green's second identity to calculate the wave loads without calculating the second-order potential solution in a closed-form; and (ii) an explicit calculation of the second-order potential using a Fourier-Bessel expansion which satisfies second-order free surface condition by means of a modified form of Weber's integral theorem, which was introduced by Hunt and Baddour (1981). Recent attempts for the correct and complete solutions to the second-order diffraction forces can be seen in Huang and Eatock Taylor (1996) and Newman (1996). It is clear that the debate is still continuing on the numerical results rather than on the formulation, as remarked by Newman (1996). On the other hand, besides the diffraction problem, Gören (1996) extended the linear theory of Sabuncu and Calisal (1981) and Yeung (1981) to include second-order effects by adopting an approach similar to the one used in Hunt and Baddour (1981).

The present work considers Gören's (1996) exterior solution as a buit-in formulation and makes use of the Boundary Element Method (BEM) in the interior region which allows the application of that procedure to more general, axisymmetric forms. The solution is obtained by matching the interior and the exterior solutions on the common boundary to satisfy the continuity of pressure and radial velocities. To test the numerical results, a computational example is given for the heave motion of a truncated circular cylinder in finite water depth.

2. Second-order Formulation

2.1 Boundary Value Problem

A vertical axisymmetric cylinder (Figure 1) is forced to make a harmonic motion with a known frequency ω, and an amplitude A. The motion amplitude is assumed to be small compared to the radius of the cylinder. The displacement of the oscillatory cylinder is prescribed as;

$$c(t) = A \sin \omega t \tag{1}$$

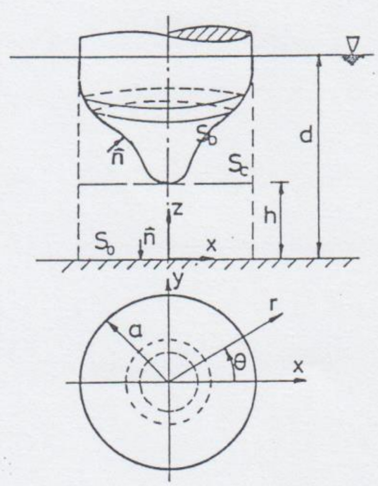

Figure 1.

With the usual assumptions necessary for the existance of a potential flow, the velocity is defined as $\mathrm{V} = \nabla\phi$, and accordingly the governing equation is Laplace's equation:

$$\nabla^2 \phi(r, \theta, z; t) = 0 \tag{2}$$

Both Cartesian $(Oxyz)$ and cylindrical coordinates (r, θ, z) are employed in Figure 1. The velocity potential, ϕ, satisfies the kinematic and dynamic boundary conditions, respectively, on the free surface at $z = \zeta(r, \theta; t) + d$:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi \cdot \nabla\phi) + g\zeta = 0 \quad ; \quad z = \zeta(r, \theta; t) + d \tag{3}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial r}\frac{\partial \zeta}{\partial r} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}\frac{\partial \zeta}{\partial \theta} - \frac{\partial \phi}{\partial z} = 0 \quad ; \quad z = \zeta(r, \theta; t) + d \qquad (4),$$

the boundary condition at the flat bottom:

$$\frac{\partial \phi}{\partial z} = 0 \quad ; \quad z = 0 \qquad (5),$$

the kinematic condition on the rigid moving body surface, $S(r, \theta, z; t)$:

$$\frac{\partial \phi}{\partial n} = -\frac{S_t}{|\nabla S|} \tag{6}$$

and a proper radiation condition at infinity which ensures the uniqueness of the solution. n is the unit normal pointing out of the fluid.

In the present analysis, a classical perturbation method is employed through the second-order. Perturbation parameter ϵ represents the ratio of the amplitude of the oscillation to the radius of the cylinder in the perturbation series:

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots \tag{7}$$

$$\zeta = \epsilon\zeta_1 + \epsilon^2\zeta_2 + \cdots \tag{8}$$

Zero-order approximation represents the fluid at rest. Expanding boundary conditions (3), (4) and (6) into a Taylor series about the mean positions of the moving surfaces, and using perturbation series (7) and (8) in these expressions and ultimately equating like powers of ϵ, lead to the reduced series of linear sub-problems. The solution for the first-order problem can be found in detail in Sabuncu and Calisal (1981) and Yeung (1981). Since the main purpose of this paper is to obtain second-order forces and wave components, the second-order boundary value problem (BVP) of the heaving cylinder is explained below

$$\nabla^2\phi_2(r, \theta, z; t) = 0 \tag{9}$$

with the boundary condition on the free surface

$$\left(\frac{\partial^2}{\partial t^2} + g\frac{\partial}{\partial z}\right)\phi_2 = -\frac{\partial}{\partial t}\left[(\frac{\partial \phi_1}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial \phi_1}{\partial \theta})^2 + (\frac{\partial \phi_1}{\partial z})^2\right] + $$
$$+\frac{1}{g}\frac{\partial \phi_1}{\partial t}\frac{\partial}{\partial z}(\frac{\partial^2\phi_1}{\partial t^2} + g\frac{\partial \phi_1}{\partial z}) \quad ; \quad z = d \tag{10}$$

and on the cylinder

$$\frac{\partial \phi_2}{\partial r} = 0 \qquad ; \quad r = a, \quad -h \leq z \leq d$$
$$\frac{\partial \phi_2}{\partial z} = -a\frac{\partial^2\phi_1}{\partial z^2}\sin\omega t \qquad ; z = h, \quad r \leq a \tag{11}$$

and on the sea bottom

$$\frac{\partial \phi_2}{\partial z} = 0 \quad ; \quad z = 0 \tag{12}$$

The radiation condition for the sub-BVP having a homogeneous free surface condition can be chosen such that the radial waves must be outgoing to infinity. The condition at infinity for the sub-BVP with nonhomogeneous free surface condition is presumed to be spontaneously imposed by the asymptotic behaviour of the r.h.s. of (10) which is quadratic in ϕ_1. The quadratic character of equation (10) and the kinematic condition on the body implies that ϕ_2 has the form:

$$\phi_2(r, \theta, z; t) = \mathrm{Re}\{\omega a^2 \sum_{m=0}^{\infty} \psi_2^{(m)}(r, z)\cos m\theta \exp[-i2\omega t]\} \tag{13}$$

where $\psi_2^{(m)}$ is the nondimentional spatial potential. Time-independent term which should be included in (13) is discarded, since it contributes nothing to the second-order quantities. For heave motion, m-summation in (13) exists only for $m = 0$.

2.2 *Method of Solution*

According to the solution procedure, the fluid domain is separated into an interior (fluid volume beneath the cylinder) and an exterior region (the remainder extending to infinity).

In the exterior region BVP is decomposed into two BVPs each having one nonhomogeneous boundary condition. Following the same notation of Gören (1996), exterior solution is split into a homogeneous and a particular solution:

$$\psi_{2e}^{(0)} = \psi_{2eh}^{(0)} + \psi_{2ep}^{(0)} \quad ; \quad r \geq a \tag{14}$$

where $\psi_{2eh}^{(0)}$ can be given in terms of an eigenfunction expansion as;

$$\psi_{2eh}^{(0)}(r,z) = E_0^{(0)} \frac{H_0(\kappa_0 r)}{H_0'(\kappa_0 a)} Z_0(\kappa_0 z) + \sum_{q=1}^{\infty} E_q^{(0)} \frac{K_0(\kappa_q r)}{K_0'(\kappa_q a)} Z_q(\kappa_q z) \tag{15}$$

$\psi_{2e}^{(0)}$ is the solution for the sub-BVP which consists of the equations (9), (11), (12) and the homogeneous form of (10). Eigenvalues κ_q are the roots of the second-order dispersion relation $\kappa \tanh(\kappa d) = 4\omega^2/g$ where κ represents κ_0 or $i\kappa_q$ for $q \geq 1$. $Z_q(z)$ are orthonormal eigenfunctions valid in the interval $[0, d]$. H_0 and K_0 are Hankel function of the first kind and the modified Bessel function of the second kind, respectively, both of order of 0. A prime denotes differentiation with respect to the argument. Differentiating of (15) with respect to r, at $r = a$, and multiplying both sides by $Z_q(\kappa_q z)$, then integrating over the interval [0,d] and using orthonormal properties of $Z_q(\kappa_q z)$ gives:

$$E_q^{(0)} = \frac{1}{\kappa_q d} \int_0^d \frac{\partial \psi_{2eh}^{(0)}}{\partial r} Z_q(\kappa_q z)\, dz \quad ; \quad q = 0, 1, \cdots \tag{16}$$

Particular solution $\psi_{2ep}^{(0)}$ must satisfy nonhomogeneous second-order free surface condition. Equation (10) can be rearranged as given by Sabuncu and Gören (1985) for the heave motion as;

$$\frac{\partial^2 \phi_{2e}}{\partial t^2} + g\frac{\partial \phi_{2e}}{\partial z} = \mathrm{Re}\left\{ i\omega^3 a^4 \frac{1}{2} 2 f_{0,0} \exp[-i2\omega t] \right\} \tag{17}$$

where

$$f_{0,0}(r) = \left(\frac{\partial \psi_{1e}^{(0)}}{\partial r} \right)^2 + \left(\frac{\partial \psi_{1e}^{(0)}}{\partial z} \right)^2 + \frac{\omega^2}{g} \psi_{1e}^{(0)} \frac{\partial \psi_{1e}^{(0)}}{\partial z} - \frac{1}{2} \psi_{1e}^{(0)} \frac{\partial^2 \psi_{1e}^{(0)}}{\partial z^2} \tag{18}$$

and $\psi_{1e}^{(0)}$ is the first-order exterior solution. By virtue of (17), particular solution should be given by a continuous spectrum which satisfies (9), (11) and (12);

$$\psi_{2ep}^{(0)}(r,z) = a \int_0^{\infty} A^{(0)}(k) C_0(kr) \cosh kz\, dk \tag{19}$$

where

$$C_0(kr) = [J_0(kr) Y_0'(ka) - Y_0(kr) J_0'(ka)] \tag{20}$$

J_0 and Y_0 are Bessel functions of first and second kind, respectively,of order of 0. Thus, following Hunt and Baddour (1981), we make use of the modified form of Weber's integral theorem

$$f_{0,0}(r) = \int_0^{\infty} \frac{C_0(kr)}{Q_0} k\, dk \int_a^{\infty} R C_0(kR) f_{0,0}(R)\, dR \tag{21}$$

in (17) to give:

$$A^{(0)}(k)[k \sinh kd - \frac{4\omega^2}{g} \cosh kd] = i\frac{kd}{Q_0}\frac{\omega^2 a}{g}\int_a^\infty RC_0(kR)f_{0,0}(R)\,dR \tag{22}$$

where $Q_0 = [J_0^{'2}(ka) + Y_0^{'2}(ka)]$.

The interior solution, $\psi_{2i}^{(0)}$, is obtained by BEM with circular ring elements distributed on the control surface, S_t, composed of a circular flat portion, S_o ($r \le a$), of the sea bottom, common surface, S_c, at $r = a$ ($0 \le z \le h$) and the cylinder surface, S_b. Discretization on the control surface is made so that N_0, N_c and N_b elements are taken on S_0, S_c and S_b, respectively, with a total of N ring elements. Parallel to the idea of Drimer and Agnon's (1994) 2D study, free surface is excluded from the boundary of the inner region. The potential value of each of these ring elements is taken to be constant, and the mid point, $P = P(r_m, 0, z_m)$, of each element is defined as the node point of that element, as is used in Calisal and Chan (1986). Rankine sources are utilized as the Green's function in the integral equation

$$2\pi\phi_{2i}^{(0)}(P) + \int_{S_t} \phi_{2i}^{(0)}(Q)\frac{\partial}{\partial n}\left(\frac{1}{r'(P,Q)}\right) dS(Q) = \int_{S_t} \frac{\partial\phi_{2i}^{(0)}(Q)}{\partial n}\frac{1}{r'(P,Q)}\, dS(P,Q) \tag{23}$$

where the point $Q = Q(r_n \cos\theta, r_n \sin\theta, z_n)$, $(n, m = 1, 2, \cdots, N)$, is the source point, and r' is the distance between P and Q and is written as

$$r' = \overline{QP} = \sqrt{(r_n \cos\theta - r_m)^2 + (r_n \sin\theta)^2 + (z_n - z_m)^2}.$$

The matching of the velocity potential and its radial derivative at the common surface ($r = a$) is achieved by the following matching conditions:

$$\psi_{2i}^{(0)} = \psi_{2eh}^{(0)} + \psi_{2ep}^{(0)} \quad ; \quad r = a, \quad 0 \le z \le h \tag{24}$$

$$\frac{\partial\psi_{2eh}^{(0)}}{\partial r} = \frac{\partial\psi_{2i}^{(0)}}{\partial r} - \frac{\partial\psi_{2ep}^{(0)}}{\partial r} \quad ; \quad r = a, \quad 0 \le z \le h \tag{25a}$$

$$\frac{\partial\psi_{2eh}^{(0)}}{\partial r} = -\frac{\partial\psi_{2ep}^{(0)}}{\partial r} \quad ; \quad r = a, \quad h \le z \le d \tag{25b}$$

(24) is used to express $\psi_{2i}^{(0)}$ at $r = a$ in terms of the exterior solution, since we consider $(\frac{\partial\psi_{2i}^{(0)}}{\partial r})_n$ unknowns on S_c. Therefore, by taking into account the boundary conditions (11) and (12) in (23) we arrive at the following N equations with $N + L + 1$ unknowns:

$$2\pi(\psi_{2i}^{(0)})_n - 4\sum_{n=1}^{N_0}(\psi_{2i}^{(0)})_n z_m \int_{\Delta r_n} r_n \frac{E(\pi/2, k')}{(a'-b')\sqrt{a'+b'}}\,dr_n - 4\sum_{n=N_0+1}^{N_0+N_c}(\frac{\partial\psi_{2i}^{(0)}}{\partial r})_n \int_{\Delta z_n} \frac{aF(\pi/2,k')}{\sqrt{a'+b'}}\,dz_n$$

$$-4\sum_{n=N_0+N_c+1}^{N}(\psi_{2i}^{(0)})_n(h - z_m)\int_{\Delta r_n} r_n\frac{E(\pi/2,k')}{(a'-b')\sqrt{a'+b'}}\,dr_n - 2\int_0^h\{E_0\frac{H_0(\kappa_0 a)}{H'_0(\kappa_0 a)}Z_0(z_n)$$

$$+\sum_{l=1}^{L}E_l\frac{K_0(\kappa_l a)}{K'_0(\kappa_l a)}Z_l(z_n)\}\{2a^2\frac{E(\pi/2,k')}{(a'-b')\sqrt{a'+b'}} + \frac{F(\pi/2,k')}{\sqrt{a'+b'}} - \frac{a'E(\pi/2,k')}{(a'-b')\sqrt{a'+b'}}\}\,dz_n$$

$$= 2a\int_0^h\int_0^\infty A(k)C_0(ka)\cosh(kz_n)\,dk\{2a^2\frac{E(\pi/2,k')}{(a'-b')\sqrt{a'+b'}} + \frac{F(\pi/2,k')}{\sqrt{a'+b'}} - \frac{a'E(\pi/2,k')}{(a'-b')\sqrt{a'+b}}\}\,dz_n$$

$$-i\frac{a}{2}\int_0^a\{\sum_{j=1}^{\infty}A_j^{(0)}(\frac{j\pi}{h})^2(-1)^{j+1}\frac{I_0(\frac{j\pi r_n}{h})}{I'_0(\frac{j\pi a}{h})} + \frac{1}{ah}\}\frac{2F(\pi/2,k')}{\sqrt{a'+b'}}r_n\,dr_n \quad ; \quad (m = 1, 2, \cdots, N) \tag{26}$$

where $F(\pi/2, k')$ and $E(\pi/2, k')$ are the complete elliptic integrals of the first and second kinds, respectively, and $a' = r_n^2 + r_m^2 + (z_n - z_m)^2$, $b' = 2r_n r_m$, $k' = \sqrt{2b'/(a'+b')}$, Gradshteyn and Rhzhik (1973). The limiting value $2\pi\psi_{2i}^{(0)}$ in (26) is expressed in terms of the exterior solution when P is on the common surface, S_c. $A_j^{(0)}$'s are the Fourier coefficients of the first-order potential solution. The number of unknown coefficients, E_l, of the exterior homogeneous solution are not necessarily equal to the number of elements, N_c, on the common surface. The rest $L+1$ equations are obtained by using (25) in (16) as:

$$-\sum_{n=N_0}^{N_0+N_c}\left(\frac{\partial\psi_{2i}^{(0)}}{\partial r}\right)_n \int_{\Delta z_n} Z_l(z_n)\,dz_n + E_l(\kappa_l d)\delta_{nl} = 0 \quad ; \quad (l = 0, 1, \cdots, L) \tag{27}$$

which completes the solution.

2.3 *Hydrodynamic Force due to the Second-order Potential*

The total dynamic pressure through the second-order at the bottom of the cylinder is given as;

$$p = -\,\epsilon\rho\left\{\frac{\partial\phi_1}{\partial t} + ga\sin\omega t\right\} - \epsilon^2\rho\left\{\frac{\partial\phi_2}{\partial t} + a\frac{\partial^2\phi_1}{\partial z\partial t}\sin\omega t\right.$$
$$\left. + \frac{1}{2}\left[\left(\frac{\partial\phi_1}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi_1}{\partial\theta}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2\right]\right\} + O(\epsilon^3) \quad ; \quad z = h, \quad r \le a \tag{28}$$

Second-order harmonic force is evaluated from the integration of the second-order pressure over the cylinder which is proportional to ϵ^2. With a nondimensional notation second-order vertical force due to the second-order potential is obtained as:

$$\frac{\epsilon^2 F_{22}}{\rho g a A^2} = Re\left\{\left(\frac{\omega^2}{g}d\right)4\pi i\left\{\frac{1}{ad}\sum_{n=N_0+N_c+1}^{N}(\psi_{2i}^{(0)})_n\int_{\nabla r_n} r_n\,dr_n\right\}\exp -i2\omega t\right\} \tag{29}$$

3. Numerical Study and Results

A truncated vertical circular cylinder is chosen to compare the results of this formulation with other available studies. First-order quantities calculated here are based on an analytical linear theory. Main difficulty lies in the evaluation of $R-$ and $k-$ integrals in (19) and (22). Numerical considerations and accuracy in this numerical integration process are examined in Hunt and Baddour (1981) and Gören (1996). The control surface bounding the interior region is discretized here so that equal number of elements are taken on S_o, S_c and S_b. Complete elliptic functions are computed by the polinomial approximations given in Abramowitz and Stegun (1972). Line integrals in (26) are evaluated numerically by means of a 10 point Gaussian-quadratures. An example of the numerical convergence, as a function of total number of elements, N, observed during numerical tests is given in Table 1.

Table 1. Numerical convergence for $\omega^2 d/g = 2.6, h/d = 0.5, a/d = 1.0$

N	$\epsilon^2 F_{22}/(\rho g a A^2)$
6	18.952
12	18.883
18	19.401
24	19.435

Truncation limit is set as 8 in eigenfunction expansion (15). When 24 elements are used for the interior region, the resultant complex matrix is a 32 by 32 matrix which yields a (64×64) real matrix. The system of linear equations is solved by a Gaussian elimination procedure.

Comparative results for the second-order vertical force due to the second-order potential are shown in Figure 2. Li's (1995) work is a numerical study based on a mixed Eulerian-Lagrangian method. Although a good agreement is seen in the low and moderate frequency region, discrepancies get larger in the higher frequency region. Figure 3 illustrates the phase shifts of the total second-order force which includes second-order contribution due to the first-order potential. In the interval $0 \leq k_0 a \leq 1.25$, the agreement is quite satisfactory, while deviations between solutions are observed, to some extend, at higher frequencies.

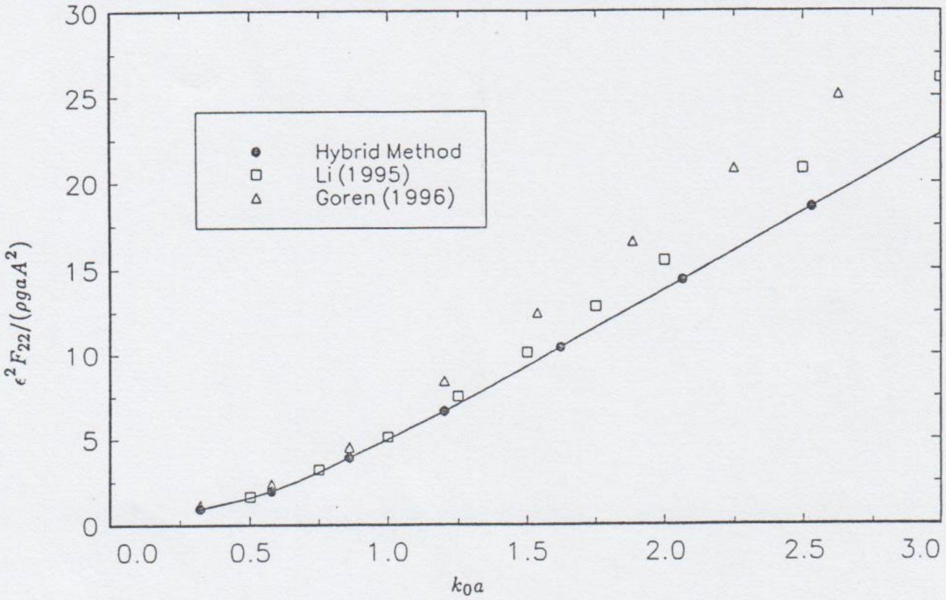

Figure 2. Comparison of second-order vertical force due to the second-order potential ($a/d = 1.000$, $h/d = 0.500$)

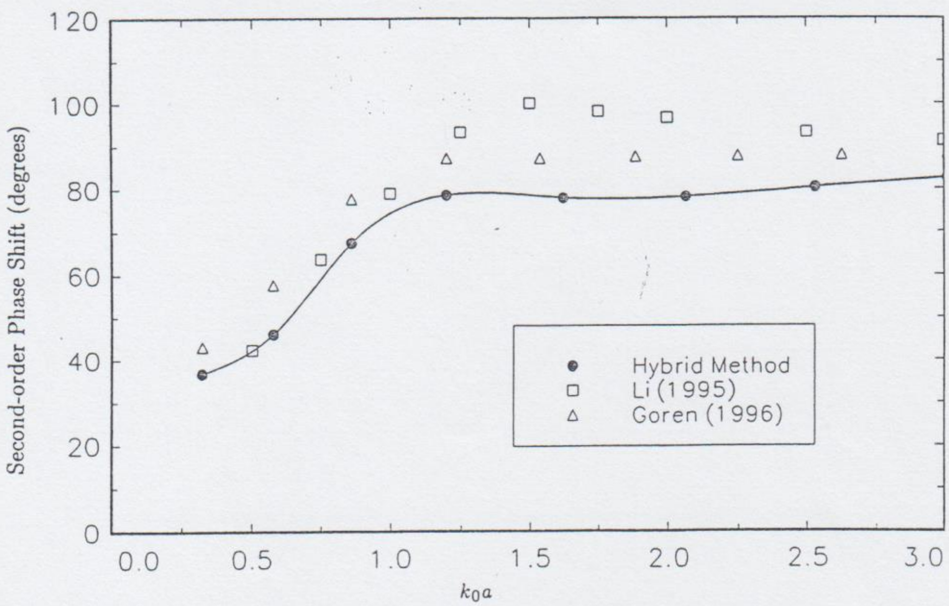

Figure 3. Second-order phase shifts of the heaving cylinder. ($a/d = 1.000$, $h/d = 0.500$)

4. Concluding Remarks

A hybrid method is presented which allows the calculation of second-order hydrodynamic forces for axisymmetric cylinders. This approach eliminates the numerical treatment of nonhomogeneous free surface condition and satisfies the conditions at infinity through the second-order. Some additional work is required on the numerical procedure to clarify the effects of the numerical parameters which are influenced by the frequency, water depth and the radius of the cylinder.

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