VORTICITY AND VORTICITY EQUATION

Definition and Properties of Vorticity

Let $\vec{u}(\vec{x},t) = \vec{u}(u_1, u_2, u_3)$ denotes a velocity field of selenoidal character.

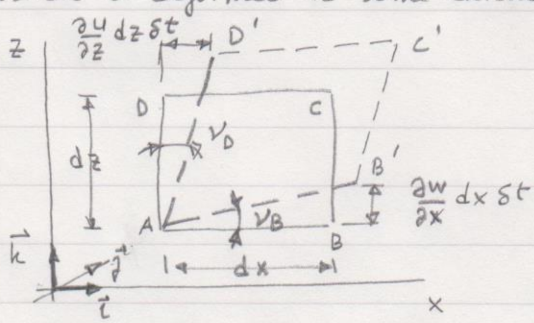
Vortcity vector can then be expressed as ; $\qquad (u_1, u_2, u_3) \equiv (u, v, w)$

$$\vec{\omega}(\vec{x},t) = \nabla \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

$$= \varepsilon_{ijk} \frac{\partial u_j}{\partial x_i} \vec{e}_k$$

What can we say about the physical meaning of $\vec{\omega}$?

Take a fluid particle to examine the relative angular displacements when the particle is deformed to some extend in 2-D :



One can write down :

$$\nu_B\, dx = \frac{\partial w}{\partial x} dx\, \delta t$$

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and $\nu_D\, dz = \dfrac{\partial u}{\partial z} dz\, \delta t \quad ; \quad \nu_D = \dfrac{\partial u}{\partial z} \delta t$

From here the relative angular displacement of the particle:

$$\nu_D - \nu_B = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\delta t \text{ ; and the angular velocity :}$$

$$\frac{\nu_D - \nu_B}{\delta t} = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \text{(or spin velocity).}$$

This is equivalent of the vorticity component $\omega_y = \left(\dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}\right)$; the only component when the flow is 2-D .

Helmholtz Decomposition of the velocity field

If velocity field $\vec{u}(\vec{x},t)$ satisfies ;

i) Continuity equation $\nabla \cdot \vec{u} = \operatorname{div} \vec{u} = 0$

ii) Fluid region is simply connected

iii) $u_n = \vec{n} \cdot \vec{u}$ is given (defined) on all the bounding surfaces

iv) The velocity vanishes at infinity

v) The normal component of $\vec{\omega}$ $(\vec{n} \cdot \vec{\omega})$ vanishes on S

vi) The vorticity field is compact (outside of the fluid region is zero or exponentially zero)

Then the velocity is uniquely given by the sum of selenoidal vector field (div-free component) and irrotational scalar field (curl-free component) as:

$$\vec{u}(\vec{x},t) = \vec{u}_v(\vec{x},t) + \nabla\Phi$$

Note that $\vec{u}_v$ is a selenoidal field which gives ;

$$\text{curl } \vec{u}_v = \nabla \times \vec{u}_v = \vec{\omega} \quad \Longrightarrow \quad \text{div } \vec{\omega} = \nabla \cdot \vec{\omega} = 0 \quad (\text{selenoidal})$$

$$\text{and} \quad \nabla \times (\nabla\Phi) = \text{curl}(\nabla\Phi) = 0 \quad (\text{irrotational}).$$

Some useful theorems

Divergence (Gauss) theorem : Let there be a finite volume $\mathcal{V}$ surrounded by closed surface S, in which normal vector $\vec{n}$ is pointing out of the fluid: (Within $\mathcal{V}$; $\vec{A}$ is a continuous vector field)

$$\iiint_{\mathcal{V}} \nabla \cdot \vec{A}\, d\mathcal{V} = \iint_S \vec{A} \cdot \vec{n}\, dS$$

There are other forms of divergence theorem such as;

$$\iiint_{\mathcal{V}} \nabla\Phi\, d\mathcal{V} = \iint_S \vec{n}\,\Phi\, dS$$

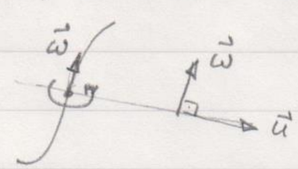
$$\iiint_{\mathcal{V}} \nabla \times \vec{A}\, d\mathcal{V} = \iint_S \vec{n} \times \vec{A}\, dS .$$

Stokes theorem : If S is bounded by a closed contour (line) C :

$$\iint_S (\nabla \times \vec{A})\, \vec{n}\, dS = \oint_C \vec{A} \cdot d\vec{\ell} \quad \text{which is circulation.}$$

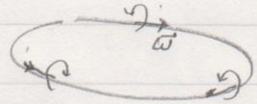
Vortex Structures

a) Vortex line ; is a line everywhere tangent to $\vec{\omega}$.



b) Vortex tube ; is a bundle of vortex lines

c) Vortex ring :



Meantime, consider a vorticity vector defined in a volume closed by surface S. In this case, by divergence theorem:

$$\iint_S \vec{\omega} \cdot \vec{n}\, dS = \iiint_{\mathcal{V}} \nabla \cdot \vec{\omega}\, d\mathcal{V} = 0 \quad , \quad \text{because } \vec{\omega} \perp \nabla \text{ and } \vec{u} .$$

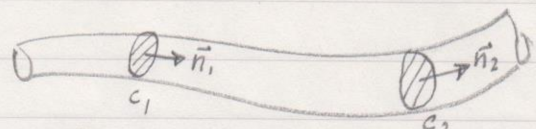
$\longrightarrow$

which means ; net vorticity flux through a closed surface S is zero.

<u>Conservation of vorticity flux</u>

Note that $\Gamma = \oint_C \vec{u}\, d\vec{l} = \iint_S \vec{\omega}\cdot\vec{n}\, dS$ by Stokes theorem.

Take a vortex tube ;



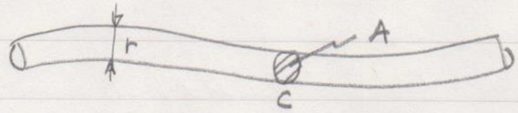
$$\Gamma_1 = \oint_{c_1} \vec{u}\, d\vec{l} = \iint_{S_1} \vec{\omega}\cdot\vec{n}_1\, dS \overset{?}{=} \iint_{S_2} \vec{\omega}\cdot\vec{n}_2\, dS = \oint_{c_2} \vec{u}\, d\vec{l} = \Gamma_2$$

Since $\Gamma_1 = \Gamma_2$; $\therefore$ vorticity flux through surfaces S_1 and S_2 are equal.

Note that a vortex structure is a material volume and remains material.

<u>Vortex streching</u>

Take a very thin vortex tube (with small radius r and length L) :



Again employing Stokes theorem :

$$\Gamma = \oint_C \vec{u}\, d\vec{l} = \iint_A \vec{\omega}\cdot\vec{n}\, dA = \omega A$$

$\because$ Tube is very thin and thus $\vec{\omega}$ is assumed to be constant through A.

We already know that tube is material by volume which means ; $A\cdot L = \pi r^2 L$ constant in time.

$$\therefore \quad \frac{\Gamma}{\text{Volume}} = \frac{\omega A}{LA} = \frac{\omega}{L} = \text{constant !}$$

This concludes that as vortex tube streches ; L increases and since the volume is constant and as a consequence ω increases.

<u>Vorticity Equation</u> (Helmholtz Equation)

Let's begin with the two basic equations ;

continuity equation $\nabla\cdot\vec{q} = 0$ and

N-S equation $\dfrac{\partial \vec{q}}{\partial t} + \vec{q}\cdot\nabla\,\vec{q} = -\dfrac{1}{\rho}\nabla p + \nu\,\nabla^2\vec{q}$

From vector analysis , one can derive (if $\vec{u}$ and $\vec{v}$ are vector fields)

$$\nabla(\vec{u}\cdot\vec{v}) = (\vec{u}\cdot\nabla)\,\vec{v} + (\vec{v}\cdot\nabla)\,\vec{u} + \vec{u}\times(\nabla\times\vec{v}) + \vec{v}\times(\nabla\times\vec{u})$$

If we take $\vec{u} = \vec{v} = \vec{q}$

$$\nabla(q^2) = 2\,(\vec{q}\cdot\nabla)\,\vec{q} + 2\,\vec{q}\times(\nabla\times\vec{q})$$. Since $\vec{\omega} = \nabla\times\vec{q}$;

$$(\vec{q}\cdot\nabla)\,\vec{q} = \vec{\omega}\times\vec{q} + \nabla(q^2/2)$$

$\longrightarrow$

∴ LHS of N-S Equation :

$$\frac{D\vec{q}}{Dt} = \frac{\partial \vec{q}}{\partial t} + \vec{\omega}\times\vec{q} + \nabla(q^2/2)$$

Thus, N-S equation turs out to be :

$$\frac{\partial \vec{q}}{\partial t} + \vec{\omega}\times\vec{q} + \nabla(q^2/2) = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{q}$$

If we take the curl of both sides of the equation, we arrive at;

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla\times(\vec{\omega}\times\vec{q}) = \nu\nabla^2\vec{\omega}$$

By making use of the vector identity;

$$\nabla\times(\vec{u}\times\vec{v}) = (\vec{v}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{v} + \vec{u}(\nabla\cdot\vec{v}) - \vec{v}(\nabla\cdot\vec{u})$$

and setting $\vec{u}=\vec{\omega}$ and $\vec{v}=\vec{q}$ in the above expression, we obtain;

$$\nabla\times(\vec{\omega}\times\vec{q}) = (\vec{q}\cdot\nabla)\vec{\omega} - (\vec{\omega}\cdot\nabla)\vec{q} + \vec{\omega}(\nabla\cdot\vec{q}) - \vec{q}(\nabla\cdot\vec{\omega})$$

As $\nabla\cdot\vec{q}=0$, $\nabla\cdot(\nabla\times\vec{u}) = \nabla\cdot\vec{\omega}=0$

∴ $\nabla\times(\vec{\omega}\times\vec{q}) = (\vec{q}\cdot\nabla)\vec{\omega} - (\vec{\omega}\cdot\nabla)\vec{q}$. Substituting this into N-S eq:

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{q}\cdot\nabla)\vec{\omega} = \underbrace{\boxed{(\vec{\omega}\cdot\nabla)\vec{q}}}_{\text{vortex streching}} + \underbrace{\boxed{\nu\nabla^2\vec{\omega}}}_{\text{viscous diffusion}}$$; Vorticity or Helmholtz Equation

Re-writing the vorticity equation;

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{q} + \nu\nabla^2\vec{\omega} \ .$$

Note that there will be no vortex streching in a 2-D flow.

<u>Inviscid Flows - Euler's Equation</u>

$$\frac{D\vec{q}}{Dt} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\vec{F} \qquad \text{where} \quad \vec{F} = \rho\vec{g}z$$

The vorticity equation counter-part is:

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{q} \ .$$

It can be shown that if $\vec{\omega}=0$ for $t\le 0$, then based on $\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{q}$; $\vec{\omega}=0$ at all times $(t\ge 0)$. In other words, if an inviscid fluid is irrotational at a given time, then it will continue to be irrotational. This is ensured also by Kelvin's theorem of conservation of circulation. (Note that conservation of circulation can be expressed as

$$\frac{D\Gamma}{Dt} = 0 \quad ; \text{ where } \quad \Gamma = \oint_C \vec{q}\,d\vec{l} \ ,$$

Bernoulli's Equation

To arrive at Bernoulli's Equation, one can start with Euler's equation with an assumption that the flow is steady but no condition on vorticity. Thus Euler's equation without time-dependent part (by taking its counter-part from Helmholtz equation and considering the conservative force field);

$$\vec{\omega}\times\vec{q} + \nabla(q^2/2) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla(\rho g z)$$

$$\pm\,\vec{q}\times\vec{\omega} = \pm\,\nabla\left(\frac{p}{\rho} + \frac{q^2}{2} + gz\right)$$

Multiplying both sides by $\vec{q}$;

$$\vec{q}\cdot(\vec{q}\times\vec{\omega}) = \vec{q}\cdot\nabla\left(\frac{p}{\rho} + \frac{q^2}{2} + gz\right) \qquad \text{where } \vec{q}\cdot(\vec{q}\times\vec{\omega}) = 0$$

$\therefore \quad \vec{q}\cdot\nabla\left(\frac{p}{\rho} + \frac{q^2}{2} + gz\right) = 0$ means that the operator $\vec{q}\cdot\nabla$ is the steady term in the substantial derivative which implies rate of change following a fluid particle with velocity $\vec{q}$. In conclusion, along any streamline the quantity in paranthesis is constant:

$$\frac{p}{\rho} + \frac{q^2}{2} + gz = C$$

The second alternative or version of Bernoulli's equation is for unsteady but for irrotational flows. In this case, we can start with the general form of Euler's equation (its vorticity counter-part):

$$\frac{\partial\vec{q}}{\partial t} + \vec{\omega}\times\vec{q} + \nabla(q^2/2) = -\frac{1}{\rho}\nabla p - \frac{1}{\rho}\nabla(\rho g z)$$

Omitting vorticity and taking into account $\vec{q} = \nabla\phi$;

$$\frac{\partial(\nabla\phi)}{\partial t} + \nabla(q^2/2) = -\frac{1}{\rho}\nabla(p - \rho g z) \quad , \quad \text{which gives;}$$

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi\cdot\nabla\phi) + \frac{p}{\rho} + gz\right) = 0$$

By means of the integration

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi\cdot\nabla\phi) + \frac{p}{\rho} + gz = C(t)$$

The constant $C(t)$ in space coordinates may be absorbed into the velocity potential such as;

$$\phi' = \phi - \int_0^t C(t)\,dt$$

This causes any problem in calculating the velocity field and in turn the pressure field.