IDEAL FLUID MOTION

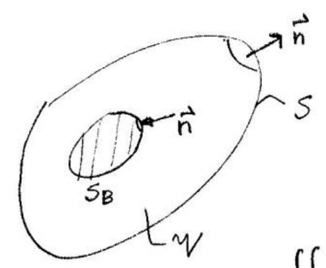
From Euler equation, we may further make a simplification by assuming the flow is irrotational. According to Kelvin's theorem (conservation of circulation) any motion that started from a state of no rotation ($\vec{\omega}=0$) will remain irrotational for all times. Then if one considers no vorticity at the start of the motion, as a consequence of Kelvin's theorem (note that Kelvin's theorem uses Euler's equation), the flow remains to be non-vortical motion. Accordingly, from Helmholtz theorem in vector analysis, when there is only curl-free velocity field, then $\vec{V}=\nabla\Phi$ only, and Φ scalar function Φ is termed as velocity potential. This class of the flows is also called potential flows.

If we consider continuity equation $\nabla\cdot\vec{V}=0$, then

$\nabla\cdot(\nabla\Phi) = \nabla^2\Phi = 0$ which is expressed in Cartesian coordinates as;

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

and is called Laplace's differential equation. This is a linear partial differential equation, which can be solved analytically with boundary conditions on the surrounding surface of the fluid volume.



Take $\vec{n}$ as the unit normal pointing out of the fluid and consider the quantity $\Phi\frac{\partial\Phi}{\partial n}$ on the surface S. Note that $\frac{\partial\Phi}{\partial n}=\vec{n}\cdot\nabla\Phi$.

By means of divergence theorem;

$$\iint_S \Phi\frac{\partial\Phi}{\partial n}\,dS = \iint_S \Phi\,\vec{n}\cdot\nabla\Phi\,dS = \iiint_{\mathcal{V}} \nabla(\Phi\cdot\nabla\Phi)\,d\mathcal{V}$$

$$= \iiint_{\mathcal{V}} [(\nabla\Phi\cdot\nabla\Phi)+\Phi\nabla^2\Phi]\,d\mathcal{V}$$

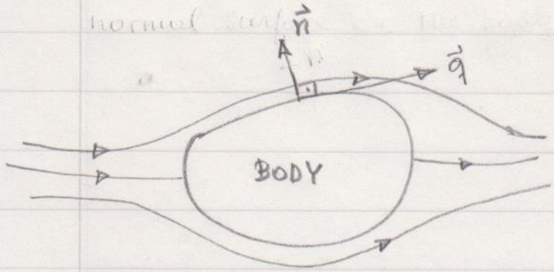
If we let $\frac{\partial\Phi}{\partial n}=0$ (which physically means that the normal velocities on the surrounding surface S is zero), then

$\nabla\Phi\cdot\nabla\Phi=0$, which means no fluid motion inside S!

There is another outcome of this analysis; it implies the uniqueness of the velocity potential Φ. This also shows the importance of boundary conditions.

Boundery Conditions

In inviscid fluids, kinematic boundery conditions on the body and/or dynamic boundery conditions on the material boundery surfaces are used.



a) Kinematic boundery condition on a solid boundery is given as the velocity tangent to the solid boundery:

$$\vec{q}\cdot\vec{n}=0 \qquad \text{where } \vec{q} \text{ denotes tangent velocity.}$$

More appropriately; normal velocity to the body surface is prescribed:

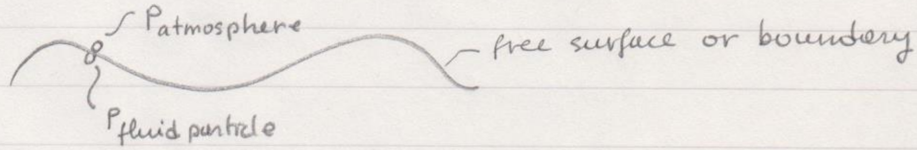
$$\frac{\partial \Phi}{\partial n}=\vec{n}\cdot\nabla\Phi=0 \qquad \text{(when the body is fixed in the flow).}$$

This is also called as the non-permeable condition. If we consider a more general case; unsteady motion of a body with translational velocity $\vec{U}(t)$ and body angular velocity $\vec{\Omega}(t)$, then the velocity potential on the body satisfies;

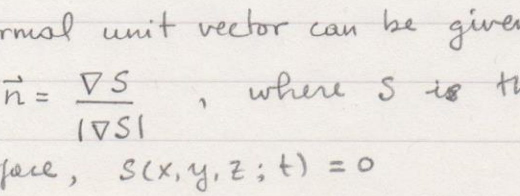
$$\frac{\partial \Phi}{\partial n}=\vec{U}\cdot\vec{n}+\vec{\Omega}\cdot(\vec{r}\times\vec{n})$$

where $\vec{r}$ is the position vector from the center of rotation.

b) Dynamic boundery condition, mostly given on a moving boundery, employs pressure or other dynamic quantity. In example, on the water surface adjacent particles of air and water are accepted to have the same pressures:



In viscous fluids, there is a boundery layer on the body due to no-slip condition. In hybrid methods, in which the flow outside the boundery layer may be considered as inviscid, interfacial boundery conditions are defined:



$\vec{n}_{BL}$: normal to boundery layer displaced surface

In this case;

$$\vec{n}_{BL}\cdot\vec{q}=0.$$

Note that normal unit vector can be given as

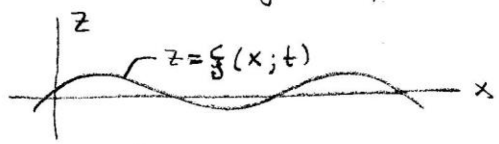
$$\vec{n}=\frac{\nabla S}{|\nabla S|}, \quad \text{where } S \text{ is the equation of the}$$

boundery surface, $S(x,y,z;t)=0$

Another approach in deriving boundary conditions is to make use of the property of material volume/surface/line. For example, if $S(x,y,z;t)=0$ denotes a material surface, then material derivative of S should be zero:

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} + w\frac{\partial S}{\partial z} = 0$$

in Cartesian coordinates with $\vec{u} = \vec{u}\,(u,v,w)$. This approach may be extended to take the material derivatives of kinematic or dynamic quantities zero on the material surfaces. A good example of this approach is to have the material derivative of the pressure on the free surface of water:



$\left(S = z - \xi(x,t) = 0 \quad \text{in 2-D}\right)$,

$$\frac{DP}{Dt} = 0 \quad ; \quad \text{on } z = \xi\,.$$

On the other hand, we may as well arrive at:

$$\frac{\partial \phi}{\partial n} = -\frac{\partial S/\partial t}{|\nabla S|}$$

To show the above expression of boundary condition, take into account of the boundary surface $S(x,y,z;t)=0$ and its normal $\vec{n} = \vec{n}\,(n_x, n_y, n_z)$.
The exact (total) derivative of S gives:

$$dS = \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz = 0$$

One can express; $dx = U_n n_x dt$, $dy = U_n n_y dt$, $dz = U_n n_z dt$

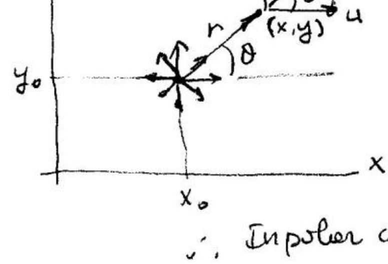
$$\therefore \quad \frac{\partial S}{\partial t} + U_n n_x \frac{\partial S}{\partial x} + U_n n_y \frac{\partial S}{\partial y} + U_n n_z \frac{\partial S}{\partial z} = 0$$

$$U_n\, \vec{n}\cdot\nabla S = -\frac{\partial S}{\partial t} \quad . \quad \text{Since } \vec{n}\cdot\nabla S = |\nabla S|$$

$$U_n = \vec{n}\cdot\vec{U} = -\frac{\partial S/\partial t}{|\nabla S|}$$

Simple Potential Flows and Singularity Distributions

The simplest case is the uniform flow in x-direction ; $\phi = Ux$. This obviously satisfies Laplace's equation. Singularities are used in modeling boundaries/bodies, as they violate the continuity equation and in turn Laplace's equation at the points of location. Let's first consider a 2-D source/sink:



$q(r)$ (for u_r; radial velocity)

Source at point (x_0, y_0) with strength m induces a radial velocity field u_r where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$

Consider strength m as the rate of emission of fluid volume per unit time. In this case; $2\pi r\, u_r = m$ due to conservation of mass. Thus; $u_r = \dfrac{m}{2\pi r}$ ($m<0$; sink)

$\therefore$ In polar coordinates: $\dfrac{\partial \phi}{\partial r} = u_r = \dfrac{m}{2\pi r} \Rightarrow \phi(r) = \displaystyle\int \frac{m}{2\pi r}dr + C = \frac{m}{2\pi}\ln r$

(by taking $C=0$).

The same potential can be derived by introducing complex potential:

$w(z) = \phi + i\psi$ (where $z = x + iy$ and ψ : stream function).

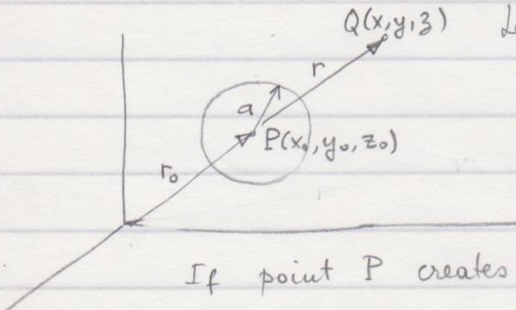
and $\frac{dw}{dz} = u - iv$. Since $u = u_r \cos\theta = \frac{m}{2\pi r}\cos\theta$ and $v = u_r \sin\theta = \frac{m}{2\pi r}\sin\theta$

Therefore $\frac{dw}{dz} = \frac{m}{2\pi r}(\cos\theta - i\sin\theta) = \frac{m e^{-i\theta}}{2\pi r} = \frac{m}{2\pi r e^{i\theta}} = \frac{m}{2\pi z}$

$$\therefore \quad \frac{dw}{dz} = \frac{m}{2\pi z} \implies w(z) = \frac{m}{2\pi}\ln z = \frac{m}{2\pi}[\phi' + i\psi']$$

$$= \frac{m}{2\pi}[\ln r + i\theta]$$

b) 3-D source / sink



Let (x_0, y_0, z_0) is the source point and (x, y, z) is the field point.

Then $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$

$r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$

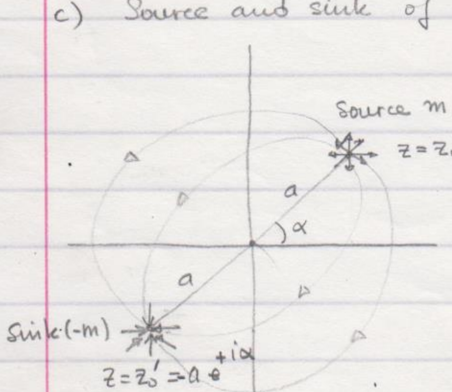
If point P creates a fluid flux of m per unit time

then; $4\pi r^2 v_r = m \implies v_r = \frac{\partial\phi}{\partial r} = \frac{m}{4\pi r^2}$

$\hookrightarrow \phi = -\frac{m}{4\pi r}$

It can be shown that $\nabla^2\phi = 0$, for $r \neq 0$.

c) Source and sink of equal strengths $\rightarrow$ doublet (or dipole) (as a limiting case).



Let's write the complex potential of a pair of source & sink:

$$w(z) = \frac{m}{2\pi}\ln(z - z_0) - \frac{m}{2\pi}\ln(z - z_0')$$

$$= \frac{m}{2\pi}\ln(z - ae^{i\alpha}) - \frac{m}{2\pi}\ln(z + ae^{+i\alpha})$$

Now, if $a \to 0$ while $m \to \infty$ to keep $ma = \mu$ finite

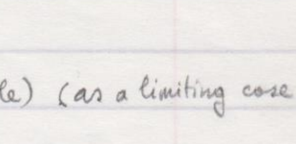
then $w(z) = \frac{m}{2\pi}\ln z\left(1 - \frac{ae^{i\alpha}}{z}\right) - \frac{m}{2\pi}\ln z\left(1 + \frac{ae^{+i\alpha}}{z}\right)$

$$= \frac{m}{2\pi}\ln\left(1 - \frac{ae^{i\alpha}}{z}\right) - \frac{m}{2\pi}\ln\left(1 + \frac{ae^{+i\alpha}}{z}\right)$$

By Taylor series expansion (of $\ln(1+x) = x - x^2/2 + x^3/3 - \dots$)

$$w(z) = -\frac{2mae^{i\alpha}}{2\pi z} - \frac{2ma^3e^{i3\alpha}}{2\pi z^3 \cdot 3} + \dots$$

Take $2ma = \mu$; $\lim_{\substack{a \to 0 \\ m \to \infty}} w(z) = -\frac{\mu e^{i\alpha}}{2\pi z} - \frac{\mu a^2 e^{i3\alpha}}{2\pi z^3 \cdot 3} = -\frac{\mu}{2\pi z}e^{i\alpha}$

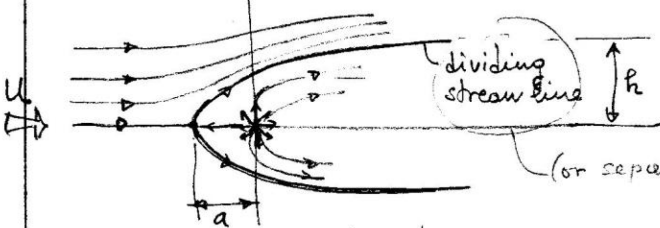


where μ is the dipole strength or moment.

In Cartesian Coord 2-D dipole: $\phi = -\frac{\mu}{2\pi}\frac{x}{x^2 + y^2}$, 3-D Dipole $\phi = -\frac{\mu}{4\pi}\frac{x}{(x^2+y^2+z^2)^{3/2}}$

Combination of source and uniform flow

Source in a uniform flow (2-D). $\Rightarrow$ Complex pot.



$$W(z) = Uz + \frac{m}{2\pi}\ln z$$
$$\hookrightarrow \phi = Ux + \frac{m}{2\pi}\ln r$$

At stagnation point $\frac{dw}{dz} = 0 \Rightarrow u - iv = U + \frac{m}{2\pi z} = 0 \Rightarrow z = -\frac{m}{2\pi U} = a$

Asympthotic height, h: From complex potential; the stream function

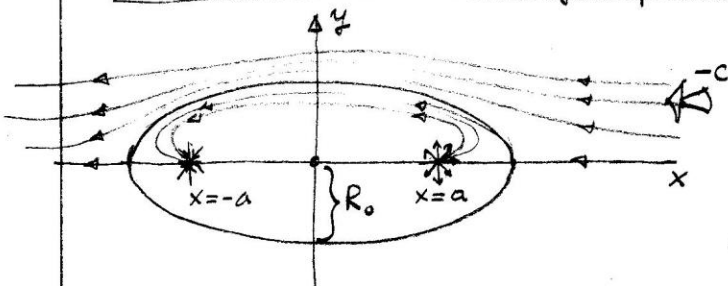
$$\psi = Uy + \frac{m}{2\pi}\theta$$

Recall that along a stream line, ψ should be constant. By using stagnation point, the dividing streamline

$$\psi_o = U\cdot(0) + \frac{m}{2\pi}(\pi) = \frac{m}{2}$$

$\therefore$ for h: $\theta = 0$. $\Rightarrow \frac{m}{2} = Uy + \frac{m}{2\pi}(0) \Rightarrow y = h = \frac{m}{2U}$

Source and Sink in a uniform flow (3-D): "Rankine's Ovoid"



Total potential

$$\Phi = \underbrace{-\frac{m}{4\pi r_1}}_{\text{source}} + \underbrace{\frac{m}{4\pi r_2}}_{\text{sink}} \underbrace{- cx}_{\text{uni. flow}}$$

where $r_1^2 = (x-a)^2 + y^2 + z^2$
$r_2^2 = (x+a)^2 + y^2 + z^2$.

$$\therefore \quad u = \frac{\partial\Phi}{\partial x} = \frac{m(x-a)}{4\pi r_1^3} - \frac{m(x+a)}{4\pi r_2^3} - c$$

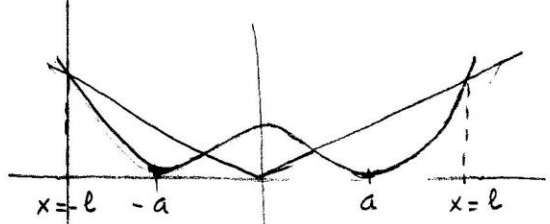
$$v = \frac{\partial\Phi}{\partial y} = \frac{my}{4\pi r_1^3} - \frac{my}{4\pi r_2^3} \quad ; \quad w = \frac{\partial\Phi}{\partial z} = \frac{mz}{4\pi r_1^3} - \frac{mz}{4\pi r_2^3}$$

To find the coordinates of the stagnation point, we must take $u = v = w = 0$.

$\therefore$ (Since $y = z = 0$) using $u = 0 \Rightarrow \frac{1}{4\pi(x-a)^2} - \frac{1}{4\pi(x+a)^2} = \frac{c}{m}$

for $x > 0 \qquad (x^2 - a^2)^2 = \frac{max}{\pi c}$

$x < 0 \qquad (x^2 - a^2)^2 = -\frac{max}{\pi c}$



For home: Show that in Rankine's Ovoid
(4.2)
$$R_o^2\sqrt{R_o^2 + a^2} = \frac{ma}{\pi c}$$