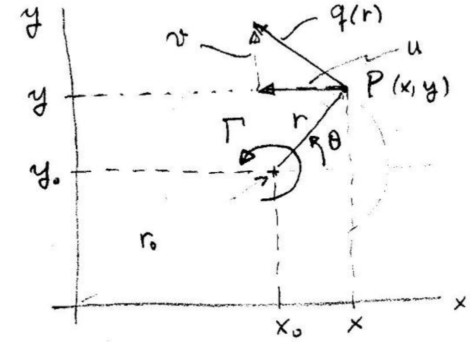
## Flow due to a vortex (2-D)




Vortex at point $(x_0, y_0)$ with strength $\Gamma$ induces a circumferential velocity field $q(r)$.

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

Due to irrotational flow (everywhere except where the vortex is located), we have

$$2\pi r\, q(r) = \Gamma \quad \text{(circulation)}$$

$$\therefore \quad q(r) = \frac{\Gamma}{2\pi r}$$

Meanwhile $$u(x,y) = -q(r)\sin\theta = -\frac{\Gamma}{2\pi r}\cdot\frac{y-y_0}{r} = -\frac{\Gamma(y-y_0)}{2\pi\left[(x-x_0)^2+(y-y_0)^2\right]}$$

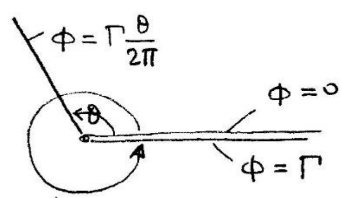
$$v(x,y) = q(r)\cos\theta = \frac{\Gamma}{2\pi r}\,\frac{x-x_0}{r} = \frac{\Gamma(x-x_0)}{2\pi\left[(x-x_0)^2+(y-y_0)^2\right]}$$

Then the velocity potential of a point vortex will be such that:

$$q(r) = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r} \Longrightarrow \frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi} \Longrightarrow \phi = \Gamma\frac{\theta}{2\pi}$$

It can be shown that $\nabla^2\phi = 0$.

Graphically:




Complex velocity potential which represents the flow due to a vortex located at $z = z_0$ can then be defined as:

$$w = -\frac{i\Gamma}{2\pi}\ln(z-z_0)$$

Prove the above expression (for home assignment). (4.3)

## Multipoles

The simplest example for a multipole is a dipole. Multipoles may be considered as the successive derivatives of singularities. For example a dipole is obtained by the 1st derivative of a source/sink:

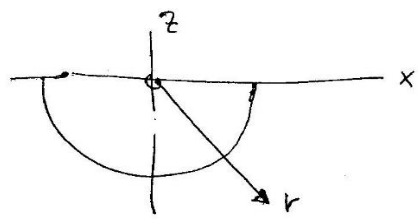
$$w(z) = \frac{m}{2\pi} \ln z$$ is a complex source potential, and

$$w(z) = -\frac{\mu}{2\pi} \frac{1}{z}$$ is a dipole potential which may be

considered as the 1st derivative of the source potential.

A famous multipole expansion is due to Ursell (1949) to investigate the heave motion of a semi-circular cylinder on the free surface:

$$\sum_{m=1}^{\infty} a_{2m} \left[ \frac{\sin 2m\theta}{r^{2m}} + \frac{K}{2m-1} \frac{\sin (2m-1)\theta}{r^{2m-1}} \right] \begin{matrix} \cos \\ \sin \end{matrix} (\omega t)$$




As it is understood from the above expansion; multipoles are located at the center of the cylinder. This is because of the fact that multipoles possess strong singularities not easily dealt with when they are distributed on the body.

## Discrete Singularity Distributions

Indeed Rankine's ovoid is a good example of a discrete singularity. In a more general way, we may formulate discrete formulations (in a uniform flow):

$$\Phi(x,y,z) = Ux - \sum_{i=1}^{N} \frac{m_i}{4\pi r_i}$$

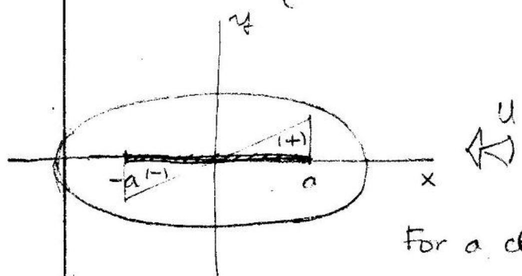
If a closed surface is required, a necessary condition in this case;

$$\sum_{i=1}^{N} m_i = 0$$

## Continuous Singularity Distributions

### i) Line distribution

Now instead of a pair of source/sink in a uniform flow; consider a line distribution of sources between $-a < x < a$.




In this case: $\Phi = -Ux - \int_{-a}^{a} \frac{m(\xi)}{\sqrt{(x-\xi)^2 + y^2 + z^2}}\, d\xi$

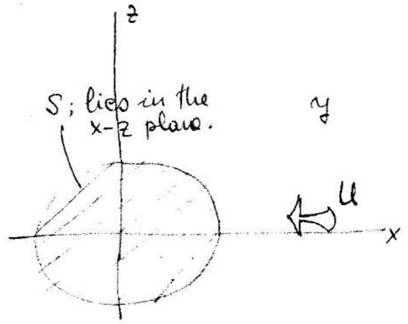
where $\xi$ denotes source pts.

For a closed surface $\int_{-a}^{a} m(\xi)\, d\xi = 0$.

If $m(\xi) = k\xi$ ; then this generates what we called a prolate ellipsoid.

### ii) Surface distribution

Consider a limited (bounded) surface distribution of sources/sinks of surface strength $m(\xi,\zeta)$, (where $(\xi,\zeta)$ is the source point).




However the problem is a 3-D problem, the emission of fluid volume directed to $\pm y$ directions.
Thus the velocity potential;

$$\Phi(x,y,z) = -Ux - \iint_S \frac{m(\xi,\zeta)}{[(x-\xi)^2 + y^2 + (z-\zeta)^2]^{1/2}}\, d\xi\, d\zeta$$

A necessary condition for a closed boundary ; $\iint_S m(\xi,\zeta)\, d\xi\, d\zeta = 0$

For example, thin-ship approximation is obtained by considering a surface source/sink distribution on the centerplane of the ship. Slender body theory can also be studied by source/sink distribution on the center plane considering the slope of the waterlines $\partial f/\partial x \cong 0$.

Here are some of the flow examples obtained by singularities:

2-D Stream of velocity $U$ flow past a circle of radius $(A/U)^{1/2}$ : $F(z) = Uz + \frac{A}{z}$

3-D Flow past a fixed sphere : $\Phi = U\,(r + \frac{1}{2}\, r_0^3/r^2) \cos\theta$

2-D Translation of ~~Body~~ circle with fluid at rest : $\Phi = -\,(U r_0^2/r) \cos\theta$

3-D " " Sphere " " " " : $\Phi = -\,(\frac{1}{2} U r_0^3/r^2) \cos\theta$

2-D Uniform flow past a circle : $\Phi = U\,(r + r_0^2/r) \cos\theta$

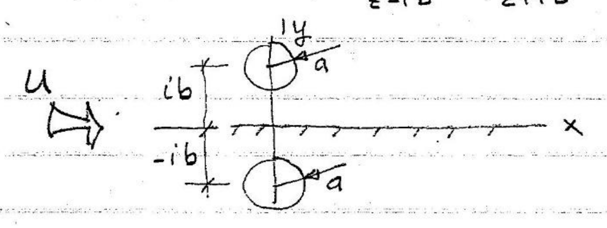
## Method of Images

In some of the problems of marine hydrodynamics, either rigid (wall) boundaries or moving (free) boundaries may be modeled by taking the images of distributed singularities with respect to the plane of symmetry. Let's investigate this approach by the following flow problems:

Ex/1 2-D flow past a circular cylinder near a plane wall.

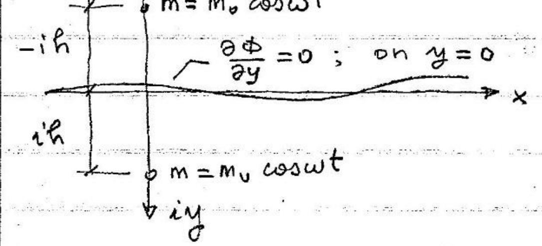
This problem is an equivalent one to the flow past a pair of circular cylinders, which is formulated by:

$$W(z) = Uz + \frac{Ua^2}{z-ib} + \frac{Ua^2}{z+ib}$$




Note that due to the interaction of the doublets, the geometry of the cylinders are distorted a little bit.

Ex/2 Slowly pulsating source under a free surface.




Then the total velocity potential in this case can be given as:

$$W(z) = m \ln(z-ih) + m \ln(z+ih)$$
$$= m_0 \cos\omega t \,[\ln(z-ih) + \ln(z+ih)]$$
$$= \Phi + i\,\Psi$$