Source-Panel Method (Hess & Smith's method)

This is indeed a numerical/computational method developed by J.L. Hess and A.M.O. Smith at Douglas Aircraft company. (See; Hess, J.L. and Smith AMO, Calculation of potential flow about arbitrary bodies, Progress in Aeronautical Sciences, Vol. 8, 1966; for more detail). The basic idea is; to discretize the body by finite number of panels and make a "constant strength" source-sink distribution over each panel and determine the source-sink strengths over each panel by satisfying kinematic boundary condition at the node points of the panels.

For simplicity we handle the 2-D problem by assuming

$$\phi = Ux + \int_{S_0} \frac{m}{2\pi} \ln|\vec{r}-\vec{r}_0| \, dS_0 = Ux + \varphi$$

where S_0 is the body surface and $\vec{r}$ denotes the displacement vector of field point and $\vec{r}_0$ relates with source point. From now on we will take $\sigma = \frac{m}{2\pi}$. Our physical model is developed in a way that we distribute singularities on the rigid body surface. On this surface we impose;

$$\frac{\partial \phi}{\partial n} = 0 \implies \frac{\partial \varphi}{\partial n} = -U n_x \qquad (\text{where } \vec{n} = n_x \vec{i} + n_y \vec{j})$$

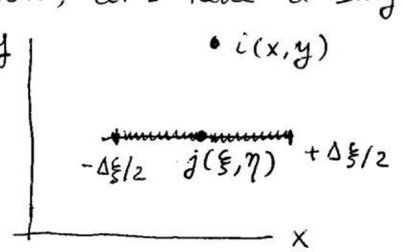
Re-writing the total potential again:

$$\phi = Ux + \int_{S_0} \sigma(\xi,\eta) \ln\left[(x-\xi)^2 + (y-\eta)^2\right]^{1/2} d\xi \, d\eta$$

· We can discretize S_0 by taking ΔS_i panels (segments) on it:

$$\varphi = \int_{S_0} \sigma \ln|\vec{r}-\vec{r}_0| \, dS_0 \cong \sum_{i=1}^{N} \int_{\Delta S_i} \sigma_i \ln\left[(x-\xi)^2 + (y-\eta)^2\right]^{1/2} d\xi \, d\eta$$

Now, let's take a single (isolated) panel (segment);



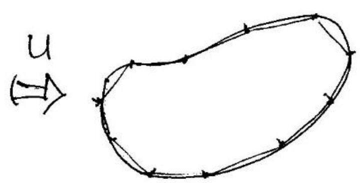
evaluate

$$\frac{\partial \varphi}{\partial y} = \int_{\xi-\Delta\xi/2}^{\xi+\Delta\xi/2} \sigma(\xi,\eta) \frac{y-\eta}{(x-\xi)^2 + (y-\eta)^2} \, d\xi$$

What happen when $x \to \xi$ and $y \to \eta$?

(This is a homework for you).

Now return back to the problem of the boundary value problem of "an arbitrary body in a uniform flow".



The body is represented and approximated by N segments (panels) on which a source/sink distribution is made. We assume $\sigma(\xi,\eta)$ is constant over each panel.

In this case the problem is reduced to the determination of panel source strengths by the use of kinematic condition on the body:

$$\frac{\partial \phi}{\partial n} = \vec{n}\cdot\nabla\phi = 0 \Rightarrow \sum_{j=1}^{N} \sigma_j \int_{\Delta S_j} \frac{(x-\xi)\vec{i} + (y-\eta)\vec{j}}{[(x-\xi)^2 + (y-\eta)^2]}\, \vec{n}_i \, dS_j = -U n_{i_x}$$

for $i = 1, 2, \ldots, N$

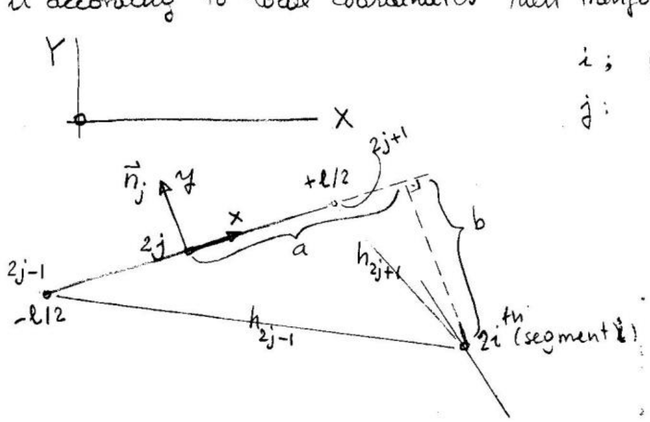
which gives us $N \times N$ linear system of equations.

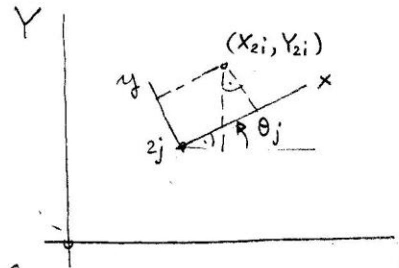
(continued on next page).

Now we have to make the integration along the segment j clear. First we can take it according to local coordinates then transform it to global coordinates.

i; field point ; l : panel length

j: source point





In local coordinates:

$$a = x_{2i} = (X_{2i} - X_{2j})\cos\theta_j + (Y_{2i} - Y_{2j})\sin\theta_j$$

$$b = y_{2i} = (Y_{2i} - Y_{2j})\cos\theta_j - (X_{2i} - X_{2j})\sin\theta_j$$

Transformation from global coord. to local coord.

$$\begin{cases} x = (X - X_{2j})\cos\theta_j + (Y - Y_{2j})\sin\theta_j \\ y = (Y - Y_{2j})\cos\theta_j - (X - X_{2j})\sin\theta_j \end{cases}$$

$$\therefore \quad h_{2j+1} = \sqrt{(a - l/2)^2 + b^2}$$

$$h_{2j-1} = \sqrt{(a + l/2)^2 + b^2}$$

. Now we can evaluate the local velocities at the node point $2i$ of element i due to the jth panel having unit source strength distribution.

$$b_{ij} = \int_{-l/2}^{l/2} \frac{a - \xi}{(a-\xi)^2 + b^2}\, d\xi = \ln \frac{h_{2j-1}}{h_{2j+1}} \triangleq \ln \frac{h_{2j-1,2i}}{h_{2j+1,2i}}$$

$$c_{ij} = \int_{-l/2}^{l/2} \frac{b}{(a-\xi)^2 + b^2}\, d\xi = \arctan\left(\frac{a + l/2}{b}\right) - \arctan\left(\frac{a - l/2}{b}\right).$$

· Now, to convert quantities b_{ij} & c_{ij} into global coordinates; use transformations:

$$X = x\cos\theta_j - y\sin\theta_j + X_{2j}$$

$$Y = y\cos\theta_j + x\sin\theta_j + Y_{2j}$$

Thus, for the global velocity components;

$$X - X_{2j} = B_{ij} = b_{ij}\cos\theta_j - c_{ij}\sin\theta_j$$

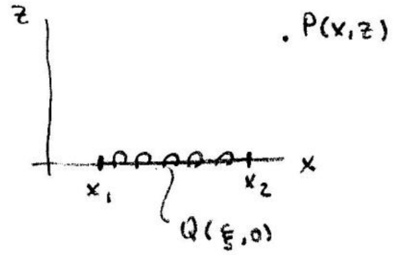
$$Y - Y_{2j} = C_{ij} = c_{ij}\cos\theta_j + b_{ij}\sin\theta_j$$

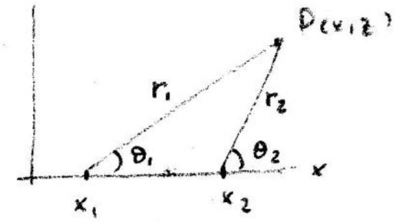
Therefore, the kinematic condition on the body turns out to be;

$$\pi\sigma_i + \sum_{\substack{j=1 \\ (i\neq j)}}^{N} \sigma_j \left(B_{ij} n_{i_x} + C_{ij} n_{i_y}\right) = -U n_{i_x} \quad ; \quad (i = 1, 2, \ldots, N).$$

* 5.2 Homework: Investigate the potential flow about a circular cylinder in a uniform flow. Let the circular cylinder devided into segments (panels). Compare the numerical results for velocity field with those of analytical results.

Lifting Surfaces Modeled by Vortex Distributions





A discrete vortex with strength Γ having a rotation in clockwise direction can be represented by the velocity potential:

$$\phi = -\frac{\Gamma}{2\pi}\theta .$$

Distribution of vortices between x_1 and x_2 gives:

$$\phi = -\frac{\Gamma}{2\pi}\int_{x_1}^{x_2}\arctan\left(\frac{z}{x-\xi}\right)d\xi$$. Integration gives:

$$\phi = -\frac{\Gamma}{2\pi}\left\{(x-x_1)\arctan\left(\frac{z}{x-x_1}\right) - (x-x_2)\arctan\left(z/(x-x_2)\right) + \frac{z}{2}\ln\left[\frac{(x-x_1)^2+z^2}{(x-x_2)^2+z^2}\right]\right\}.$$

Thus the velocity components in Cartesian coordinates:

$$u = \frac{\partial\phi}{\partial x} = -\frac{\Gamma}{2\pi}\left\{\arctan\left(\frac{z}{x-x_1}\right) - \arctan\left(\frac{z}{x-x_2}\right)\right\}$$

$$w = \frac{\partial\phi}{\partial z} = -\frac{\Gamma}{2\pi}\left\{\frac{1}{2}\ln\left[\frac{(x-x_1)^2+z^2}{(x-x_2)^2+z^2}\right]\right\}$$

Note that from geometry depicted in the above figures:

$$\theta_j = \arctan\left(\frac{z}{x-x_j}\right)$$

$$r_j = \sqrt{(x-x_j)^2 + z^2}$$

$$\therefore \quad u = -\frac{\Gamma}{2\pi}(\theta_1 - \theta_2)$$

$$w = -\frac{\Gamma}{2\pi}\left\{\frac{1}{2}\ln\left(\frac{r_1^2}{r_2^2}\right)\right\}$$

What happen as the field point approaches to the source pt at $z = \pm 0$; $x_1 \le x \le x_2$? In this limiting case when P approaches from above

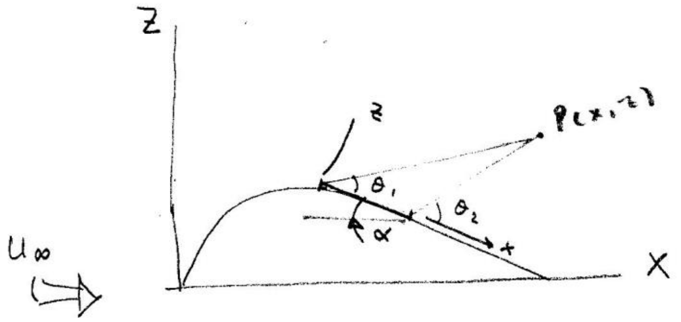
$$\phi(x,0+) = \lim_{\substack{z\to +0 \\ (\theta_1=0;\ \theta_2=\pi)}} \phi(x,z) = -\frac{\Gamma}{2\pi}\left[(x-x_1)\cdot 0 - (x-x_2)\pi\right] = \frac{\Gamma}{2}(x-x_2)$$

and

$$\phi(x,0-) = \lim_{\substack{z\to -0 \\ (\theta_1=0,\ \theta_2=-\pi)}} \phi(x,z) = -\frac{\Gamma}{2}(x-x_2).$$

Meantime $u(x,\pm 0) = \pm\frac{\Gamma}{2}$

N adet paneli dis

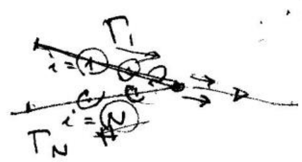


$$\Phi = \phi + U x$$

$$\left(\frac{\partial \Phi}{\partial n}\right)_i = 0 \qquad (i = 1, \dots, N) \qquad (*)$$

$$\left(\frac{\partial \Phi}{\partial l}\right)_i = 0 \qquad (i = 1, 2, \dots N) \qquad (**)$$

Trailing edge stagnation point requires special treatment



∴ Kutta condition at the trailing edge : $\Gamma_1 + \Gamma_N = 0$

∴ (*) gives with Kutta condition as the last equation :

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ a_{N-1,1} & a_{N-1,2} & \dots & a_{N-1,N} \\ 1 & 0 & \dots & 1 \end{bmatrix} \begin{Bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_{N-1} \\ \Gamma_N \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{N-1} \\ 0 \end{Bmatrix}$$

where $a_{ij} = u_{ij} \cos\alpha_i + w_{ij} \sin\alpha_i$ and $b_i = -U_\infty \cos\alpha_i$.

Note that, for example ; u_{ij} ;

$$u_{ij} = -\frac{\Gamma_j}{2\pi}\left(\theta_{i,j1} - \theta_{i,j2}\right) \quad \text{where} \quad \theta_{i,j} = \arctan\left(\frac{z_i}{x_i - x_j}\right)$$

* Transformation procedure from global to local and from local to global will rest upon the student.

Meantime , circulation around a vortex line element

$$d\Gamma = \Gamma\, ds = (V_e - V_i)\, dS \qquad \text{Since } V_i = 0 \text{, then } \Gamma = V_e \;!$$

Pressure coefficient, which can be obtained from Bernoulli's equation ;

$$C_p = 1 - \left(\frac{V}{U_\infty}\right)^2$$