Take $\nabla\Phi = \vec{q}$ and $\vec{q} = -\frac{m_i}{r^2}\vec{n} + \vec{q}_i$; $\vec{q}_i$ velocity at ith point (source) due to other sources, except itself.

$$\therefore \sum_{i=1}^{n} \vec{F}_i = \sum_i \rho \iint_{S_i} \left[\left(-\frac{m_i}{r^2}\vec{n} + \vec{q}_i\right)^2 \vec{n} - \frac{1}{2}\left(-\frac{m_i}{r^2}\vec{n} + q_i\right)^2 \vec{n} \right] dS$$

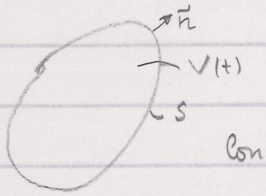
note that $\iint_{S_i} \vec{n}\, dS = 0$; therefore the only term which exists

$$\sum_{i=1}^{n} \vec{F}_i = \sum_i \rho \frac{m_i}{r^2}\vec{q}_i \iint_{S_i} dS = \sum_i 4\pi r^2 \rho \frac{m_i \vec{q}_i}{r^2}$$

$$= \sum_i 4\pi\rho m_i \vec{q}_i$$ which proves Lagally's theorem

Here $\vec{q}_i$ is the induced velocity due to the singularities at point singularity i, except the effect of ith singularity.
$\vec{F}_i$ is the force on ith singularity due to all of the singularities.

Transport Theorem : $f(\underset{\sim}{x}, t)$ is defined in a general volume surrounded by $S(t)$ which has a normal velocity U_n, and take the specific integral:



$$I(t) = \iiint_{V(t)} f(\underset{\sim}{x}, t)\, dV$$

Consider a difference in I ; $\Delta I = I(t+\Delta t) - I(t)$. Thus

$$\Delta I = \iiint_{V(t+\Delta t)} f(\underset{\sim}{x}, t+\Delta t)\, dV - \iiint_{V(t)} f(\underset{\sim}{x}, t)\, dV$$. Now expand $f(\underset{\sim}{x}, t+\Delta t)$ linearly:

$$f(\underset{\sim}{x}, t+\Delta t) = f(\underset{\sim}{x}, t) + (t+\Delta t - t)\frac{\partial f}{\partial t}(\underset{\sim}{x}, t) + O(\Delta t)^2$$

$$\therefore \Delta I = \iiint_{V(t+\Delta t)} \left(f + \Delta t \frac{\partial f}{\partial t}\right) dV - \iiint_{V(t)} f\, dV$$

$$= \iiint_{V(t)} \left(\cancel{f} + \Delta t \frac{\partial f}{\partial t}\right) dV + \iiint_{V(\Delta t)} \left(f + \Delta t \frac{\partial f}{\partial t}\right) dV - \iiint_{V(t)} f\, dV = \Delta t \iiint_{V(t)} \frac{\partial f}{\partial t}\, dV + \iiint_{V(\Delta t)} f\, dV + O[\Delta t^2]$$

$V(\Delta t)$ is the volume between $S(t)$ and $S(t+\Delta t)$. Since the normal vel. is U_n

$$\therefore \Delta I = \Delta t \iiint_V \frac{\partial f}{\partial t}\, dV + \iint_S (U_n \Delta t) f\, dS + O[(\Delta t)^2]$$

Devide both sides by Δt and take the limit as $\Delta t \to 0$

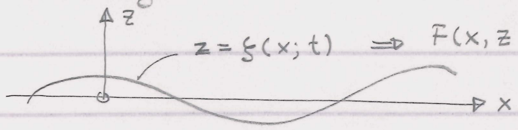
$$\frac{dI}{dt} = \iiint_V \frac{\partial f}{\partial t}\, dV + \iint_S U_n f\, dS$$ which gives the Transport Theorem.

If we consider V as a material volume and utilize divergence theorem

$$\frac{d}{dt}\iiint_V f\, dV = \iiint_V \frac{\partial f}{\partial t}\, dV + \iint_S (u_i n_i) f\, dS = \iiint_V \left\{ \frac{\partial f}{\partial t} + \nabla(f\vec{u}) \right\} dV$$

FREE SURFACE EFFECTS

a) *Linearized* B.V.P.



$\Rightarrow F(x,z;t) = z - \xi(x,t)$

Since the free surface F is a material surface, $\dfrac{DF}{Dt} = 0$

$$\therefore \quad \frac{\partial F}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial F}{\partial z} = 0 \qquad \text{at } z=\xi(x,t)$$

$$-\frac{\partial \xi}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial \phi}{\partial z} = 0 \quad ; \text{ at } z=\xi(x,t) \qquad \text{(Namely kinematic f.s. cond.)}.$$

By using Bernoulli's equation ; dynamic free surface cond. is obtained ;

$$\frac{p}{\rho} + \frac{1}{2}(\nabla\phi)^2 + g\xi + \frac{\partial \phi}{\partial t} = C(t) = \frac{P_a}{\rho} + 0 \longrightarrow \text{at infinity}.$$

$\therefore$ Evaluating the L.H.S. at free surface $(p=P_a)$ gives ;

$$\xi = -\frac{1}{g}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi \cdot \nabla\phi)\right) \quad ; \text{at} \quad z=\xi.$$

If we neglect quadratic and higher-order terms (assuming that ξ ~~and~~ its slopes is infinitesimally small), kinematic and dynamic boundary conds., respectively, turn out to be ;

$$-\frac{\partial \xi}{\partial t} + \frac{\partial \phi}{\partial z} = 0 \quad ; \text{ at } z=0$$

$$\xi = -\frac{1}{g}\frac{\partial \phi}{\partial t} \quad ; \text{ at } z=0.$$ Now take the time derivative of both sides, and substitude ~~insert~~ the obtained ξ_t in the former ;

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \quad ; \text{ at } z=0$$ is obtained as the linearized (2-D) free surface condition. The same free surface condition can also be obtained – as proposed by Newman – by considering the material derivative of p be vanish at $z=\xi$:

$$\frac{Dp}{Dt} = 0 \qquad \text{at } z=\xi(x,t).$$

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial p}{\partial z} = 0$$

$$= \left(\frac{\partial}{\partial t} + \nabla\phi\cdot\nabla\right)\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + g\xi\right) = 0 \quad ; \text{ at } z=\xi$$

$$\therefore \quad \frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \xi}{\partial t} + \nabla\phi\cdot\nabla\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla(\nabla\phi\cdot\nabla\phi) + g\,\nabla\phi\cdot\nabla\xi = 0 \quad ; \text{at } z=\xi$$

Note that from kinematic b.c. $\partial\xi/\partial t = \dfrac{\partial \phi}{\partial z}$ then by neglecting higher-order ~~quadratic~~ terms:

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z=0.$$