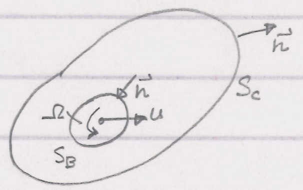
HYDRODYNAMIC FORCES AND MOMENTS

Integrals of pressure over the body surface

$$\vec{F} = \iint_{S_B} p \cdot \vec{n} \, dS$$



$$\vec{M} = \iint_{S_B} p \, (\vec{r} \times \vec{n}) \, dS$$

Here $\vec{r}$ is the position vector from the point of rotation to the body surface. In a linearized sense;

$$\vec{F} = -\rho \iint_{S_B} \frac{\partial \phi}{\partial t} \vec{n} \, dS \quad \text{and} \quad \vec{M} = -\rho \iint_{S_B} \frac{\partial \phi}{\partial t} (\vec{r} \times \vec{n}) \, dS .$$

If we consider a general case of unsteady motion and $\vec{U}$ is the translational velocity and $\vec{\Omega}(t)$ is the rotational (or angular) velocity, then the velocity potential satisfies the boundary condition on the body;

$$\frac{\partial \phi}{\partial n} = \vec{U} \cdot \vec{n} + \vec{\Omega} \cdot (\vec{r} \times \vec{n})$$

where $\vec{U} = (U_1, U_2, U_3)$ and $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3) \equiv (U_4, U_5, U_6)$

By virtue of the above boundary condition we propose: $\phi = U_i \phi_i$

Recall that U_1, U_2, U_3 denote surge, sway ~~heave~~, heave and $\Omega_1, \Omega_2, \Omega_3$ denote roll, pitch, yaw, respectively. Then

$$\frac{\partial \phi_i}{\partial n} = n_i \quad \text{for } i = 1, 2, 3$$

$$\frac{\partial \phi_i}{\partial n} = (\vec{r} \times \vec{n})_i \quad \text{for } i = 4, 5, 6 .$$

Here ϕ_i is also called spatial velocity potential $\phi_i = \phi_i(x, y, z)$. Now let's have a look at $\vec{F}$:

$$\vec{F} = -\rho \frac{d}{dt} \iint_{S_B} U_i \phi_i \vec{n} \, ds = -\rho \dot{U}_i \iint_{S_B} \phi_i \vec{n} \, ds - \rho U_i \iint_{S_B} \phi_i \frac{d\vec{n}}{dt} \, ds$$

If $\vec{n}$ is defined on a surface with a rotational motion, then vector analysis gives

$$\frac{d\vec{n}}{dt} = \vec{\Omega} \times \vec{n}$$

Thus the last term becomes: $-\rho U_i \vec{\Omega} \times \iint_{S_B} \phi_i \vec{n} \, ds$

This result is interpreted as; the force is decomposed into two components; one related with translational motion and the second one is related with rotational motion. Anyway both terms contain;

$$\iint_{S_B} \phi_i n_j \, ds = \iint_{S_B} \phi_i \frac{\partial \phi_j}{\partial n} \, dS$$

; contribution to the jth component of Force due do the motion in ith mode.

Since $\iint_{S_B} \phi_i \frac{\partial \phi_j}{\partial n} ds$ is the proportion of F_j to $\dot{U}_i$, and it to F_j, it must be a mass, namely added-mass in a fluid of a moving body.

$$m_{ji} = \rho \iint_{S_B} \phi_i \frac{\partial \phi_j}{\partial n}\, dS$$

is the added-mass of particular volume of fluid particles accelerated in the jth direction due to the body motion in ith mode.

(7.1) prove $m_{ij} = m_{ji}$

by using Green's theorem.

There is also a relation between added-mass and kinetic energy of the fluid:

$$m_{ij} = \rho \iint_{S_B} \phi_j \frac{\partial \phi_i}{\partial n}\, dS = \rho \iint_{S} \phi_j\, \vec{n}\cdot\nabla\phi_i\, dS \qquad (\text{Recall } T = \tfrac{1}{2} m V^2)$$

(By making use of divergence theorem).

$$= \rho \iiint_V \nabla\phi_j \cdot \nabla\phi_i\, dS$$

On the other hand;

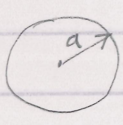
$$T = \frac{1}{2}\rho \iiint_V \nabla\phi\cdot\nabla\phi\, dV = \frac{1}{2}\rho \iiint_V (U_i \nabla\phi_i)\cdot(U_j\nabla\phi_j)\, dV$$

$$= \frac{1}{2}\rho \iiint_V U_i U_j (\nabla\phi_i\cdot\nabla\phi_j)\, dV$$

which means $T = \frac{1}{2} U_i U_j m_{ij}$.

Added masses some simple form →

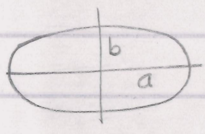
Added masses for some 2-D bodies.



$m_{11} = \rho\pi a^2$

$m_{22} = \rho\pi a^2$

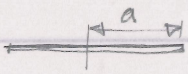
$m_{66} = 0$



$m_{11} = \rho\pi b^2$

$m_{22} = \rho\pi a^2$

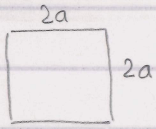
$m_{66} = \frac{1}{8}\rho\pi(a^2 - b^2)^2$



$m_{11} = 0$

$m_{22} = \rho\pi a^2$

$m_{66} = \frac{1}{8}\rho\pi a^4$



$m_{11} = 4.754\,\rho a^2$

$m_{22} = 4.754\,\rho a^2$

$m_{66} = 0.725\,\rho a^4$.

(7.2) Home Assignment: Consider a spherical body - radius a - is accelerating with velocity $U(t)$ in an infinite fluid. Show that added mass of this sphere is $m_{11} = \frac{2}{3}\rho\pi a^3$. (Hint: use the expression for kinetic energy of the sphere

$$T = \frac{1}{2}\rho \iiint_V \nabla\Phi\cdot\nabla\Phi\, dV = \frac{1}{2}\rho \iint_S \Phi\, \vec{n}\cdot\nabla\Phi\, dS \quad .).$$

Lagally's theorem →

<u>And Reading Assignment</u>: J. Newman, "Marine Hydrodynamics", 4.16 The Body Mass Force.

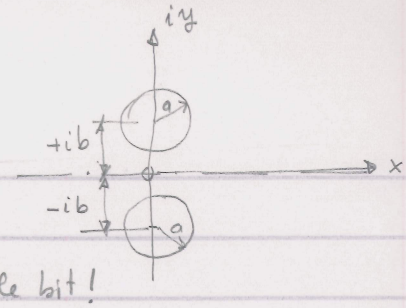
Images →

Method of Images

In some cases either rigid (wall) boundaries or moving (free) boundaries may be modeled by taking the images of distributed singularities. Let's investigate this approach by giving some simple problems.

Ex/1. 2-D flow past a circular cylinder near a plane wall. This problem is an equivalent one to the flow past a pair of circular cylinders:

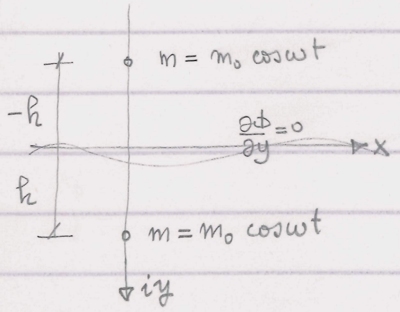
$$f(z) = Uz + \frac{Ua^2}{z-ib} + \frac{Ua^2}{z+ib} \quad ; \quad \overset{U}{\Rightarrow}$$



However the cylinders (dipoles) are distorted a little bit!

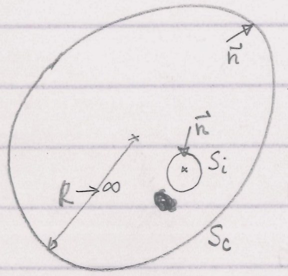
8/2. Slowly pulsating source under a free surface

If we consider the problem 2-D in a complex plane $(z = x+iy)$;



$$f(z) = m \ln(z-ih) + m\ln(z+ih)$$
$$= \left[\ln(z-ih) + \ln(z+ih)\right] m_0 \cos\omega t = \Phi + i\Psi.$$

Lagally Theorem in 3-D



$$\vec{F} = + \iint p\,\vec{n}\,dS$$
$$= -\rho \iint \left(\frac{\partial\Phi}{\partial t} + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi\right)\vec{n}\,dS$$

Gauss's theorem: $\iiint_V \nabla\Phi\,dV = \iint_S \Phi\,\vec{n}\,dS$

Now take the rate of change of the fluid momentum in $S_i + S_c$

$$\rho\frac{d}{dt}\iint_{S_c+S_i}\Phi\,\vec{n}\,dS = \rho\frac{d}{dt}\iiint_V \nabla\Phi\,dV$$

$$= \rho\iiint_V \nabla\left(\frac{\partial\Phi}{\partial t}\right)dV + \rho\iint_{S_c+S_i}\nabla\Phi\,(\vec{u}\cdot\vec{n})\,dS$$ by virtue of transport theorem (see Newman, pp. 57-58)

$$= \rho\iint_{S_c+S_i}\left[\frac{\partial\Phi}{\partial t}\vec{n} + \nabla\Phi\,(\vec{u}\cdot\vec{n})\right]dS.$$

on S_c $\vec{u}\cdot\vec{n}\to 0$. Thus

$$\rho\frac{d}{dt}\iint_{S_i}\Phi\cdot\vec{n}\,dS = \rho\iint_{S_i}\left[\frac{\partial\Phi}{\partial t}\vec{n} + \frac{\partial\Phi}{\partial n}\nabla\Phi\right]dS$$

Therefore $$\vec{F} = -\rho\frac{d}{dt}\iint_{S_i}\Phi\,\vec{n}\,dS + \rho\iint_{S_i}\left(\frac{\partial\Phi}{\partial n}\nabla\Phi - \frac{1}{2}\nabla\Phi\cdot\nabla\Phi\,\vec{n}\right)dS$$

If we consider uniform flow, then

$$\vec{F} = \rho\iint_{S_i}\left(\frac{\partial\Phi}{\partial n}\nabla\Phi - \frac{1}{2}\nabla\Phi\cdot\nabla\Phi\,\vec{n}\right)dS.$$ If there are 'n' singularities

in V; $$\sum_{i=1}^{n}\vec{F}_i = \sum_i \rho\iint_{S_i}\left[\vec{n}(\nabla\Phi)^2 - \frac{1}{2}\nabla\Phi(\nabla\Phi\cdot\vec{n})\right]dS$$

Take a singularity as $\Phi_i = \frac{m_i}{r}$