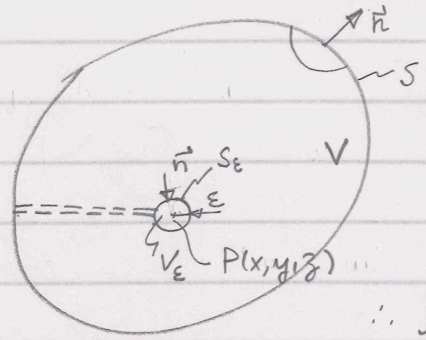
The Method of Green's Theorem



Let ϕ and ψ are defined in V and be the solutions of Laplace's equation. V is bounded by closed surface S. Consider now the integral $\int_S [\psi \frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}]dS$; where $\frac{\partial\phi}{\partial n} = \vec{n}\cdot\nabla\phi$

$$\therefore \int_S [\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}]dS = \int_S \vec{n}[\psi\nabla\phi - \phi\nabla\psi]dS.$$

By using Divergence theorem :

$$\int_S \vec{n}[\psi\nabla\phi - \phi\nabla\psi]ds = \int_V \nabla[\psi\nabla\phi - \phi\nabla\psi]\,dV$$
$$= \int_V [\nabla\psi.\nabla\phi + \psi\nabla^2\phi - \nabla\phi.\nabla\psi - \phi\nabla^2\psi]\,dV = 0$$

In general we would like to represent bodies in the fluid by distributing singularitres on the boundaries. But in this case the above integral needs a carefull treatment. Assume that ψ represent a source with unit stregth:

$$\psi = \frac{1}{4\pi r} = \frac{1}{4\pi}[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{-1/2}$$

Here $P(x,y,z)$ denotes the field pnt. and $Q(\xi,\eta,\zeta)$ denotes source pnt.

ψ, in this case, can be regarded as a particular solution, called Green function. The problem with the source distribution within the fluid volume V is that sources located within the region V are the points where the Laplaces equation is violated. Therefore the elementary volumes V_ε surrounded by closed surface S_ε are introduced to eliminate singularities.

$\therefore$ Using the volume integral first ;

$$\frac{1}{4\pi}\int_{V-V_\varepsilon}\nabla[\frac{1}{r}\nabla\phi - \phi\nabla(\frac{1}{r})]\,dV = \frac{1}{4\pi}\int_S[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}(\frac{1}{r})]dS + \frac{1}{4\pi}\int_{S_\varepsilon}[-\frac{1}{\varepsilon}\frac{\partial\phi}{\partial r} - \phi\frac{1}{\varepsilon^2}]dS_\varepsilon = 0$$

Now we can take limit of the second-term in RHS :

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{S_\varepsilon}[-\frac{1}{\varepsilon}\frac{\partial\phi}{\partial r} - \phi\frac{1}{\varepsilon^2}]dS_\varepsilon = \lim_{\varepsilon\to 0}\frac{1}{4\pi}[-\frac{1}{\varepsilon}\frac{\partial\phi}{\partial r} - \phi\frac{1}{\varepsilon^2}]4\pi\varepsilon^2 = -\phi(P)$$

where we accept that $(\frac{\partial\phi}{\partial r})$ and ϕ are assumed constant within S_ε when $\varepsilon \to 0$.

$$\therefore \frac{1}{4\pi}\int_S[\frac{1}{r(P,Q)}\frac{\partial\phi(Q)}{\partial n(Q)} - \phi(Q)\frac{\partial}{\partial n(Q)}(\frac{1}{r(P,Q)})]ds(Q) = \phi(P)$$

This is a very important expression ; i) we reduce the 3-D (volume) problem to a surface problem (BIM method), ii) by using singularitres, such as $(\frac{1}{r})$ and $\frac{\partial}{\partial n}(1/r)$, we can obtain ϕ in the fluid region in terms of the boundary values defined at the boundaries.

In a more general way, Green's theorem may be expressed ;

$$\int_S \left[G(P,Q) \frac{\partial \Phi}{\partial n}(Q) - \Phi(Q)\, G_n(P,Q) \right] dS(Q) = \begin{cases} 0\,; & \text{when P outside S} \\ 2\pi\, \Phi(P)\,; & \text{when P is on the S} \\ 4\pi\, \Phi(P)\,; & \text{when P is inside.} \end{cases}$$

In this expression $G(P,Q)$ denotes a general Green function which may satisfy a boundary condition in addition to Laplaces equation.

(6.1) Home assignment: Write down the Green theorem formulation for a 2D-rectangular heaving body which makes very high-frequency and small-amplitude oscillations. (Hint: free surface boundary condition can be taken as $\Phi = 0$ at the free surface in heave motion).

Seperation of Variables

Let's obtain circular (cylindrical) progressive waves in 3-D.

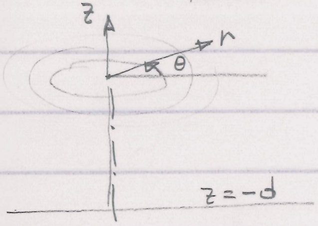
The Boundary Value Problemlem that $\Phi(r,\theta,z;t)$ satisfies in cylindrical coordinates.

$\nabla^2 \Phi = 0$ Laplace's eq. (continuity eq.)

$\frac{\partial \Phi}{\partial z} = 0$ at $z=-d$ (Bottom cond.)

$\Phi_{tt} + g\Phi_z = 0$ at $z=0$ (Linearized free surface cond.), and

Radiation cond. which states that waves must be outgoing.



In cylindrical coordinates:

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

We suggest a general solution for Φ in cylindrical coord.

$\Phi(r,\theta,z,t) = R(r)\cdot\Theta(\theta)\, Z(z)\, T(t)$. Then Laplacian gives:

$$\nabla^2\Phi = R''\cdot\Theta Z T + \frac{1}{r}R'\Theta Z T + \frac{1}{r^2}\Theta'' R Z T + Z'' R \Theta T = 0$$

$$\therefore\ \underbrace{R\Theta Z T}_{\neq 0 \Rightarrow} \underbrace{\left[\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} \right]}_{=0} = 0$$

$$\frac{Z''}{Z} = k^2 = -\frac{1}{R}\left(R'' + \frac{R'}{r}\right) - \frac{1}{r^2}\frac{\Theta''}{\Theta} \quad ; \text{ From here}$$

$$\frac{Z''}{Z} = k^2 \Rightarrow Z = Ae^{kz} + Be^{-kz}$$

Using bottom boundary condition gives;

$$\frac{\partial\Phi}{\partial z} = 0\ (z=-d) \Rightarrow R\Theta T\left(Ake^{-kd} - Bke^{kd}\right) = 0 \Rightarrow B = Ae^{-2kd}$$

$$\therefore Z = Ae^{kz} + Ae^{-2kd}e^{-kz} = Ae^{-kd}\,2\left(\frac{e^{kz}e^{kd} + e^{-kz}\cdot e^{-kd}}{2}\right)$$

$$= A'\cosh k(z+d)$$

$\Rightarrow$ At the same time we set $\frac{\Theta''}{\Theta} = -m^2 < 0$; which gives $\Theta = B'\cos(m\theta)$

$\Longrightarrow$

thus: $\frac{1}{R}\left(R''+\frac{1}{r}R'\right)-\frac{1}{r^2}m^2+k^2=0$

$$R''+\frac{1}{r}R'-\frac{m^2}{r^2}R+k^2R=0$$

$$R''+\frac{1}{r}R'+\left(k^2-\frac{m^2}{r^2}\right)R=0$$

Changing the variable by $kr=\rho$, gives

$$\frac{dR}{dr}=\frac{dR}{d\rho}\frac{d\rho}{dr}=R_\rho\cdot k$$

$$\frac{d^2R}{dr^2}=\frac{d}{d\rho}(k\cdot R_\rho)\,k=k^2R_{\rho\rho}$$

$$\therefore\quad k^2\frac{d^2R}{d\rho^2}+\frac{k}{r}\frac{dR}{d\rho}k+\left(k^2-\frac{m^2}{r^2}\right)R=0$$

$$\frac{d^2R}{d\rho}+\frac{1}{\rho}\frac{dR}{d\rho}+\left(1-\frac{m^2}{\rho^2}\right)R=0$$

; This is what we call Bessel diff. eq. of order m.

a general solution to $R(r)$

$$R=A\,J_m(kr)+B\,Y_m(kr)$$

J_m: Bessel func. of the first kind.

Y_m: Bessel func. of the 2nd kind

In order to have out going cylindrical waves, we have to set $B=iA$, then

$$R(r)=A\left[J_m(kr)+iY_m(kr)\right]=A\,H_m^{(1)}(kr)$$

$\rightarrow$ Hankel func. of the 1st kind of order m

There is also a solution of $H_m^{(2)}=J_m(kr)-iY_m(kr)$ $\rightarrow$ Hankel func. of the 2nd kind which represents waves coming from infinity towards origin.

> The remaining condition to be satisfied is the linearized free surface condition;

$\Phi_{tt}+g\Phi_z=0$ gives $\frac{T''}{T}+g\frac{Z'}{Z}=0$ (at $z=0$)

$$\frac{T''}{T}+gk\tanh kd=0$$

By taking $\omega^2=gk\tanh kd$ $\Rightarrow$ $T''+\omega^2T=0$ which gives

$$T=F\cos\omega t+E\sin\omega t$$

In a complex form the resultant solution;

$$\phi=A\,H_m^{(1)}(kr)\cos m\theta\cosh k(z+d)\,e^{-i\omega t}$$

or

$$\Phi=A\left[J_m(kr)\cos\omega t+Y_m(kr)\sin\omega t\right]\cos(m\theta)\cosh k(z+d).$$

The asymptotic behaviour of ϕ at $r\to\infty$:

$$\lim_{r\to\infty}\Phi=\mathrm{Re}\sqrt{\frac{2}{\pi kr}}\,e^{i(kr-\omega t-\frac{m\pi}{2}-\pi/4)}\cos(m\theta)\cosh k(z+d)$$

$$=k\sqrt{\frac{2}{\pi kr}}\cosh k(z+d)\cos(m\theta)\cos\left(kr-\omega t-\frac{m\pi}{2}-\frac{\pi}{4}\right).$$

cylindrical propagating waves

HW 6.2 $\rightarrow$

Homework : 6.2

Determine the velocity potential (2-D) of the flow due to a moving circular cylinder with a velocity u in the positive x-direction.
(Recall that the solution can be obtained by using a dipole as $\phi = -(Ua^2/r)\cdot\cos\theta$)
Take "a" as the radius of the cylinder.