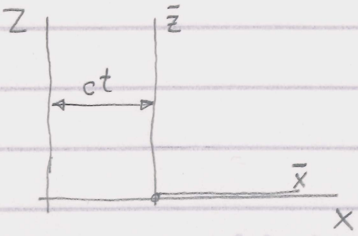
b) Condition in a moving coordinate axes



$$\bar{x} = x - ct$$
$$\bar{z} = z$$

where the $o\bar{x}\bar{z}$ axis moves in $\bar{x}$ axis with velocity c.

$$\therefore \quad \phi(x, z; t) = \phi(\bar{x}+ct, \bar{z}; t)$$
$$= \bar{\Phi}(\bar{x}, \bar{z}; t) = \bar{\Phi}(x-ct, z; t) \quad \text{is valid.}$$

Here;
$$\frac{\partial \phi}{\partial x} = \frac{\partial \bar{\Phi}}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x} = \frac{\partial \bar{\Phi}}{\partial \bar{x}}$$
$$\frac{\partial \phi}{\partial z} = \frac{\partial \bar{\Phi}}{\partial \bar{z}}$$
$$\frac{\partial \phi}{\partial t} = \frac{\partial \bar{\Phi}}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial t} + \frac{\partial \bar{\Phi}}{\partial t} = \frac{\partial \bar{\Phi}}{\partial t} - c \frac{\partial \bar{\Phi}}{\partial \bar{x}}$$

Now kinematic b.c. with respect to moving coordinate system:

$$\frac{DF}{Dt} = -\frac{\partial \bar{\xi}}{\partial t} + c\frac{\partial \bar{\xi}}{\partial \bar{x}} - \frac{\partial \bar{\Phi}}{\partial \bar{x}}\frac{\partial \bar{\xi}}{\partial \bar{x}} + \frac{\partial \bar{\Phi}}{\partial \bar{z}} = 0 \quad ; \quad (\bar{z} = \bar{\xi})$$
$$\therefore \quad \frac{\partial \bar{\xi}}{\partial t} + \frac{\partial \bar{\Phi}}{\partial \bar{x}}\frac{\partial \bar{\xi}}{\partial \bar{x}} - c\frac{\partial \bar{\xi}}{\partial \bar{x}} - \frac{\partial \bar{\Phi}}{\partial \bar{z}} = 0 \quad ; \quad (\bar{z} = \bar{\xi})$$

and using Bernoulli's equation at the free surface $(p = p_a)$ gives;

$$\frac{1}{2}(\nabla\bar{\Phi})^2 + g\bar{\xi} + \frac{\partial \bar{\Phi}}{\partial t} - c\frac{\partial \bar{\Phi}}{\partial \bar{x}} = 0 \quad ; \quad (\bar{z} = \bar{\xi})$$

Again by neglecting higher-order terms, we obtain;

$$\left.\begin{aligned} &\frac{\partial \bar{\xi}}{\partial t} - c\frac{\partial \bar{\xi}}{\partial \bar{x}} - \frac{\partial \bar{\Phi}}{\partial \bar{z}} = 0 \quad \text{and} \\ &g\bar{\xi} + \frac{\partial \bar{\Phi}}{\partial t} - c\frac{\partial \bar{\Phi}}{\partial \bar{x}} = 0 \end{aligned}\right\} z = 0.$$

$\therefore$ $\bar{\xi}$ can be expressed in terms of $\bar{\Phi}$, so that we arrive at the free surface boundary cond for a moving coordinate system:

Since
$$\frac{\partial}{\partial t}(g\bar{\xi}) = g\left(\frac{\partial \bar{\xi}}{\partial t} - c\frac{\partial \bar{\xi}}{\partial \bar{x}}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial \bar{\Phi}}{\partial t}\right) + \frac{\partial}{\partial t}\left(c\frac{\partial \bar{\Phi}}{\partial \bar{x}}\right) \quad ; \quad z = 0$$
$$= -\frac{\partial^2 \bar{\Phi}}{\partial t^2} + c\frac{\partial^2 \bar{\Phi}}{\partial \bar{x}\partial t} + c\frac{\partial^2 \bar{\Phi}}{\partial t \partial \bar{x}} - c^2\frac{\partial^2 \bar{\Phi}}{\partial \bar{x}^2}$$

$\therefore$ Substituting in the former one;

$$-\frac{1}{g}\frac{\partial^2 \bar{\Phi}}{\partial t^2} + 2\frac{c}{g}\frac{\partial^2 \bar{\Phi}}{\partial \bar{x}\partial t} - \frac{c^2}{g}\frac{\partial^2 \bar{\Phi}}{\partial \bar{x}^2} - \frac{\partial \bar{\Phi}}{\partial \bar{z}} = 0$$

Thus;

$$\frac{\partial^2 \bar{\Phi}}{\partial t^2} - 2c\frac{\partial^2 \bar{\Phi}}{\partial \bar{x}\partial t} + c^2\frac{\partial^2 \bar{\Phi}}{\partial \bar{x}^2} + g\frac{\partial \bar{\Phi}}{\partial \bar{z}} = 0 \quad ; \quad (z = 0).$$

8.1 For home: try to obtain the free surface boundary cond. for moving coord axes if the axes (or motion) is in an acceleration as $c = c(t)$.

2nd Cond: $\Phi \propto e^{\mp ikx}$ as $x \to \pm\infty$ in 2-D

$\Phi \propto R^{-1/2} e^{-ikR}$ as $R \to \infty$

2-D regular waves (plane waves)

The related boundary value problem:

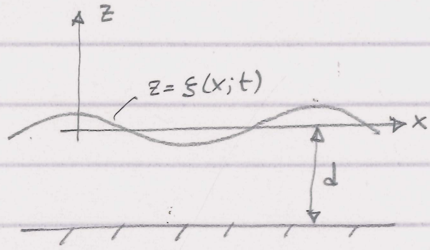
We assume the fluid is inviscid and the flow is irrotational, hence $\vec{V} = \nabla\Phi$

$\therefore$ Governing equation $\nabla^2\Phi = 0$

Linearized free surface b.c. $\Phi_{tt} + g\,\Phi_z = 0$ at $z=0$

Bottom b.c. $\Phi_z = 0$ at $z=-d$ or $z \to -\infty$ and a radiation c. at infinity.

The global coordinates are depicted as follows:



This problem can be solved by the separation of variables, for example.

Let $\Phi(x,z;t) = X(x)\,Z(z)\,T(t)$

Reading assignment: Sabuncu, T. "Geni Hareketleri", İTÜ, 1983. pp. 9–12.

From the above reference $Z = A\cosh k(z+d)$

$X = B\cos(kx+\alpha)$ where α is a phase angle.

at this step free surface condition must be satisfied; which gives

$$\frac{1}{T}\frac{d^2T}{dt^2} + g\,\frac{1}{Z}\frac{dZ}{dz} = 0 \qquad \text{at } z=0$$

$$\frac{1}{T}\frac{d^2T}{dt^2} + \underbrace{gk\tanh kd}_{\omega^2} = 0$$

Thus the solution of this diff. eq:

$T = C\sin(\omega t + \varepsilon)$. Here $\omega^2 = gk\tanh kd$ is called ~~the~~ dispersion relation. There is only one real solution (k_0) to this relation and infinite number of imaginary roots, (ik_q). The end product of the potential solution is

$$\Phi = \frac{g\,\xi_a}{\omega}\,\frac{\cosh k(z+d)}{\cosh kd}\,\sin(kx-\omega t)$$

which represents a progressive wave in +x-direction.

Here $k = \frac{2\pi}{\lambda}$ is the wave number and $\omega = \frac{2\pi}{T}$ is the circular frequency of the wave motion. The phase velocity or celerity c can be obtained by considering $kx - \omega t = \text{constant}$ which gives $c = \frac{dx}{dt} = \frac{\omega}{k} = \frac{\lambda}{T}$

In most cases, the wave system is modeled by a discrete spectrum of waves

$$\xi = \sum_{n=1}^{N} \xi_{a_n}\sin(k_n x - \omega_n t)$$

, where ω_n and k_n are related to each other.

This means that every component of waves travel with a different celerity $c_n = \frac{\omega_n}{k_n}$, which results in a continuous change of wave pattern or profile. In order to investigate this mechanism take a couple of waves having nearly equal wave numbers and frequency : $\xi_1 = \xi_a \cos(kx-\omega t)$ and $\xi_2 = \xi_a \cos[(k+dk)x-(\omega+d\omega)t]$

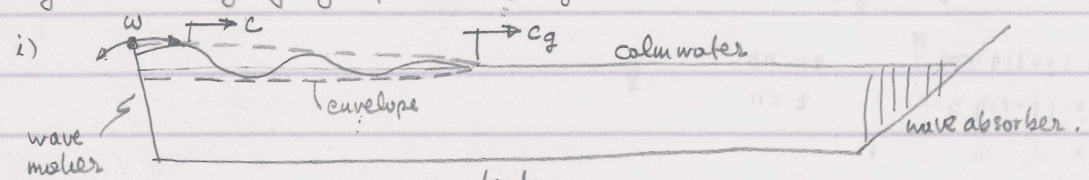
If we superpose these two components we can arrive at the group velocity

$$c_g = \frac{d\omega}{dk}.$$

(8.2) Show that the group velocity $c_g = \frac{1}{2} c \left[1 + \frac{2kd}{\sinh 2kd}\right]$

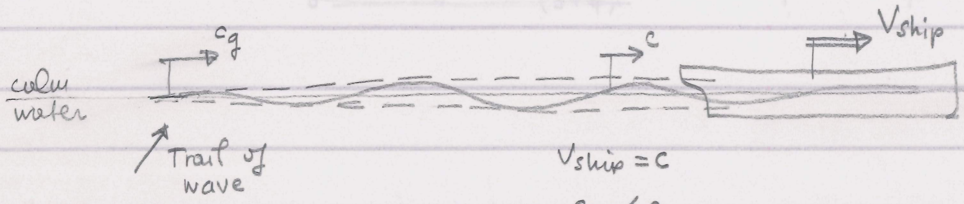
Physical meaning of group velocity c_g :

i)



$c_g \leq c$

ii)



$V_{ship} = c$

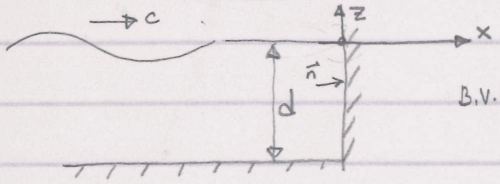
$c_g < c$

Note that energy moves with group velocity :

$$E_t = \frac{1}{2} \rho g \xi_a^2 \lambda$$

$$\overline{W} = \frac{1}{2} \rho g \xi_a^2 c_g = \overline{E} c_g$$

The problem of reflection of waves from a vertical barrier



$$\Phi = \Phi_i + \Phi_r$$

B.V.P

$$\begin{cases} \nabla^2 \Phi = 0 \\ \Phi_{tt} + g\,\Phi_z = 0 & (z=0) \\ \Phi_z = 0 & (z=-d) \\ \Phi_x = 0 & (x=0) \end{cases}$$

It is known that $\Phi_i = \frac{g\xi_a}{\omega} \frac{\cosh k(z+d)}{\cosh kd} \sin(kx-\omega t)$

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi_i}{\partial x} + \frac{\partial \Phi_r}{\partial x} = 0 \qquad (x=0)$$

$$\therefore \quad \left.\frac{\partial \Phi_r}{\partial x}\right|_{x=0} = -\frac{g\xi_a}{\omega} k \frac{\cosh k(z+d)}{\cosh kd} \cos(-\omega t)$$

Integrating both sides with respect to x gives ; $(\cos(-\omega t) = \cos \omega t)$

$$\Phi_r = -\frac{g\xi_a}{\omega} \frac{\cosh k(z+d)}{\cosh kd} \sin(kx+\omega t)$$

$\Phi_i \propto \sin(kx-\omega t)$; $\Phi_r \propto \sin(kx+\omega t)$

$$\Phi_i + \Phi_r : \quad \sin(kx-\omega t) - \sin(kx+\omega t) = \sin kx \cos\omega t - \sin\omega t \cos kx$$
$$- \sin kx \cos\omega t - \sin\omega t \cos kx$$
$$= -2 \cos kx \sin\omega t$$

$$\therefore \quad \Phi = -\frac{2\zeta_a g}{\omega}\,\frac{\cosh k(z+d)}{\cosh kd}\,\cos kx \,\sin\omega t$$

The force acting on the wall:

$$\vec F = \iint_S p\cdot\vec n\, dS \qquad p = -\rho\frac{\partial\Phi}{\partial t} \quad ; \quad \vec n = \vec i$$

$$F_x = -\int_{-d}^{0} \rho\frac{\partial\Phi}{\partial t}\,dz = +\rho\int_{-d}^{0} 2\zeta_a g\,\frac{\cosh k(z+d)}{\cosh kd}\cos kx \cos\omega t\,dz\,\Big|_{x=0}$$

$$= \frac{2\rho\zeta_a g}{\cosh kd}\int_{-d}^{0}\cosh k(z+d)\,dz \cdot \cos\omega t$$

$c = \frac{\omega}{k}$

$\omega^2 = gk\tanh kd$

$$= \frac{2\zeta_a}{k}\rho g\,\frac{\sinh kd}{\cosh kd}\cdot\cos\omega t$$

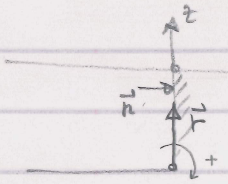
$c = \sqrt{\frac{g}{k}\tanh kd}$

$c^2 = \frac{g}{k}\tanh kd$

$$= 2\zeta_a\rho c^2\cos\omega t$$

where c is the celerity of the incident wave system.

Moment according to the base point



$$\vec M = \iint_{S_B} p\,(\vec r\times\vec n)\,dS \qquad \left(\begin{matrix}\text{Here}\\ \text{on } x=0\end{matrix}\right) \; \vec r = (z+d)\vec k, \;\; \vec n = \vec i$$

$$M_x\vec i + M_y\vec j + M_z\vec k = -\rho\int_{-d}^{0}\frac{\partial\Phi}{\partial t}\Big|_{x=0}\cdot[(z+d)\,\vec k\times\vec i\,]\,dz$$

$\omega^2 = gk$

$$M_y = -\rho\int_{-d}^{0}\frac{\partial\Phi}{\partial t}\Big|_{x=0}(z+d)\,dz$$

$$= +\rho\int_{-d}^{0} 2\zeta_a g\,\frac{\cosh k(z+d)}{\cosh kd}\cos kx\cos\omega t\,(z+d)\,dz$$

$$= \rho g\frac{2\zeta_a}{k}d\tanh kd\cos\omega t + 2\zeta_a\rho g\,\frac{\cos\omega t}{\cosh kd}\int_{-d}^{0} z\cdot\cosh k(z+d)\,dz$$

$$= \rho g\frac{2\zeta_a}{k}d\tanh kd\cos\omega t + \rho g\,2\zeta_a\,\frac{(1+\cosh kd)}{k^2\cosh kd}\cos\omega t$$