Nonlinear Effects and Linearization

Nonlinearity arises from free surface condition. Exact form of free surface condition makes it very difficult to solve the problem. It can either be solved by numerical-iterative techniques or by reducing the full nonlinear problem into linear subproblems - generally obtained by perturbation analysis. In this analysis we introduce a small perturbation parameter ε into the potential solution : $\Phi = \Phi(x,y,z;t;\varepsilon)$

and assume $\quad \Phi(x,y,z;t;\varepsilon) = \varepsilon\,\Phi^{(1)}(x,y,z;t) + \varepsilon^2\,\Phi(x,y,z;t) + O(\varepsilon^3)$

ε can be determined according to the character of the problem, i.e. $\varepsilon = \xi_a/\lambda$.

$(z=0)$. Physically we assume small perturbations about a known ~~solu~~ (or a basic) solution. In the case of progressive waves the known solution (basic solution) corresponds to still water; ε^0, which is $\Phi^{(0)}$. The same power series in ε is valid for ξ and p :

$$\xi = \xi^{(0)} + \varepsilon\,\xi^{(1)} + \varepsilon^2\,\xi^{(2)} + O(\varepsilon^3)$$

$$p = p^{(0)} + \varepsilon\,p^{(1)} + \varepsilon^2 p^{(2)} + O(\varepsilon^3).$$

Now the quantities of the problem are expanded into Taylor series about the mean positions $(\xi^{(0)})$ of the moving boundaries :

$$\Phi(x,y,\xi;t;\varepsilon) = \Phi(x,y,\xi^{(0)};t;\varepsilon) + \frac{\xi-\xi^{(0)}}{1!}\Phi_z(x,y,\xi^{(0)};t;\varepsilon) + \frac{(\xi-\xi^{(0)})^2}{2!}\Phi_{zz}(x,y,\xi^{(0)};t;\varepsilon)$$
$$+ \cdots$$

By substituting power series; and taking $\xi^{(0)} = 0$;

$$\Phi(x,y,\xi;t;\varepsilon) = \varepsilon\,\Phi^{(1)}(x,y,0;t) + \varepsilon^2\,\Phi^{(2)}(x,y,0;t) + \varepsilon^2\,\xi^{(1)}(x;t)\,\Phi_z^{(1)}(x,y,0;t) + O(\varepsilon^3)$$

Now recall the dynamic free surface condition ;

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi\cdot\nabla\Phi) + g\xi = 0 \qquad \text{at } z=\xi$$

Subsequently we insert the Taylor expansion in dynamic condition and equate the like powers of ε in the equation

$$\varepsilon\,\Phi_t^{(1)} + \varepsilon^2\Phi_t^{(2)} + \varepsilon^2\,\xi^{(1)}\,\Phi_{tz}^{(1)} + \varepsilon^2\,\frac{1}{2}\left[(\Phi_x^{(1)})^2 + (\Phi_y^{(1)})^2 + (\Phi_z^{(1)})^2\right] +$$
$$+ \varepsilon g\,\xi^{(1)} + \varepsilon^2 g\,\xi^{(2)} + O(\varepsilon^3) = 0 \qquad ; \qquad z = \xi^{(0)} = 0$$

thus ;

$$\varepsilon^1 : \quad \Phi_t^{(1)} + g\,\xi^{(1)} = 0 \qquad (\text{at } z=0) \qquad (a)$$

$$\varepsilon^2 : \quad \Phi_t^{(2)} + g\,\xi^{(2)} + \xi^{(1)}\Phi_{tz}^{(1)} + \frac{1}{2}\left[(\Phi_x^{(1)})^2 + (\Phi_y^{(2)})^2 + (\Phi_z^{(3)})^2\right] = 0 \qquad (\text{at } z=0) \quad (b)$$

Recall that the kinematic condition at free surface $z = \xi(x;t)$

$$\frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}\frac{\partial\Phi}{\partial x} + \frac{\partial\xi}{\partial y}\frac{\partial\Phi}{\partial y} - \frac{\partial\Phi}{\partial z} = 0 \qquad ; z=\xi.$$

If, again, we expand the quantities in the boundary condition into Taylor series with the assumption that $\phi = \sum_i \varepsilon^i \phi^{(i)}$;

$$\varepsilon \xi_t^{(1)} + \varepsilon^2 \xi_t^{(2)} + \varepsilon^2 \xi_x^{(1)} \phi_x^{(1)} + \varepsilon^2 \xi_y^{(1)} \phi_y^{(1)} - \varepsilon \phi_z^{(1)} - \varepsilon^2 \phi_z^{(2)} - \varepsilon^2 \xi^{(1)} \phi_{zz}^{(1)} + O(\varepsilon^3) = 0$$

The equating the equal powers of ε :

$$\varepsilon : \quad \xi_t^{(1)} - \phi_z^{(1)} = 0 \qquad (z=0) \qquad (c)$$

$$\varepsilon^2 : \quad \xi_t^{(2)} - \phi_z^{(2)} + \xi_x^{(1)} \phi_x^{(1)} + \xi_y^{(1)} \phi_y^{(1)} - \xi^{(1)} \phi_{zz}^{(1)} = 0 \qquad (z=0). \qquad (d)$$

Now order by order ;

using (a) & (c) we obtain

$$\phi_{tt}^{(1)} + g\,\phi_z^{(1)} = 0 \qquad \text{at } z=0 . \qquad (A)$$

and using time derivative of (b) in (d) and first-order quantities of $\xi^{(1)}$

we obtain
$$\phi_{tt}^{(2)} + g\,\phi_z^{(2)} = -\frac{\partial}{\partial t}(\nabla\phi^{(1)})^2 + \frac{1}{g}\,\phi_t^{(1)} \frac{\partial}{\partial z}\left(\phi_{tt}^{(1)} + g\,\phi_z^{(1)}\right) \qquad (\text{at } z=0). \quad (B)$$

Condition (A) gives $\quad \phi^{(1)} = + \dfrac{g\,\xi_a}{\omega} \dfrac{\cosh k(z+d)}{\cosh kd} \sin(kx - \omega t)$

whereas second-order condition (B) gives

$$\phi^{(2)} = \frac{g\,\xi_a^2}{\omega}\, k_0\, \frac{3}{8}\, \frac{\cosh 2k(z+d)}{\cosh kd\, \sinh^3 kd} \sin(2kx - 2\omega t)$$

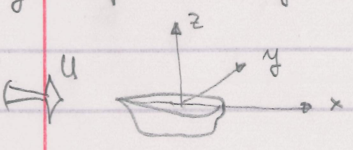
9.1 Home assignment :
What can you say about second-order potential of progressive waves and second-order wave elevation when the depth is infinite?

Note that in finite depth waters ; celerity is the same up to third-order, that is $c = \sqrt{\frac{g}{k}\tanh kd}$ is valid through the second-order. But in deep water waves celerity needs to be corrected as

$$c = \omega/k = \sqrt{\frac{g}{k}}\left(1 + k^2 \xi_a^2\right)^{1/2} + O(k^3 \xi_a^3) = \sqrt{\frac{g}{k}}\left[1 + (2\pi)^2 \varepsilon^2\right]^{1/2} + O\left((2\pi)^3 \varepsilon^3\right)$$

This is also called amplitude dispersion.

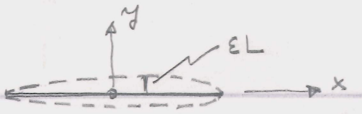
> Now let's try to obtain 1st & 2nd order kinematic boundary conditions on the wetted surface of a ship in a uniform flow :



$y = f(x,z)$

that is ; $F = y - f(x,z) = 0$

We suggest that the total vel. potential as $\Phi = Ux + \phi$. Therefore here the basic or known flow is the uniform flow around flat plate. And ϕ denotes perturbation potential due to geometric perturbations from flat plate :



since ε may be taken as B/L.

First obtain exact b.c. on the body: $\frac{\partial \Phi}{\partial n} = 0$ on $F = 0$ where $\vec{n} = \frac{\nabla F}{|\nabla F|}$

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

$$= -(U + \Phi_x) f_x + \Phi_y - \Phi_z f_z = 0 \qquad \text{on the body surface } F$$

$$\Phi = \varepsilon^1 \Phi^{(1)} + \varepsilon^2 \Phi^{(2)} + O(\varepsilon^3)$$

$$y = f(x, z; \varepsilon) = \varepsilon f^{(1)}(x, z).$$

Now we have to expand the potential terms in the above b.c. into Taylor series about $y = f^{(0)}(x,z) = 0$. That is;

$$\Phi_x(x, f, z) = \Phi_x(x, 0, z) + f\,\Phi_{xy}(x, 0, z) + \frac{f^2}{2!}\Phi_{xyy}(x, 0, z) + \dots$$

If we use power series in Φ;

$$\Phi_x(x, f, z) = \varepsilon \Phi_x^{(1)} + \varepsilon^2 \Phi_x^{(2)} + \left((\varepsilon f^{(1)}) \cdot (\varepsilon \Phi_{xy}^{(1)} + \varepsilon^2 \Phi_{xy}^{(2)}) + \dots \right.$$

$$= \varepsilon \Phi_x^{(1)} + \varepsilon^2 (\Phi_x^{(2)} + f^{(1)} \Phi_{xy}^{(1)}) + O(\varepsilon^3)$$

and similarly;

$$\Phi_y(x, f, z) = \varepsilon \Phi_y^{(1)} + \varepsilon^2 (\Phi_y^{(2)} + f^{(1)} \Phi_{yy}^{(1)}) + O(\varepsilon^3)$$

$$\Phi_z(x, f, z) = \varepsilon \Phi_z^{(1)} + \varepsilon^2 (\Phi_z^{(2)} + f^{(1)} \Phi_{zy}^{(1)}) + O(\varepsilon^3).$$

Using the series expansions in the boundary condition;

$$-\left[U + \varepsilon \Phi_x^{(1)} + \varepsilon^2 (\Phi_x^{(2)} + f^{(1)} \Phi_{xy}^{(1)})\right] \varepsilon f_x^{(1)} + \varepsilon \Phi_y^{(1)} + \varepsilon^2 (\Phi_y^{(2)} + f^{(1)} \Phi_{yy}^{(1)}) -$$

$$- \left[\varepsilon \Phi_z^{(1)} + \varepsilon^2 (\Phi_z^{(2)} + f^{(1)} \Phi_{zy}^{(1)})\right] \varepsilon f_z^{(1)} = 0 \qquad \text{on } y = 0.$$

and equating like powers of ε;

$$\varepsilon: \quad \Phi_y^{(1)} = U f_x^{(1)} \qquad \text{on } y = 0.$$

$$\varepsilon^2: \; -\Phi_x^{(1)} f_x^{(1)} + \Phi_y^{(2)} + f^{(1)} \Phi_{yy}^{(1)} - \Phi_z^{(1)} f_z^{(1)} = 0 \qquad \text{or;}$$

$$\Phi_y^{(2)} = \Phi_x^{(1)} f_x^{(1)} - \Phi_{yy}^{(1)} f^{(1)} + \Phi_z^{(1)} f_z^{(1)} \qquad \text{on } y = 0.$$

(9.2) For Home: Obtain first- and second-order pressures on the surface of the ship for the above given problem.
