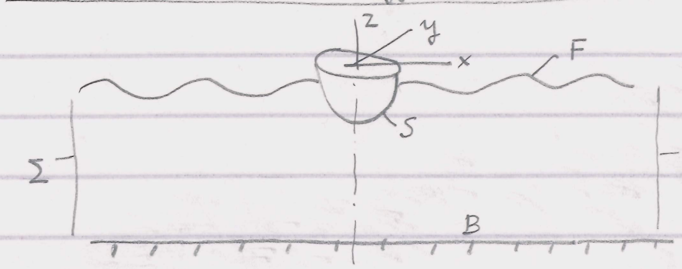
Green's Theorem in Diffraction Problem



Diffraction prob. $\Phi = \Phi_I + \Phi_D$

$$\Phi_I = \frac{g\,\xi_a}{\omega}\,\frac{\cosh k(z+d)}{\cosh kd}\cos(kx-\omega t)$$

and

$$\Phi_D = Re\ \varphi_D\, e^{-i\omega t}$$

$\therefore$ B.V.P.
1. $\nabla^2 \Phi_D = 0$
2. $\Phi_{tt} + g\,\Phi_z = 0 \quad (z=0)$
3. $\Phi_{D_z} = 0 \quad (z=-d)$
4. $\lim\limits_{R\to\infty} R^{1/2}[\Phi_{D_R} - ik\,\Phi_D] = 0$
5. $\frac{\partial \Phi}{\partial n} = 0$ on S.

We choose a Green function in the form

$$G(P,Q) = [\frac{1}{r_1(P,Q)} + H_1(P,Q)]\cos\omega t + H_2(P,Q)\sin\omega t$$

; where $P = P(x,y,z)$ field pnt.
$Q = Q(\xi,\eta,\zeta)$ source pnt.

Green's function should also satisfy ;

$$\nabla^2 G(P,Q) = 0$$

$$G(P,Q)_\zeta - \frac{\omega^2}{g}G(P,Q) = 0 \qquad (\zeta = 0)$$

$$G_\zeta\ (P;\xi,\eta,-d) = 0 \qquad (\zeta = -d)$$

$$\lim_{R\to\infty} R^{1/2}[G_R - ikG] = 0$$

in full detail :

$$G(P,Q;t) = [\frac{1}{r_1} + \frac{1}{r_2} + \oint_0^\infty \frac{2(k+\nu)\,e^{-kd}\cos k(z+d)\cos k(\zeta+d)\,J_0(kr)}{k\sinh kd - \nu\cosh kd}\,dk\,]\cos\omega t +$$

$$+ \frac{2\pi\,(k_0+\nu)\,e^{-k_0 d}\sin k_0 d\,\cos k_0(z+d)\cos k_0(\zeta+d)}{\nu d + \sinh^2 k_0 d}\,J_0(k_0 r)\sin\omega t$$

$$\nu = \frac{\omega^2}{g} \quad ; \quad k_0 \tanh k_0 d = \nu$$

$$r_1 = [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2}$$

$$r_2 = [(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2]^{1/2}$$

Fluid domain ∂V is surrounded by

$$\partial V = \Sigma \cup B \cup F \cup S .$$

Thus the Green's identity for φ_D :

$$\varphi_D(P) = \frac{1}{2\pi}\int_{\partial V}[G(P,Q)\frac{\partial \varphi_D(Q)}{\partial n} - \varphi_D(Q)\frac{\partial G}{\partial n}(P,Q)]\,dS(Q)$$

$$2\pi\,\varphi_D(P) = \int_S [G\,\varphi_{D_n} - G_n\varphi_D]\,dS + \underbrace{\int_F [\varphi_{D_\zeta}G - G_\zeta\varphi_D]\,d\xi\,d\eta}_{=0}$$

$$+ \underbrace{\int_B [\varphi_{D_\zeta}G - G_\zeta\varphi_D]\,d\xi\,d\eta}_{=0} + \int_{-d}^{0} d\zeta \int_0^{2\pi} d\theta\, R\,\underbrace{[\varphi_{D_R}G - \varphi_D G_R]}_{\to 0} = \int_S [\varphi_{D_n}G - G_n\varphi_D]\,dS$$

$\to$

Since, for example ;

$$R_0\left[\varphi_{D_R} G - G_R \varphi_D\right] = R_0\left[\varphi_{D_R} G - ik\varphi_D G + ik\varphi_D G - \varphi_D G_R\right]$$

$$= R_0^{1/2} G \underbrace{R^{1/2}\left[\varphi_{D_R} - ik\varphi_D\right]} - R_0^{1/2}\varphi_D \underbrace{R_0^{1/2}\left[G_R - ikG\right]}$$

Radiation cond. (=0).

If field pnts P are taken on the boundary surface S, and note that $\varphi = \varphi_D + \varphi_I$:

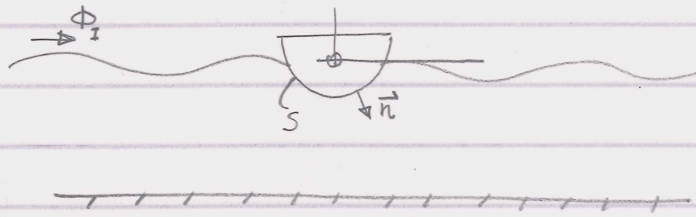
$$\varphi_D(P) + \frac{1}{2\pi}\int_S \varphi_D(Q)\, G_n(P,Q)\, ds = -\frac{1}{2\pi}\int_S \varphi_I(Q)\, G_n(P,Q)\, ds(Q)$$

In order to solve the above equation numerically, surface S should be discretized. Thus, for the ith element ;

$$\varphi_D(P_i) + \frac{1}{2\pi}\sum_{j=1}^{N} \varphi_D(P_j)\, G_\nu(P_i, P_j)\,\Delta S = -\frac{1}{2\pi}\sum_{j=1}^{N} \varphi_I(P_j)\, G_\nu(P_i, P_j)\,\Delta S \quad ; \; i = 1, 2, \ldots, N.$$

H.W. (11.1) Show that $\int_F \left[\varphi_{D_\xi} G - \varphi_D G_\xi\right] d\xi\, d\eta = 0$

Haskind's relation



$$\Phi_I = \frac{g\xi_a}{\omega}\frac{\cosh k_0 z}{\cosh k_0 d} e^{i(k_0 x - \omega t)} = \varphi_I(x,y,z)\, e^{-i\omega t}$$

Diffraction potential: $\Phi_D(x,y,z;t) = \varphi_D\, e^{-i\omega t}$

$\therefore$ Total potential $\quad \Phi = \Phi_I + \Phi_D$

Kin. cond. on the body (S_B): $\vec{n}\,\nabla\Phi = \vec{n}\,\nabla(\Phi_I + \Phi_D) = \frac{\partial}{\partial n}(\Phi_I + \Phi_D) = 0$

Now consider radiation problem in which the body is forced to make oscillatory motion: $\quad v = v_j\, e^{-i\omega t}$ $\quad$ (x_1, x_2, x_3)

Then radiation potential;

$$\Phi_j = v_j\, \varphi_j(x,y,z)\, e^{-i\omega t} \quad ; \quad j = 1,2,3,4,5,6$$

Also; $\quad \Phi = e^{-i\omega t}\sum_{j=1}^{6} v_j\, \varphi_j(x,y,z)$

In this case kin. cond. on the moving body;

$$\vec{n}\,\nabla\Phi = \vec{v}\cdot\vec{n} + \vec{\Omega}\cdot(\vec{r}\times\vec{n}) \qquad \text{where} \quad \vec{v} = \vec{v}(v_1, v_2, v_3), \quad \vec{\Omega} = \vec{\Omega}(v_4, v_5, v_6)$$

Recall that $\quad \frac{\partial\varphi_j}{\partial n} = n_j \quad$ where $\quad \vec{n} = \vec{n}(n_1, n_2, n_3)$

$$\left.\begin{aligned} n_4 &= (y n_3 - z n_2) \\ n_5 &= (z n_1 - x n_3) \\ n_6 &= (x n_2 - y n_1) \end{aligned}\right\} \text{HW } 11.2$$

Now apply Green's theorem;

$$\iint_S \left(\varphi_j \frac{\partial\varphi_D}{\partial n} - \varphi_D \frac{\partial\varphi_j}{\partial n}\right) dS = 0$$

$$\iint_S \varphi_D \frac{\partial\varphi_j}{\partial n}\, dS = \iint_S \varphi_j \frac{\partial\varphi_D}{\partial n}\, dS \, , \quad \left(\text{By making use of } \frac{\partial\varphi_D}{\partial n} = -\frac{\partial\varphi_I}{\partial n}\right)$$

$$= -\iint_S \varphi_j \frac{\partial\varphi_I}{\partial n}\, dS$$

Meanwhile, wave forces can be calculated by;

$$X_j = -\iint_S p\, n_j\, dS \qquad \text{where} \quad p = -\rho\frac{\partial\Phi}{\partial t} = i\omega\rho\,\varphi\, e^{-i\omega t}$$

$$\therefore \quad X_j = -i\omega\rho\, e^{-i\omega t}\iint_S (\varphi_I + \varphi_D)\, n_j\, dS$$

$$X_j = -i\omega\rho e^{-i\omega t} \iint_S (\varphi_I + \varphi_0) \frac{\partial \varphi_j}{\partial n} \, ds$$

$$\therefore \quad \boxed{X_j = -i\omega\rho e^{-i\omega t} \iint_S \left(\varphi_I \frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial \varphi_I}{\partial n} \right) ds} \quad \text{Haskind's relation}$$

If we consider the sum of $S + S_\infty$

$$\iint_{S+S_\infty} \left(\varphi_I \frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial \varphi_I}{\partial n} \right) ds = 0$$

$$\therefore \quad \iint_S \left(\varphi_I \frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial \varphi_I}{\partial n} \right) ds = - \iint_{S_\infty} \left(\varphi_I \frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial \varphi_I}{\partial n} \right) ds$$

$$= X_j \frac{1}{-i\omega\rho e^{-i\omega t}}$$

That is $$X_j = -i\omega\rho e^{-i\omega t} \lim_{R \to \infty} \int_0^{2\pi} \int_{-d}^{0} \left(\varphi_I \frac{\partial \varphi_j}{\partial R} - \varphi_j \frac{\partial \varphi_I}{\partial R} \right) R \, d\theta \, dz .$$