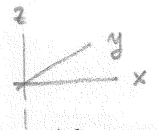
Moving Source under Free Surface

Determination of Green's function for the problem of:

$$\begin{cases} \nabla^2 G = 0 \\ G_{xx} + k_0 G_z = 0 \; ; \; z = 0 \quad \text{where } k_0 = \dfrac{g}{U^2} \\ G_z = 0 \quad \text{as } z \to -\infty \\ \left.\begin{aligned} G &= O\left[(x^2+y^2)^{-1/3}\right] ; \; r \to \infty \\ G &= o\,(1) \qquad ; \; r \to -\infty \end{aligned}\right\} \text{Rad. Cond.} \end{cases}$$

for a coordinate system moving with velocity U.



$*\ (\xi = 0,\ \eta = 0,\ z = -\zeta)$

For $G(x,y,z;\xi,\eta,\zeta) = -\dfrac{1}{[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{1/2}} + H(x,y,z;\xi,\eta,\zeta)$

Utilize the known integral;

$$\frac{1}{(x^2+y^2+z^2)^{1/2}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_0^\infty e^{k[i\overbrace{(x\cos\theta + y\sin\theta)}^{\varpi} - |z|]}\, dk.$$

Consider a simple source at $x=0$, $y=0$ and $z=\zeta$ where $\zeta<0$.

$$\therefore \quad G = -\frac{1}{[x^2+y^2+(z+\zeta)^2]^{1/2}} = -\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_0^\infty e^{k[i\varpi-(z+\zeta)]}\, dk$$

But this does not satisfy free surface condition. Now add a term to satisfy free surface condition:

$$G = -\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_0^\infty e^{k[i\varpi-(z+\zeta)]}\, dk + \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_0^\infty A(k,\theta)\, e^{k[i\varpi+z]}\, dk$$

At this stage $A(k,\theta)$ is unknown; but applying free surface condition and bottom boundary condition, we obtain (by utilizing fourier integrals);

$$A(k,\theta) = -\frac{k+k_0\sec^2\theta}{k-k_0\sec^2\theta}\, e^{k\zeta} = -\left(1+\frac{2k_0\sec^2\theta}{k-k_0\sec^2\theta}\right) e^{k\zeta}$$

$$\therefore \quad G = -\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_0^\infty e^{k[i\varpi-(z+\zeta)]}\, dk + \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_0^\infty e^{k[i\varpi+(z+\zeta)]}\, dk +$$

$$+ \frac{k_0}{\pi}\int_{-\pi}^{\pi} \sec^2\theta\, d\theta \int_0^\infty \frac{e^{k[i\varpi+(z-\zeta)]}}{k-k_0\sec^2\theta}\, dk.$$

$$= -\frac{1}{r_1} + \frac{1}{r_2} + \frac{k_0}{\pi}\int_{-\pi}^{\pi} \sec^2\theta\, d\theta \int_0^\infty \frac{e^{k[i\varpi+(z-\zeta)]}}{k-k_0\sec^2\theta}\, dk \quad ; \text{ where}$$

$$r_1 = \sqrt{x^2+y^2+(z-\zeta)^2} \quad ; \quad r_2 = \sqrt{x^2+y^2+(z+\zeta)^2}$$

Now, the remaining condition is the radiation condition which states that

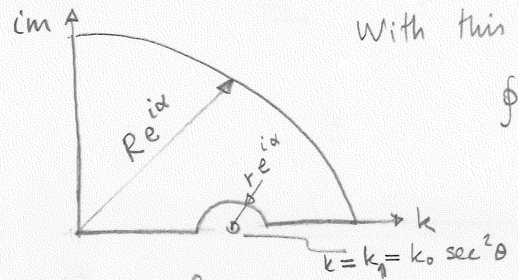
$G \to 0$ for $x \to \infty$ and G is bounded as $x \to -\infty$. In order to take the limits of $\sec\theta$ must be evaluated in one quadrant so that integration interval $[-\pi, \pi]$ should be converted, for example, to $[0, \pi/2]$:

$$G = -\frac{1}{r_1} + \frac{1}{r_2} + \frac{k_0}{\pi}\int_0^{\pi/2} \sec^2\theta \, d\theta \int_0^{\infty} \left\{ \frac{\exp[ik(x\cos\theta + y\sin\theta)] + \exp[ik(x\cos\theta - y\sin\theta)]}{k - k_0\sec^2\theta} dk + \right.$$

$$\left. + \frac{\exp[-ik(x\cos\theta + y\sin\theta)] + \exp[-ik(x\cos\theta - y\sin\theta)]}{k - k_0\sec^2\theta} \right\} \exp[k(z+\xi)]\, dk.$$

The above integrals can be taken by counter integral technique which is of the following type: $\displaystyle\int \frac{e^{\eta[(z+\xi) + i\varpi]}}{\eta - k_0\sec^2\theta} d\eta$

k is taken as complex variable ($\eta = k + im$). The integrand has a pole at $\eta = k_0\sec^2\theta$



With this type of contour: $\oint f(\eta)\, d\eta = 0$ where $r \to 0$ and $R \to \infty$

Following this countour we have

$$\lim_{r\to 0}\left[\int_0^{k_1 - r} f(k)\, dk - \int_0^{\pi} f[(k_1 + re^{i\alpha})]\, ire^{i\alpha} d\alpha + \int_{k_1+r}^{\infty} f(k)\, dk\right] +$$

$$\lim_{R\to\infty}\left[\int_0^{\pi/2} f(Re^{i\alpha})\, iRe^{i\alpha} d\alpha + \int_{\infty}^{0} f(im)\, i\, dm\right] = 0$$

$$\therefore \int_0^{\infty} \frac{e^{k[i\varpi + (z+\xi)]}}{k - k_0\sec^2\theta} dk = \int_0^{\infty} \frac{e^{[im(z+\xi) - m\varpi]}}{[m + ik_0\sec^2\theta]} dm + i\pi e^{k_0\sec^2\theta[(z+\xi) - i\varpi]}$$

Therefore the term in Green's function is expressed as

$$\int_{-\pi}^{\pi} \sec^2\theta\, d\theta \int_0^{\infty} \frac{e^{[i\varpi + (z+\xi)]}}{k - k_0\sec^2\theta} dk =$$

$$4\int_0^{\pi/2} \sec^2\theta\, d\theta \int_0^{\infty} \frac{e^{[-mx\cos\theta]}\cosh(my\sin\theta)\,[\cos m(z+\xi) + k_0\sec^2\theta \sin m(z+\xi)]}{m^2 + k_0^2\sec^2\theta} dm$$

$$-4\pi\int_0^{\pi/2} e^{k_0\sec^2\theta(z-\xi)} \sin(k_0 x\sec\theta)\cos(k_0 y\sec^2\theta\sin\theta)\sec^2\theta\, d\theta.$$

As $x \to -\infty$, only the last term is not vanished, but this contradicts with the radiation condition, so that we change the sign of it and add to the initial solution.

So the resultant Green function;

$$G(x,y,z;\xi,\eta,\zeta) = -\frac{1}{r_1} + \frac{1}{r_2} + \frac{4k_0}{\pi}\int_0^{\pi/2} \sec^2\theta \, d\theta \int_0^{\infty} \frac{\exp[k(z-\zeta)]\cos(kx\cos\theta)\cos(ky\sin\theta)}{k-k_0\sec^2\theta}\, dk$$

$$+ 4k_0 \int_0^{\pi/2} \exp[k_0\sec^2\theta(z-\zeta)]\sin[k_0 x\sec\theta]\cos[k_0 y\sec^2\theta\sin\theta]\sec^2\theta \, d\theta$$

where the source point is located at $(0, 0, \zeta)$ for $\zeta < 0$.

A Numerical Approach for 2.D Ship-Wave Problem.

First lay down the boundary value Problem:

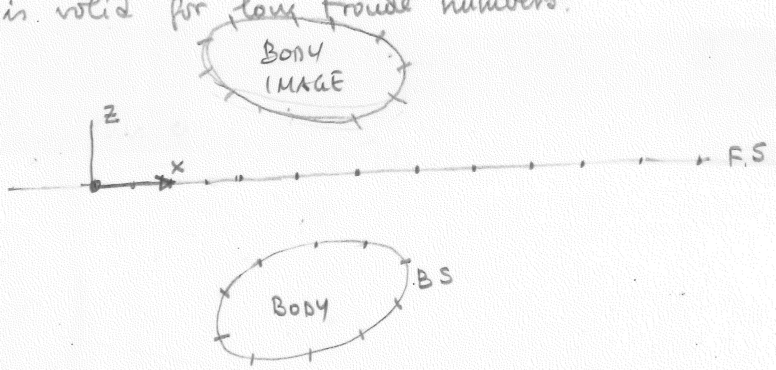
$$\nabla^2\Phi = 0$$

$\Phi_n = 0$ on the body

$g\zeta + \frac{1}{2}(\Phi_x^2 + \Phi_z^2 - U_\infty^2) = 0$
$\Phi_x \zeta_x - \Phi_z = 0$ } on the free surface.

$\nabla\Phi = (U_\infty, 0, 0)$ at $-\infty$
$\Phi = 0\,(1)$ at $+\infty$ } Rad. cond.

We propose a double model solution for this purpose which is valid for low Froude numbers.



In this case the velocity potential can be given as;

$$\Phi(x,z) = U_\infty x + \oint_{BS} s(x',z')(\ln r + \ln \bar{r})\, dl' + \int_{\substack{FS \\ (z=0)}} s(x',y')\ln r \, dl'$$

where $r = [(x-x')^2 + (z-z')^2]^{1/2}$

$\bar{r} = [(x-x')^2 + (z+z')^2]^{1/2}$ for the image.

This class of potential function satisfies Laplace's Eq. and bottom b. cond. We have to impose BS & FS conditions while assuring the radiation condition. Linearized FS condition is taken as;

$$U_\infty^2 \Phi_{xx} + g\Phi_z = 0 \quad ; z = 0.$$

We know from previous lectures that B_{ij} and C_{ij} are the x and z direction-velocities, respectively, at point i due to j^{th} panel having unit source strength distribution (line segment in 2-D). $\overline{B}_{ij}$ and $\overline{C}_{ij}$ represent body image influence coefficients and B'_{ij} and C'_{ij} represent the influences of F.S. panels. Therefore kinematic BS boundary condition;

$$\nabla\Phi\cdot\vec{n}_i = U_\infty n_{i_x} + s_i\left[\pi + \overline{B}_{ii}n_{i_x} + \overline{C}_{ii}n_{i_y}\right] + \sum_{\substack{j=1\\(i\neq j)}}^{L} s_j\left[(B_{ij}+\overline{B}_{ij})n_{i_x} + (C_{ij}+\overline{C}_{ij})n_{i_y}\right]$$

$$+\sum_{j=L+1}^{M} s_j\left[B'_{ij}n_{i_x} + C'_{ij}n_{i_y}\right] = 0 \qquad ; \qquad i = 1, 2, \ldots, L$$

Now the F.S. boundary condition requires the derivative of $u = \Phi_x$; u_x. This is performed by Dawson's 4-point backward scheme which is also supposed to satisfy radiation condition as:

$$(u_x)_i = \mathcal{A}_i u_i + \mathcal{B}_i u_{i-1} + \mathcal{C}_i u_{i-2} + \mathcal{D}_i u_{i-3}$$

where $Q_i = (x_{i-1}-x_i)(x_{i-2}-x_i)(x_{i-3}-x_i)(x_{i-2}-x_{i-1})(x_{i-3}-x_{i-2})(x_{i-3}-x_{i-1})(x_{i-3}+x_{i-2}+x_{i-1}-3x_i)$

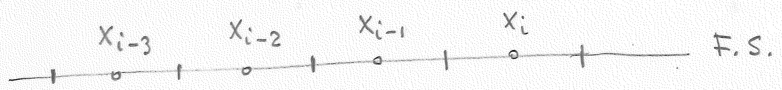
$$\mathcal{D}_i = (x_{i-1}-x_i)^2(x_{i-2}-x_i)^2(x_{i-2}-x_{i-1})(x_{i-2}+x_{i-1}-2x_i)/Q_i$$

$$\mathcal{C}_i = -(x_{i-1}-x_i)^2(x_{i-3}-x_i)^2(x_{i-3}-x_{i-1})(x_{i-3}+x_{i-1}-2x_i)/Q_i$$

$$\mathcal{B}_i = (x_{i-3}-x_i)^2(x_{i-2}-x_i)^2(x_{i-3}-x_{i-2})(x_{i-3}+x_{i-2}-2x_i)/Q_i$$

$$\mathcal{A}_i = -(\mathcal{B}_i + \mathcal{C}_i + \mathcal{D}_i)$$

when the points x_i's are the node points of the elements at F.S. such that



By using the coefficients of the differentiation scheme, we are now able to write the F.S. condition numerically:

$$U_\infty^2\left\{\sum_{j=1}^{L} s_j\left[\mathcal{A}_i(B_{ij}+\overline{B}_{ij}) + \mathcal{B}_i(B_{i-1,j}+\overline{B}_{i-1,j}) + \mathcal{C}_i(B_{i-2,j}+\overline{B}_{i-2,j}) + \mathcal{D}_i(B_{i-3,j}+\overline{B}_{i-3,j})\right]\right.$$

$$\left.+\sum_{j=L+1}^{M} s_j\left[\mathcal{A}_i B'_{ij} + \mathcal{B}_i B'_{i-1,j} + \mathcal{C}_i B'_{i-2,j} + \mathcal{D}_i B'_{i-3,j}\right]\right\} - g\pi s_i = 0 \qquad ; \quad i = L+1, L+2, \ldots, M.$$

Once the system of MxM equations is solved, then all s_i's and accordingly Φ is solved so that pressure and velocity field can be calculated.

For example the linearized pressure;

$$p = -\rho U_\infty \Phi_x$$