BASIC FLOW DESCRIPTIONS

Lagrangian Description : When a flow is modeled by fluid particles which carry flow properties, such as density, pressure, velocity etc., and if we follow the advance of the particles (its properties may change in time meanwhile), then we end up with Lagrangian description. The method of Smoothed Particle Hydrodynamics which represents the fluid flow by a large number of particles with properties $\rho_i(t)$, $\vec{v}_i(t)$, $p_i(t)$ strictly uses Lagrangian description and follow the trajectories of the fluid particles. In that sense it is simple to model the flow - as conservation of mass and momentum apply to each particle -, but its computational cost is too high!

Eulerian Description : Instead of tracking each fluid particle in a fluid flow, Eulerian description is focused on the evolution of the flow properties at every point in space as time advances, such as $\vec{v}(\vec{x},t)$, $\rho(\vec{x},t)$, $p(\vec{x},t)$.

For example ; a wave probe fixed in space is indeed an Eulerian measuring device, whereas a neutrally bouyant wave probe is a Lagrangian measuring device, i.e. fixed probe provides records in $\xi(\vec{x},t)$ - which defines wave elevation at position $\vec{x}$ and time t, and bouyant probe provides data in pairs ; $\xi(t)$ and $\vec{x}(t)$. A streamline on which the fluid velocity $\vec{v}$ at a given time is tangent is an Eulerian definition. A path line which is the trajectory of a given particle in time is a Lagrangian definition.

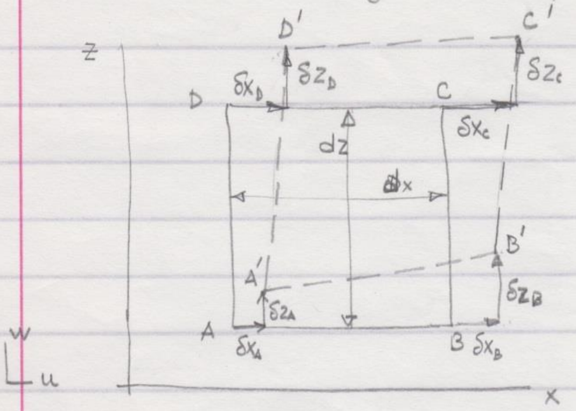
Continuous Flow : A continuous flow requires the following ;

$\vec{v}(\vec{x},t)$ should be finite and a continuous function of $\vec{x}$ and t, that is $\nabla\vec{v}$ and $\frac{\partial\vec{v}}{\partial t}$ are finite. Thus (without proof), if the flow is continuous two particles that are neighbors will always remain neighbors.

In a finite fluid volume - if no segment of fluid can be joined or broken apart - ; it is called material volume and will remain material. The interface between two material volumes is a material surface and will remain material. A free surface - between air and water - is an example of a material surface.

GOVERNING EQ. FOR INCOMPRESSIBLE FLUIDS

First let's investigate the displacement of an elementary 2-D fluid volume



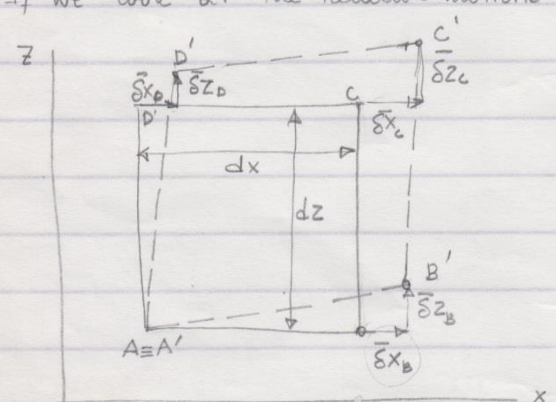
The idea in the present analysis is that fluid volume ABCD at time t moves to new location $A'B'C'D'$ at time $t+\delta t$.

For example point A moves to A';

$$x_{A'} = x_A + \delta x_A = x_A + u_A \delta t$$

$$z_{A'} = z_A + \delta z_A = z_A + w_A \delta t \quad , \text{etc.}$$

If we look at the relative motions when considering $A \equiv A'$



$$\bar{\delta} x_B = \delta x_B - \delta x_A = u_B \delta t - u_A \delta t = (u_B - u_A)\delta t$$

$$\bar{\delta} z_B = \delta z_B - \delta z_A = w_B \delta t - w_A \delta t = (w_B - w_A)\delta t$$

Similar expressions for points C & D.

Take the definition of partial derivatives into account;

$$\bar{\delta} x_B \approx \frac{\partial u}{\partial x} dx\, \delta t \qquad \text{since} \quad u_B \approx u_A + \frac{\partial u}{\partial x}(x_B - x_A) = u_A + \frac{\partial u}{\partial x} dx$$

Similarly;

$$\bar{\delta} z_B \approx \frac{\partial w}{\partial x} dx\, \delta t$$

But

$$\bar{\delta} x_C \approx \left[\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial z} dz \right] \delta t$$

?

$$\text{since} \quad u_C \approx u_A + \frac{\partial u}{\partial x}(x_C - x_A) + \frac{\partial u}{\partial z}(z_C - z_A)$$

$$\approx u_A + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial z} dz \qquad \text{and}$$

$$\bar{\delta} z_C \approx \left[\frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial z} dz \right] \delta t$$

So on;

$$\bar{\delta} x_D \approx \frac{\partial u}{\partial z} dz\, \delta t$$

$$\bar{\delta} z_D \approx \frac{\partial w}{\partial z} dz\, \delta t$$

∴ From the above expressions;

$$\bar{\delta} x_C \approx \bar{\delta} x_B + \bar{\delta} x_D$$

$$\bar{\delta} z_C \approx \bar{\delta} z_B + \bar{\delta} z_D$$

Up to now we use no physical restriction. Thus the 1st physical restriction is:

(ii) Conservation of mass (or volume) of the rectangle ABCD.

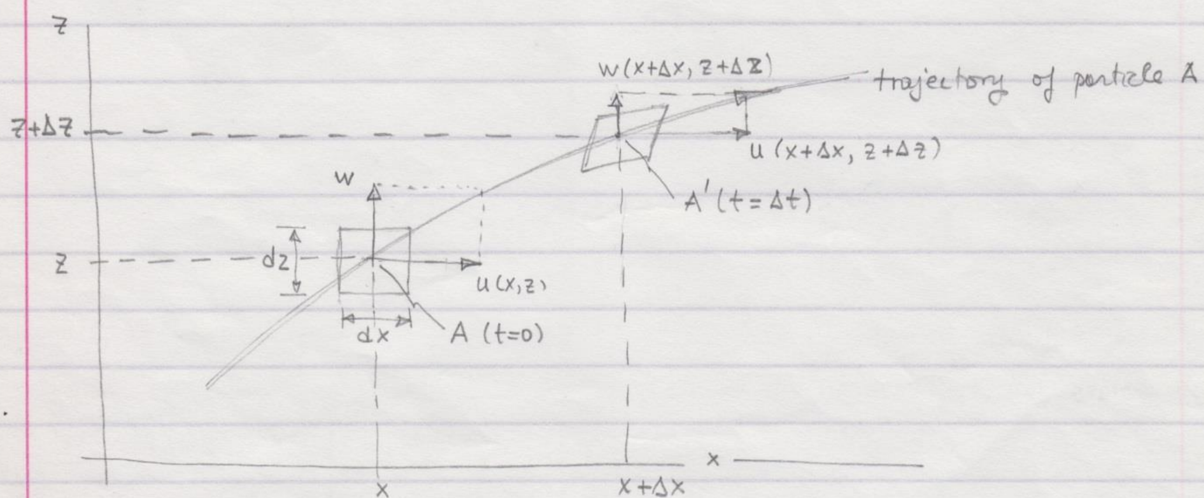
"For Home": Assume $\bar{\delta}x_i, \bar{\delta}z_i \ll dx, dz$ that is δt is small.

(2.1) Approximate the area $A'B'C'D'$ as $E' \approx A'D' \cdot A'B'$

Neglect nonlinear terms such as $\bar{\delta}x_i \cdot \bar{\delta}z_i$

and show: $\nabla \cdot \vec{V} = 0$ (Conservation of mass equation / continuity eq.).

ACCELERATED MOTION OF A FLUID PARTICLE (2-D solution)



Particle A having a mass of $dm = \rho\, dx\, dz$, where ρ is the density, is initially at (x, z) at time $t=0$.

Acceleration $\vec{a} = a_x \vec{i} + a_z \vec{k} \approx \dfrac{\Delta \vec{q}}{\Delta t} = \dfrac{\Delta(u\vec{i} + w\vec{k})}{\Delta t} = \dfrac{\Delta u}{\Delta t}\vec{i} + \dfrac{\Delta w}{\Delta t}\vec{k}$

In detail: $a_x = \dfrac{\Delta u}{\Delta t} = \dfrac{u(x+\Delta x, z+\Delta z) - u(x,z)}{\Delta t}$

$$= \frac{\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial z}\Delta z}{\Delta t}$$

since $\Delta x = u \cdot \Delta t$ (or $u = \Delta x / \Delta t$)

$\Delta z = w \cdot \Delta t$

$\therefore \quad a_x = u \dfrac{\partial u}{\partial x} + w \dfrac{\partial u}{\partial z}$

Similarly $\quad a_z = u \dfrac{\partial w}{\partial x} + w \dfrac{\partial w}{\partial z}$

Here if u and w are also functions of time (unsteady flow) then we arrive at:

$$a_x = \boxed{\frac{\partial u}{\partial t}} + \boxed{u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}}$$

$$a_z = \boxed{\frac{\partial w}{\partial t}} + \boxed{u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}}$$

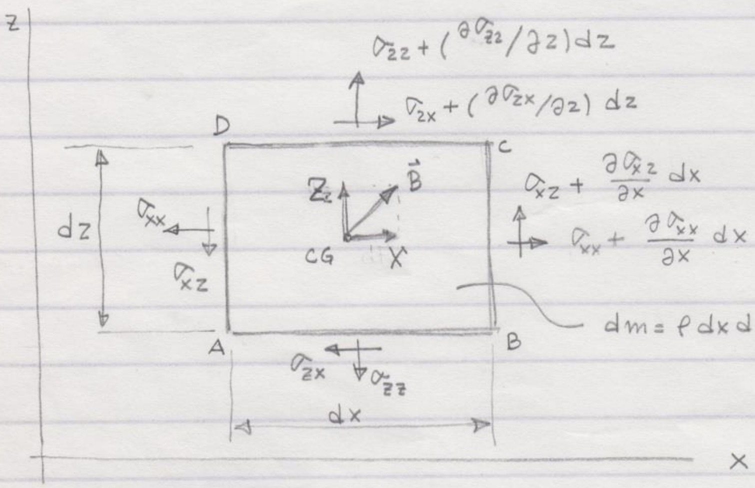
unsteady terms — convective terms

$$\Rightarrow \quad \vec{a} = \frac{D\vec{q}}{Dt} = \underbrace{\frac{\partial \vec{q}}{\partial t}}_{\text{local rate of change}} + \underbrace{\vec{q}\cdot\nabla\vec{q}}_{\text{convective rate of change.}}$$

where $\vec{q} = u\vec{i} + w\vec{k}$

Operator $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \vec{q}\cdot\nabla$ is called substantial derivative or material derivative.

FORCES ON FLUID PARTICLE (2-D)



Here for example the stress σ_{zx} acts in the x direction upon a surface of constant z

$\vec{B}$: body force per unit volume

$dm = \rho\,dx\,dz$ (mass of the fluid particle)

Force along x : $dF_x = \left[\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{zx}}{\partial z} + X\right]dx\,dz$

" z : $dF_z = \left[\dfrac{\partial\sigma_{zz}}{\partial z} + \dfrac{\partial\sigma_{xz}}{\partial x} + Z\right]dx\,dz$

When the medium is solid then $dF_x = dF_z = 0$. But in the case of fluids, we need to apply Newton's Law:

$$dF_x = dm\cdot a_x$$

$$dF_z = dm\cdot a_z$$

$$\therefore \quad \rho\frac{Du}{Dt} = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right] = \left[\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{zx}}{\partial z} + X\right]$$

$$\rho\frac{Dw}{Dt} = \rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right] = \left[\frac{\partial\sigma_{zz}}{\partial z} + \frac{\partial\sigma_{xz}}{\partial x} + Z\right]$$

Here if the Body Force is derived from a conservative field, then

$X = 0$ and $Z = -\rho g$

Meanwhile conservation of angular momentum leads to

$$\sigma_{xz} = \sigma_{zx}$$

Newton's Definition of Viscosity

Translated from "Principia", published in 1687:

"The resistance which arises from the lack of slipperiness of the parts of the liquid, other things being equal, is proportional to the velocity with which the parts of the liquid are separated from one another."

Newtonian Viscosity, the viscosity of a Newtonian or ideal liquid, is one where the viscosity is constant as the rate of shear increases, ie the shear stress is directly proportional to the shear rate:

$$\sigma = \eta\dot{\gamma}$$

where

σ = **shear stress**

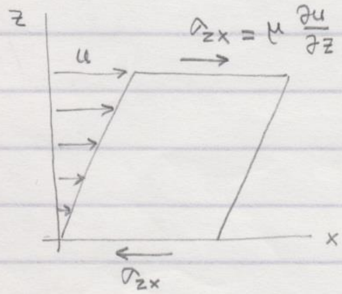
$\dot{\gamma}$ = **shear rate**

η = **Newtonian viscosity**

CONSTITUTIVE RELATIONS in NEWTONIAN FLUIDS/or
(Stress relations in Newtonian Fluids)

We must have at hand a relation between stress tensor σ_{ij} $(1 \to x,\ 2 \to y,\ 3 \to z)$ and kinematic properties of the fluid. If the fluid is at rest or the fluid is inviscid, Then: $\sigma_{ij} = -p\,\delta_{ij}$

Viscous stresses is assumed to be a linear function of velocity gradients.



$\sigma_{zx} = \mu \dfrac{\partial u}{\partial z}$ μ: coefficient of viscosity (or dynamic viscosity).

$$\therefore \quad \sigma_{ij} = -p\,\delta_{ij} + 2\mu\,\dot{\varepsilon}_{ij}$$
$$= -p\,\delta_{ij} + 2\mu \Big\{ \underbrace{\tfrac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big]}_{\text{rate of strain}} \Big\}$$

in 3-D
$$\sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} + \mu \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & 2\frac{\partial w}{\partial z} \end{bmatrix}$$

$\therefore$ By using these relations, we can go back to Newton's equations to obtain the RHS of the equation: $-\dfrac{\partial p}{\partial x} + 2\mu \dfrac{\partial^2 u}{\partial x^2} + \mu\left(\dfrac{\partial^2 u}{\partial z^2} + \dfrac{\partial^2 w}{\partial z\,\partial x}\right)$

from continuity eq: $\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \Rightarrow \dfrac{\partial w}{\partial z} = -\dfrac{\partial u}{\partial x}$ $\quad\therefore\ \dfrac{\partial^2 w}{\partial x\,\partial z} = -\dfrac{\partial^2 u}{\partial x^2}$

$$\therefore \quad -\frac{\partial p}{\partial x} + \mu \nabla^2 u$$

Now we can write down Navier-Stokes Equations: (in 3-D)

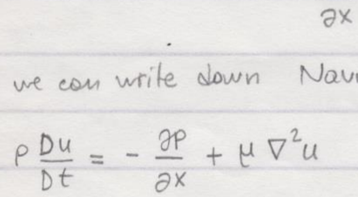
$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u$$
$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \mu \nabla^2 v$$
$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \mu \nabla^2 w$$

in vector form $\rho \dfrac{D\vec{q}}{Dt} = -\nabla p + \mu \nabla^2 \vec{q}$ or

$$\frac{D\vec{q}}{Dt} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{q}$$

ν: kinematic viscosity.

for Home:
2.2



By nondimensionalising $\nabla = \dfrac{1}{L}\nabla'$

$p' = \dfrac{p}{\rho V_\infty^2}$, $\vec{q}\,' = \dfrac{\vec{q}}{V_\infty}$

$t' = \dfrac{t}{(L/V_\infty)}$

Then show that $\dfrac{Dq'}{Dt'} = -\nabla' p' + \dfrac{1}{Re}\nabla'^2 \vec{q}\,'$