TEL502E – Homework 1

Due 25.02.2014

1. (a) Suppose that X is a non-negative random variable with a pdf $f_X(t)$ (that is, $f_X(t) = 0$ for $t < 0$). Show that, for any $n > 0$ and $s > 0$,

$$P\big(\{X \geq s\}\big) \leq \frac{\mathbb{E}(X^n)}{s^n}.$$

   (b) Using part (a), show that for an arbitrary random variable Y with $\mathbb{E}(Y) = \mu$,

$$P\big(\{\mu - \epsilon \leq Y \leq \mu + \epsilon\}\big) \geq 1 - \frac{\mathrm{var}(Y)}{\epsilon^2}.$$

   (c) Suppose that $X_1, X_2, \ldots$ is a sequence of iid random variables with $\mathbb{E}(X_i) = \mu$, $\mathrm{var}(X_i) = \sigma^2$. Also let,

$$Z_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

   Compute $\mathbb{E}(Z_n)$ and $\mathrm{var}(Z_n)$.

   (d) Show that

$$\lim_{n \to \infty} P\big(\{\mu - \epsilon < Z_n < \mu + \epsilon\}\big) = 1,$$

   for any $\epsilon > 0$.

2. (a) Show that if $\mathrm{var}(Y) = 0$, then $P\big(\{Y = \mathbb{E}(Y)\}\big) = 1$.

   (b) Show that if $\mathbb{E}(Y^2) = 0$, then $P\big(\{Y = 0\}\big) = 1$.

3. Suppose X is a discrete random variable, taking values on the set of integers $\mathbb{Z}$. Suppose we are testing whether X is distributed according to the probability mass function (PMF) $P_0(t)$ (this is the null hypothesis) or it's distributed according to the PMF $P_1(t)$ (this is the alternative hypothesis). We somehow form the acceptance region $C \subset \mathbb{Z}$ such that if a realization of X, say x falls in C, we accept the null hypothesis, and reject it otherwise. Also, let $p_I(C)$ and $p_{II}(C)$ denote the probabilities of type-I and type-II errors of this test. Below, the parts (a) and (b) are independent of each other.

   (a) Suppose we discover that for some $r \in (\mathbb{Z} \backslash C)$ and $a_1, a_2, \ldots a_n \in C$,

      (i) $P_0(r) = \sum_{i=1}^{n} P_0(a_i)$, and

      (ii) $\big(P_0(r)/P_1(r)\big) > \big(P_0(a_i)/P_1(a_i)\big)$ for $i = 1, 2, \ldots, n$.

   Based on this observation, we decide to update the acceptance region and use $D = C \cup \{r\} \backslash \{a_1, \ldots, a_n\}$ as the acceptance region (i.e., we remove a_i's and include r in the new acceptance region). Let $p_I(D)$ and $p_{II}(D)$ denote the type-I and type-II error probabilities for this updated test. Show that $p_I(D) \leq p_I(C)$, and $p_{II}(D) < p_{II}(C)$.

   (b) Suppose we find that for any $r \in (\mathbb{Z} \cap C^c)$, and $a \in C$, the inequality

$$\frac{P_0(r)}{P_1(r)} < \frac{P_0(a)}{P_1(a)} \tag{1}$$

   is satisfied. Consider now another test than the one described above with an acceptance region given as D, whose type-I and type-II error probabilities are given as $p_I(D)$ and $p_{II}(D)$ respectively. Show that if $p_I(D) \leq p_I(C)$, then $p_I(D) > p_{II}(D)$.