TEL502E – Homework 1

Due 25.02.2014

1. (a) Suppose that X is a non-negative random variable with a pdf $f_X(t)$ (that is, $f_X(t) = 0$ for $t < 0$). Show that, for any $n > 0$ and $s > 0$,

$$P\big(\{X \geq s\}\big) \leq \frac{\mathbb{E}(X^n)}{s^n}.$$

(b) Using part (a), show that for an arbitrary random variable Y with $\mathbb{E}(Y) = \mu$,

$$P\big(\{\mu - \epsilon \leq Y \leq \mu + \epsilon\}\big) \geq 1 - \frac{\mathrm{var}(Y)}{\epsilon^2}.$$

(c) Suppose that $X_1, X_2, \ldots$ is a sequence of iid random variables with $\mathbb{E}(X_i) = \mu$, $\mathrm{var}(X_i) = \sigma^2$. Also let,

$$Z_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Compute $\mathbb{E}(Z_n)$ and $\mathrm{var}(Z_n)$.

(d) Show that

$$\lim_{n \to \infty} P\big(\{\mu - \epsilon < Z_n < \mu + \epsilon\}\big) = 1,$$

for any $\epsilon > 0$.

Solution. (a) Keeping in mind that $s > 0$, we have,

$$\begin{aligned}
P\big(\{X \geq s\}\big) &= \int_s^\infty f_X(t)\, dt \\
&\leq \int_0^s \frac{t^n}{s} f_X(t)\, dt + \int_s^\infty f_X(t)\, dt \\
&\leq \int_0^s \frac{t^n}{s} f_X(t)\, dt + \int_s^\infty \frac{t^n}{s^n} f_X(t)\, dt \\
&= \frac{\mathbb{E}(X^n)}{s^n}.
\end{aligned}$$

This inequality is known as Markov's inequality.

(b) Using Y suppose we define a new random variable as $Z = |Y - \mu|$. Then, using Markov's inequality with $n = 2$, we have,

$$P\big(\{Z \geq \epsilon\}\big) \leq \frac{\mathbb{E}(Z^2)}{\epsilon^2} = \frac{\mathrm{var}(Y)}{\epsilon^2}.$$

Observe now that

$$P\big(\{Z \geq \epsilon\}\big) + P\big(\{Z < \epsilon\}\big) = 1,$$

since the two events partition the sample space. This implies,

$$P\big(\{Z < \epsilon\}\big) \geq 1 - \frac{\mathrm{var}(Y)}{\epsilon^2}$$

But now observe that

$$\{Z < \epsilon\} = \{|Y - \mu| < \epsilon\} = \{-\epsilon < Y - \mu < \epsilon\} = \{\mu - \epsilon \leq Y \leq \mu + \epsilon\}.$$

Thus the claim follows. This inequality (or an equivalent version) is known as Chebyshev's inequality.

(c) First,

$$\mathbb{Z}_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(X_i) = \mu.$$

Now, note when the random variables are independent, we can add their variances. Thus,

$$\mathrm{var}(Z_n) = \sum_{i=1}^{n} \mathrm{var}\big(X_i/n\big) = \sum_{i=1}^{n} \frac{\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

(d) Since $\mathbb{E}(Z_n) = \mu$, we can use the result of part (b). That gives,

$$P\big(\{\mu - \epsilon < Z_n < \mu + \epsilon\}\big) \geq 1 - \frac{\sigma^2}{\epsilon^2\, n}.$$

Letting $n \to \infty$, the right hand side converges to 1 and the claim follows.

2. (a) Show that if $\mathrm{var}(Y) = 0$, then $P\big(\{Y = \mathbb{E}(Y)\}\big) = 1$.
   (b) Show that if $\mathbb{E}(Y^2) = 0$, then $P\big(\{Y = 0\}\big) = 1$.

Solution. (a) Let A be the event of interest defined as,

$$A = \{Y = \mathbb{E}(Y)\}.$$

Instead of $P(A)$, we will compute the $P(A^c)$. Now observe that,

$$A^c = \{|Y - \mathbb{E}(Y)| > 0\} = \cup_{n=1}^{\infty} \underbrace{\{|Y - \mathbb{E}(Y)| > 1/n\}}_{B_n}.$$

But by part (b) of Question-1, we have that $P(B_n) = 0$. Therefore,

$$P(A^c) \leq \sum_{n=1}^{\infty} P(B_n) = 0.$$

Since $P(A^c) \geq 0$ by definition, it follows that $P(A^c) = 0$. Thus, $P(A) = 1 - P(A^c) = 1$.

(b) Since $\mathrm{var}(Y) = \mathbb{E}(Y^2) - \big(\mathbb{E}(Y)\big)^2 \geq 0$, the condition '$\mathbb{E}(Y^2) = 0$' implies that $\mathbb{E}(Y) = 0$. The desired equality follows therefore follows from part (a).

3. Suppose X is a discrete random variable, taking values on the set of integers $\mathbb{Z}$. Suppose we are testing whether X is distributed according to the probability mass function (PMF) $P_0(t)$ (this is the null hypothesis) or it's distributed according to the PMF $P_1(t)$ (this is the alternative hypothesis). We somehow form the acceptance region $C \subset \mathbb{Z}$ such that if a realization of X, say x falls in C, we accept the null hypothesis, and reject it otherwise. Also, let $p_I(C)$ and $p_{II}(C)$ denote the probabilities of type-I and type-II errors of this test. Below, the parts (a) and (b) are independent of each other.

   (a) Suppose we discover that for some $r \in (\mathbb{Z} \backslash C)$ and $a_1, a_2, \ldots a_n \in C$,
      (i) $P_0(r) = \sum_{i=1}^{n} P_0(a_i)$, and
      (ii) $\big(P_0(r)/P_1(r)\big) > \big(P_0(a_i)/P_1(a_i)\big)$ for $i = 1, 2, \ldots, n$.

   Based on this observation, we decide to update the acceptance region and use $D = C \cup \{r\} \backslash \{a_1, \ldots, a_n\}$ as the acceptance region (i.e., we remove a_i's and include r in the new acceptance region). Let $p_I(D)$ and $p_{II}(D)$ denote the type-I and type-II error probabilities for this updated test. Show that $p_I(D) \leq p_I(C)$, and $p_{II}(D) < p_{II}(C)$.

   (b) Suppose we find that for any $r \in (\mathbb{Z} \cap C^c)$, and $a \in C$, the inequality

$$\frac{P_0(r)}{P_1(r)} < \frac{P_0(a)}{P_1(a)} \tag{1}$$

   is satisfied. Consider now another test than the one described above with an acceptance region given as D, whose type-I and type-II error probabilities are given as $p_I(D)$ and $p_{II}(D)$ respectively. Show that if $p_I(D) \leq p_I(C)$, then $p_I(D) > p_{II}(D)$.

Solution. Notice that, in this setting, for an acceptance region denoted as C, the type-I and type-II error probabilities are given by

$$p_I(C) = \sum_{x \in \mathbb{Z} \cap C^c} P_0(x), \quad p_{II}(C) = \sum_{x \in C} P_1(x).$$

(a) First, observe that, by condition (i), we have,

$$p_I(D) - p_I(C) = \sum_{z \in \mathbb{Z} \cap D^c} P_0(z) - \sum_{z \in \mathbb{Z} \cap C^c} P_0(z) = \sum_{i=1}^{n} P_0(a_i) - P_0(r) = 0.$$

Rewriting (ii) as,

$$\frac{P_1(a_i)}{P_1(r)} > \frac{P_0(a_i)}{P_0(r)}, \text{ for } i = 1, 2, \dots, n,$$

and summing over i, we obtain,

$$\frac{\sum_{i=1}^{n} P_1(a_i)}{P_1(r)} > \frac{\sum_{i=1}^{n} P_0(a_i)}{P_0(r)} = 1,$$

where we made use of (i) again. Now observe that,

$$p_{II}(D) - p_{II}(C) = \sum_{z \in D} P_1(z) - \sum_{z \in C} P_1(z) = P_1(r) - \sum_{i=1}^{n} P_1(a_i) < 0.$$

(b) Suppose $p_I(D) \leq p_I(c)$. This implies,

$$p_I(D) - p_I(C) = \sum_{x \in \mathbb{Z} \cap D^c} P_0(x) - \sum_{x \in \mathbb{Z} \cap C^c} P_0(x) = \sum_{x \in D^c \cap C} P_0(x) - \sum_{x \in C^c \cap D} P_0(x) \leq 0,$$

or

$$\frac{\sum_{x \in C^c \cap D} P_0(x)}{\sum_{x \in D^c \cap C} P_0(x)} \geq 1. \tag{2}$$

Now observe similarly that

$$p_{II}(D) - p_{II}(C) = \sum_{x \in D} P_1(x) - \sum_{x \in C} P_1(x) = \sum_{x \in D \cap C^c} P_1(x) - \sum_{x \in C \cap D^c} P_1(x).$$

Thus, if we can show that

$$\frac{\sum_{x \in D \cap C^c} P_1(x)}{\sum_{x \in C \cap D^c} P_1(x)} > 1, \tag{3}$$

we are done.
For this, we first rewrite (1) in a different form. Note that if $x \in C^c$ and $c \in C$, then

$$P_1(c)\, P_0(x) < P_0(c)\, P_1(x).$$

Fixing $c \in C$, we obtain,

$$P_1(c) \left(\sum_{x \in D \cap C^c} P_0(x) \right) < P_0(c) \left(\sum_{x \in D \cap C^c} P_1(x) \right).$$

Now taking the terms inside the parentheses as fixed, we can write,

$$\left(\sum_{c \in D^c \cap C} P_1(c) \right) \left(\sum_{x \in D \cap C^c} P_0(x) \right) < \left(\sum_{c \in D^c \cap C} P_0(c) \right) \left(\sum_{x \in D \cap C^c} P_1(x) \right).$$

Rewriting and using (2), we obtain (3) :

$$\frac{\sum_{x \in D \cap C^c} P_1(x)}{\sum_{x \in D^c \cap C} P_1(x)} > \frac{\sum_{x \in D \cap C^c} P_0(x)}{\sum_{x \in D^c \cap C} P_0(x)} \geq 1.$$

Notice that throughout, I assumed that $P_i(x)$ is non-zero as long as $x \in \mathbb{Z}$. I leave it to you to consider how to modify the argument if $P_i(x) = 0$ from some x.