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Stability and bifurcation of a class of discrete-time neural networks

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Abstract

In this paper, a class of discrete-time system modelling a network with two neurons is considered. Its linear stability is investigated and Neimark–Sacker bifurcation (also called Hopf bifurcation for map) is demonstrated by analyzing the corresponding characteristic equation. In particular, the explicit formula for determining the direction of Neimark–Sacker bifurcation and the stability of periodic solution is obtained by using the normal form method and the center manifold theory for discrete time system developed by Kuznetsov. The theoretical analysis is verified by numerical simulations.

Keywords: Neural network; Stability; Neimark–Sacker bifurcation; Discrete-time system

1. Introduction

In the past few decades, neural networks have received intensive interest due to their wide applications, such as, pattern recognition, associative memory and combinational optimization, and its dynamical behavior plays an important role. Many works [1–15] have been published to investigate the dynamics of neural networks since Hopfield [16] constructed a simplified neural network model. Neural networks with one or two neurons are prototypes to understand the dynamics of large-scale networks, many results have been made for such simplified networks [17,3–7,9,10,13,14,18].

In 2003, Yuan and Huang [19] studied the asymptotical behavior of the following difference system:

$$
\begin{aligned}
x_1(n+1) &= \beta x_1(n) + a_{11} f(x_1(n)) + a_{12} f(x_2(n)), \\
x_2(n+1) &= \beta x_2(n) + a_{21} f(x_1(n)) + a_{22} f(x_2(n)), \quad n = 0, 1, 2, \ldots,
\end{aligned}
$$

where $\beta \in (0,1)$ is a constant and $f: R \to R$ is the activation function given by the piecewise constant McCulloch–Pitts nonlinearity

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$$f(u) = \begin{cases} -1, & u > \sigma, \\ +1, & u \leqslant \sigma, \end{cases}$$

where $\sigma \in R$ is a constant, and acts as the threshold. In 2004, Yuan et al. [9] considered the following system:

$$\begin{aligned} x_1(n+1) &= \beta x_1(n) + (1-\beta)f(\alpha x_1(n)) + (1-\beta)f(\gamma_1 x_2(n)), \\ x_2(n+1) &= \beta x_2(n) - (1-\beta)f(\gamma_2 x_1(n)) + (1-\beta)f(\alpha x_2(n)), \quad n = 0,1,2,\ldots, \end{aligned}$$

where $\beta \in (0,1)$ is internal decay of the neurons, the constant $\alpha > 0$ and $\gamma_i\ (i = 1,2)$ denote the gain parameters, $f: R \to R$ is a continuous transfer function and $f(0) = 0$. They discussed the global and local stability of the equilibrium, which gave some sufficient conditions to guarantee the existence of bifurcation, meanwhile got a formula to determine the direction and stability of bifurcation. In 2005, Yuan et al. [10] introduced a more general model based on [9]. They studied the following discrete-time neural network model with self-connection in the following form

$$\begin{aligned} x_1(n+1) &= \beta x_1(n) + a_{11}f_1(x_1(n)) + a_{12}f_2(x_2(n)), \\ x_2(n+1) &= \beta x_2(n) + a_{21}f_1(x_1(n)) + a_{22}f_2(x_2(n)), \quad n = 0,1,2,\ldots, \end{aligned} \tag{1.0}$$

where $x_i\ (i = 1,2)$ denotes the state of the ith neuron, $\beta \in (0,1)$ is internal decay of the neurons, the constants $a_{ij}\ (i,j = 1,2)$ denotes the connection weights, $f_i: R \to R\ (i = 1,2)$ are continuous transfer functions and $f_i(0) = 0\ (i = 1,2)$. Some sufficient conditions were given to guarantee the stability of the equilibrium and the existence of bifurcation. The direction of bifurcation and the stability of bifurcating periodic solution were discussed. However, in a real neural network every neuron is usually independent and has its characteristics. In most cases, the internal decay of neurons is different, so it is necessary to study the dynamical behavior of neural networks with different internal decay of the neurons.

In this paper, we consider the following more general model:

$$\begin{aligned} x_1(n+1) &= \alpha x_1(n) + a_{11}f_1(x_1(n)) + a_{12}f_2(x_2(n)), \\ x_2(n+1) &= \beta x_2(n) + a_{21}f_1(x_1(n)) + a_{22}f_2(x_2(n)), \quad n = 0,1,2,\ldots, \end{aligned} \tag{1.1}$$

where $\alpha \in (0,1)$, $\beta \in (0,1)$ are internal decay of neurons, $a_{ij}\ (i = 1,2)$ denote the connection weights, $f_i: R \to R$ $(i = 1,2)$ are continuous transfer functions and $f_i(0) = 0\ (i = 1,2)$.

The discrete-time system (1.1) can be regarded as a discrete form of the differential system

$$\begin{aligned} \dot{x}_1(t) &= -\mu_1 x_1(t) + \gamma_{11}f_1(x_1(t)) + \gamma_{12}f_2(x_2(t)), \\ \dot{x}_2(t) &= -\mu_2 x_1(t) + \gamma_{21}f_1(x_1(t)) + \gamma_{22}f_2(x_2(t)), \end{aligned} \tag{1.2}$$

or the system with a piecewise constant arguments

$$\begin{aligned} \dot{x}_1(t) &= -\mu_1 x_1(t) + \gamma_{11}f_1(x_1([t])) + \gamma_{12}f_2(x_2([t])), \\ \dot{x}_2(t) &= -\mu_2 x_1(t) + \gamma_{21}f_1(x_1([t])) + \gamma_{22}f_2(x_2([t])), \end{aligned} \tag{1.3}$$

where $\mu_1 > 0$, $\mu_2 > 0$ and $[\cdot]$ denotes the greatest integer function. This system has wide application in certain biomedical areas and much progress has been made in the study of such system (1.3) with the piecewise arguments [20]. For the method of discrete analogy, we refer to [7,21,19,22].

Obviously, system (1.1) includes the discrete version of systems (1.2) and (1.3), which is an improved model, compared with the model in [10]. The discussion of the stability of equilibrium is too complex and we study it by setting a proper parameter. In order to analyze bifurcation, we find a simpler bifurcation parameter and derive more general results.

Bifurcation analysis concerning the continuous dynamical systems was investigated in [13–15]. However, in this paper, based on the techniques developed by Kuznetsov [11], we not only investigate the stability of equilibrium and the existence of Neimark–Sacker bifurcation of system (1.1), but also the direction of the Neimark–Sacker bifurcation and the stability of the bifurcating periodic solution. We find that when the bifurcation parameter exceeds a critical value, the Neimark–Sacker bifurcation will occur and its direction and stability are determined completely by an algebraic expression of $a(D^*)$.

The organization of this paper is as follows: In Section 2, we will discuss the stability of the trivial solutions and the existence of Neimark–Sacker bifurcation. In Section 3, a formula for determining the direction of Neimark–Sacker bifurcation and the stability of bifurcating periodic solution will be given by using the normal form method and the center manifold theory for discrete time system developed by Kuznetsov. In Section 4, numerical simulations aimed at justifying the theoretical analysis will be reported.

2. Stability and existence of Neimark–Sacker bifurcation

In this section, we discuss the local stability of the equilibrium $(0,0)$ of system (1.1). In [5,7], the transfer function f of the models which they discussed is $f(u) = \tanh(cu)$, here we only need the following assumption:

$$(\mathrm{H}_1) \quad f_i \in C^1(R) \quad \text{and} \quad f_i(0) = 0, \qquad i = 1, 2.$$

For the sake of simplicity and the need of discussion, the following parameters are defined:

$$T_1 = \frac{1}{2}(\alpha + a_{11}f_1'(0)), \quad T_2 = \frac{1}{2}(\beta + a_{22}f_2'(0)), \quad D = -a_{12}a_{21}f_1'(0)f_2'(0),$$
$$T = \frac{1}{2}(a_{11}f_1'(0) + a_{22}f_2'(0)).$$

Theorem 1. *The zero solution of* (1.1) *is asymptotically stable if* (H_1) *is satisfied and* $(T_1, T_2, D) \in X_0$, *where* $X_0 = X_1 \cup X_2 \cup X_3$,

$$X_1 = \left\{(T_1, T_2, D) \in R^3; D > -1 + 2(T_1 + T_2) - 4T_1T_2; -\frac{\alpha+\beta}{2} < T < 1 - \frac{\alpha+\beta}{2}; (T_1 - T_2)^2 > D\right\},$$
$$X_2 = \left\{(T_1, T_2, D) \in R^3; D > -1 - 2(T_1 + T_2) - 4T_1T_2; -1 - \frac{\alpha+\beta}{2} < T < -\frac{\alpha+\beta}{2}; (T_1 - T_2)^2 > D\right\},$$
$$X_3 = \{(T_1, T_2, D) \in R^3; D < 1 - 4T_1T_2; (T_1 - T_2)^2 < D\}.$$

Proof. The associated characteristic matrix for the linearization of (1.1) at $(0,0)$ is

$$A = \begin{pmatrix} \alpha + a_{11}f_1'(0) - \lambda & a_{12}f_2'(0), \\ a_{21}f_1'(0) & \beta + a_{22}f_2'(0) - \lambda \end{pmatrix}.$$

Let $|A| = 0$, we get the associated characteristic equation of (1.1). That is

$$\lambda^2 - (\alpha + \beta + a_{11}f_1'(0) + a_{22}f_2'(0))\lambda + (\alpha + a_{11}f_1'(0))(\beta + a_{22}f_2'(0)) - a_{12}a_{21}f_1'(0)f_2'(0) = 0, \tag{2.1}$$

$$\begin{aligned}\Delta &= (\alpha + \beta + a_{11}f_1'(0) + a_{22}f_2'(0))^2 - 4(\alpha + a_{11}f_1'(0))(\beta + a_{22}f_2'(0)) \\ &\quad + 4a_{12}a_{21}f_1'(0)f_2'(0) \\ &= \alpha^2 - 2\alpha\beta + \beta^2 + 2\alpha a_{11}f_1'(0) - 2\alpha a_{22}f_2'(0) - 2\beta a_{11}f_1'(0) + 2\beta a_{22}f_2'(0) \\ &\quad + a_{11}^2 f_1'(0)^2 - 2a_{11}a_{22}f_1'(0)f_2'(0) + a_{22}^2 f_2'(0)^2 + 4a_{12}a_{21}f_1'(0)f_2'(0) \\ &= (\alpha - \beta + a_{11}f_1'(0) - a_{22}f_2'(0))^2 + 4a_{12}a_{21}f_1'(0)f_2'(0).\end{aligned}$$

Next we discuss it in two cases:

Case 1. $(T_1 - T_2)^2 > D$. In this case, the roots of characteristic equation (2.1) are given by

$$\lambda_{1,2} = T_1 + T_2 \pm \sqrt{(T_1 - T_2)^2 - D}. \tag{2.2}$$

Note that the eigenvalues $|\lambda_{1,2}| < 1$ if and only if

$$(T_1, T_2, D) \in X_1 \cup X_2, \tag{2.3}$$

where

$$X_1 = \left\{(T_1, T_2, D) \in R^3; D > -1 + 2(T_1 + T_2) - 4T_1T_2; -\frac{\alpha+\beta}{2} < T < 1 - \frac{\alpha+\beta}{2}; (T_1 - T_2)^2 > D\right\},$$
$$X_2 = \left\{(T_1, T_2, D) \in R^3; D > -1 - 2(T_1 + T_2) - 4T_1T_2; -1 - \frac{\alpha+\beta}{2} < T < -\frac{\alpha+\beta}{2}; (T_1 - T_2)^2 > D\right\}.$$

Case 2. $(T_1 - T_2)^2 < D$. In this case the characteristic equation (2.1) has a pair of conjugate complex roots

$$\lambda_{1,2} = T_1 + T_2 \pm \sqrt{D - (T_1 - T_2)^2}i. \tag{2.4}$$

The modulus of the eigenvalues $|\lambda_{1,2}| < 1$ if and only if

$$(T_1, T_2, D) \in X_3 = \{(T_1, T_2, D) \in R^3; D < 1 - 4T_1T_2; (T_1 - T_2)^2 < D\}. \tag{2.5}$$

Combining with Cases 1 and 2, we get $X_0 = X_1 \cup X_2 \cup X_3$. Thus the eigenvalues $\lambda_{1,2}$ of characteristic equation (2.1) are inside the unit circle for $(T_1, T_2, D) \in X_0$, and thus the zero solution of (1.1) is asymptotically stable.

Now, we choose D as the bifurcation parameter to study the Neimark–Sacker bifurcation of $(0,0)$ in this paper. In case of $(T_1 - T_2)^2 < D$, the characteristic equation (2.1) has a pair of conjugate complex roots, let

$$\lambda(D) = T_1 + T_2 + \sqrt{D - (T_1 - T_2)^2}i, \tag{2.6}$$

then the eigenvalues in (2.1) are $\lambda(D)$ and $\overline{\lambda(D)}$. The modulus of the eigenvalue is

$$|\lambda| = \sqrt{(T_1 + T_2)^2 + D - (T_1 - T_2)^2} = \sqrt{D + 4T_1T_2}. \tag{2.7}$$

$|\lambda| = 1$ if and only if

$$D = D^* = 1 - 4T_1T_2. \tag{2.8}$$

Obviously,

$$|\lambda| < 1 \text{ for } (T_1 - T_2)^2 < D < D^*.$$

Then we can conclude that D^* is a critical value which destroy the stability of $(0,0)$. □

Lemma 1. *Suppose that* (H_1) *is satisfied and* $-\frac{\alpha+\beta}{2} < T < 1 - \frac{\alpha+\beta}{2}$. *Then*

(i) $\left(\frac{d}{dD}|\lambda(D)|\right)_{D=D^*} > 0;$

(ii) $\lambda^k(D^*) \neq 1$, $k = 1, 2, 3, 4$,

where $\lambda(D)$ *and* D^* *are given by* (2.6) *and* (2.8), *respectively*.

Proof. We know $|\lambda(D)| = \sqrt{D + 4T_1T_2}$, By direct calculation, we obtain that

$$\left(\frac{d}{dD}|\lambda(D)|\right)_{D=D^*} = \frac{1}{2} > 0,$$

which means property (i) is true. Next we deal with the property of (ii). From $-\frac{\alpha+\beta}{2} < T < 1 - \frac{\alpha+\beta}{2}$, we know $0 < T_1 + T_2 < 1$. Clearly, $\lambda^k(D^*) = 1$ for some $k \in \{1, 2, 3, 4\}$ if and only if the argument $\lambda(D^*) \in \{0, \pm\pi/2, \pm 2\pi/3, \pi\}$. Since

$$|\lambda(D^*)| = 1, \quad \mathrm{Re}\lambda(D^*) = T_1 + T_2 > 0, \quad \mathrm{Im}\lambda(D^*) = \sqrt{1 - (T_1 + T_2)^2} > 0,$$

it follows that $\arg\lambda(D^*) \bar{\in} \{0, \pm\pi/2, \pm 2\pi/3, \pi\}$. The proof is completed. □

By the Hopf bifurcation for maps [12], we have the following theorem.

Theorem 2. *Let the assumption* (H_1) *hold and* $-\frac{\alpha+\beta}{2} < T < 1 - \frac{\alpha+\beta}{2}$. *Then we have*

(i) *if* $(T_1 - T_2)^2 < D < D^*$, *then the equilibrium* $(0,0)$ *of* (1.1) *is asymptotically stable*;
(ii) *if* $(T_1 - T_2)^2 < D$ *and* $D > D^*$, *then the equilibrium* $(0,0)$ *of* (1.1) *is unstable*;
(iii) *if* $D = D^*$, *system* (1.1) *undergoes a Neimark–Sacker bifurcation, that is, system* (1.1) *has a periodic solution bifurcating from the equilibrium* $(0,0)$ *near* $D = D^*$, *where* D^* *is given by* (2.8).

3. Direction and stability of the Neimark–Sacker bifurcation

In the previous section, we have obtained some sufficient conditions to ensure that system (1.1) undergoes a Neimark–Sacker bifurcation at the equilibrium $(0,0)$ when D takes the critical value D^*. In this section, a formula for determining the direction of Neimark–Sacker bifurcation and the stability of bifurcating periodic solutions of system (1.1) at $D = D^*$ shall be presented by employing the normal form method and the center manifold theory for discrete time system developed by Kuznetsov [11]. For most of the models in the literature [5,7,8], the transfer function f is $f(u) = \tanh(cu)$, here we assume that the transfer functions in (1.1) satisfy

$$(\mathrm{H}_2) \quad f_i \in C^3(R,R), \quad f_i(0) = f_i''(0) = 0, \quad f_i'(0)f_i'''(0) \neq 0.$$

Now system (1.1) can be rewritten as

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = \begin{pmatrix} \alpha + a_{11}f_1'(0) & a_{12}f_2'(0) \\ a_{21}f_1'(0) & \beta + a_{22}f_2'(0) \end{pmatrix} \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} + \begin{pmatrix} F_1(x,D) \\ F_2(x,D) \end{pmatrix}, \tag{3.1}$$

where $x = (x_1, x_2)^{\mathrm{T}} \in R^2$.

From assumption (H_2), we know that $F_i (i = 1,2)$ in (3.1) can be expanded as

$$F_1(x,D) = \frac{a_{11}}{6} f_1'''(0)x_1^3 + \frac{a_{12}}{6} f_2'''(0)x_2^3 + \mathrm{O}(\|x\|^4),$$
$$F_2(x,D) = \frac{a_{21}}{6} f_1'''(0)x_1^3 + \frac{a_{22}}{6} f_2'''(0)x_2^3 + \mathrm{O}(\|x\|^4).$$

Denote

$$B = B(D) = \begin{pmatrix} \alpha + a_{11}f_1'(0) & a_{12}f_2'(0) \\ a_{21}f_1'(0) & \beta + a_{22}f_2'(0) \end{pmatrix}. \tag{3.2}$$

For convenience of calculation, we introduce two notations:

$$r_1 = T_2 - T_1 + \sqrt{D - (T_1 - T_2)^2} i, \tag{3.3}$$
$$r_2 = T_1 - T_2 + \sqrt{D - (T_1 - T_2)^2} i. \tag{3.4}$$

From the definition of T_1, T_2, we can obtain

$$\overline{r_1} = -r_2, \quad |r_j|^2 = -r_1 r_2 = D = -a_{12}a_{21}f_1'(0)f_2'(0), \quad j = 1,2. \tag{3.5}$$

Here for $j = 1,2$, $r_j \neq 0$ and $a_{12}a_{21}f_1'(0)f_2'(0) < 0$.

In fact, if $r_1 = 0$ or $r_2 = 0$, From (3.5), we have $r_1 r_2 = D = 0$, then $(T_1 - T_2)^2 \geqslant D$, and this is a contradiction to $(T_1 - T_2)^2 < D$. Hence $r_j \neq 0 (j = 1,2)$ and it follows from (3.5) that $a_{12}a_{21}f_1'(0)f_2'(0) < 0$.

Let $q(D) \in C^2$ be the eigenvector of $B(D)$ corresponding to $\lambda(D)$ given by (2.6). Then

$$B(D)q(D) = \lambda(D)q(D).$$

Also let $p(D) \in C^2$ be the eigenvector of the transposed matrix $B^{\mathrm{T}}(D)$ corresponding to its eigenvalue $\overline{\lambda(D)}$,

$$B^{\mathrm{T}}(D)p(D) = \overline{\lambda(D)}p(D).$$

By direct computation we obtain that

$$q \sim \left(1, \quad \frac{a_{21}f_1'(0)}{r_2}\right)^{\mathrm{T}}, \quad p \sim \left(1, \quad \frac{a_{12}f_2'(0)}{\overline{r_2}}\right)^{\mathrm{T}},$$

where $r_j (j = 1, 2)$ are given by (3.3), (3.4). In order to normalize the $q = (1, a_{21}f_1'(0)/r_2)^{\mathrm{T}}$ and p, we obtain

$$p = \frac{\overline{r_2}}{\overline{r_2} - r_2}\left(1, \quad \frac{a_{12}f_2'(0)}{\overline{r_2}}\right)^{\mathrm{T}}.$$

It is easy to see $\langle p, q\rangle = 1$, where $\langle\cdot,\cdot\rangle$ means the standard scalar product in $C^2 : \langle p, q\rangle = \overline{p_1}q_1 + \overline{p_2}q_2$. Any vector $x \in R^2$ can be represented for D near D^* as

$$x = zq(D) + \overline{zq(D)}$$

for some complex z, obviously

$$z = \langle p(D), x\rangle.$$

Thus, system (3.1) can be transformed for D near D^* into the following form:

$$\dot{z} = \lambda(D)z + g(z, \bar{z}, D), \tag{3.6}$$

where $\lambda(D)$ can be written as $\lambda(D) = (1 + \varphi(D))\mathrm{e}^{\mathrm{i}\theta(D)}(\varphi(D)$ is a smooth function with $\varphi(D^*) = 0)$ and

$$g(z, \bar{z}, D) = \sum_{k+l\geqslant 2} \frac{1}{k!l!} g_{kl}(D)z^k\bar{z}^l. \tag{3.7}$$

We know that $F_i(i = 1, 2)$ in (3.1) can be expanded as

$$F_1(\xi, D) = \frac{a_{11}}{6}f_1'''(0)\xi_1^3 + \frac{a_{12}}{6}f_2'''(0)\xi_2^3 + \mathrm{O}(\|\xi\|^4),$$

$$F_2(\xi, D) = \frac{a_{21}}{6}f_1'''(0)\xi_1^3 + \frac{a_{22}}{6}f_2'''(0)\xi_2^3 + \mathrm{O}(\|\xi\|^4).$$

It follows that

$$B_i(x, y) := \sum_{j,k}^{2} \left.\frac{\partial^2 F_i(\xi, D^*)}{\partial\xi_j\partial\xi_k}\right|_{\xi=0} x_jx_k = 0, \quad i = 1, 2, \tag{3.8}$$

$$\begin{aligned} C_i(x, y, u) &:= \sum_{j,k,l}^{2} \left.\frac{\partial^3 F_i(\xi, D^*)}{\partial\xi_j\partial\xi_k\partial\xi_l}\right|_{\xi=0} x_jx_ku_l \\ &= a_{i1}f_1'''(0)x_1y_1u_1 + a_{i2}f_2'''(0)x_2y_2u_2, \quad i = 1, 2. \end{aligned} \tag{3.9}$$

By (3.7)–(3.9) and the formulas

$$g_{20}(D^*) = \langle p, B(q, q)\rangle, \quad g_{11}(D^*) = \langle p, B(q, \bar{q})\rangle, \quad g_{02}(D^*) = \langle p, B(\bar{q}, \bar{q})\rangle,$$
$$g_{21}(D^*) = \langle p, C(q, q, \bar{q})\rangle.$$

We obtain

$$
\begin{aligned}
g_{20}(D^*) &= g_{11}(D^*) = g_{02}(D^*) = 0,\\
g_{21}(D^*) &= \overline{p_1}C_1(q,q,\bar{q}) + \overline{p_2}C_2(q,q,\bar{q})\\
&= \frac{r_2}{r_2-\overline{r_2}}\left\{a_{11}f_1'''(0) - \frac{a_{21}^2 f_1'(0)^2 f_2'''(0)}{r_2 f_2'(0)} + \frac{a_{12}a_{21}f_2'(0)f_1'''(0)}{r_2} - \frac{a_{21}^2 a_{22} f_1'(0)^2 f_2'''(0)}{r_2^2}\right\}\\
&= \frac{1}{a_{12}(r_1+r_2)f_2'(0)}\{a_{11}a_{12}r_2 f_2'(0)f_1'''(0) + a_{21}a_{22}\overline{r_2}f_1'(0)f_2'''(0)\\
&\quad + a_{12}^2 a_{21} f_2'(0)^2 f_1'''(0) - a_{12}a_{21}^2 f_1'(0)^2 f_2'''(0)\}\\
&= \frac{1}{2a_{12}\sqrt{D^*-(T_1-T_2)^2}f_2'(0)i}\left\{a_{11}a_{12}f_2'(0)f_1'''(0)\left(T_1 - T_2\sqrt{D^*-(T_1-T_2)^2}i\right)\right.\\
&\quad + a_{21}a_{22}f_1'(0)f_2'''(0)\left(T_1 - T_2\sqrt{D^*-(T_1-T_2)^2}i\right)\\
&\quad \left. + a_{12}^2 a_{21} f_2'(0)^2 f_1'''(0) - a_{12}a_{21}^2 f_1'(0)^2 f_2'''(0)\right\}\\
&= \frac{1}{2a_{12}f_2'(0)}(a_{11}a_{12}f_2'(0)f_1'''(0) - a_{21}a_{22}f_1'(0)f_2'''(0))\\
&\quad + \frac{1}{2a_{12}\sqrt{D^*-(T_1-T_2)^2}f_2'(0)i}\{(a_{11}a_{12}f_2'(0)f_1'''(0) + a_{21}a_{22}f_1'(0)f_2'''(0))(T_1-T_2)\\
&\quad + a_{12}^2 a_{21} f_2'(0)^2 f_1'''(0) - a_{12}a_{21}^2 f_1'(0)^2 f_2'''(0)\}.
\end{aligned}
$$

Together with $e^{-i\theta(D^*)} = \overline{\lambda(D^*)}$ and $D = -a_{12}a_{21}f_1'(0)f_2'(0)$, we have

$$
\begin{aligned}
a(D^*) &= \mathrm{Re}\left(\frac{e^{-i\theta(D^*)}g_{21}}{2}\right) - \mathrm{Re}\left(\frac{(1-2e^{i\theta(D^*)})e^{-2i\theta(D^*)}}{2(1-e^{i\theta(D^*)})}g_{20}g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2\\
&= \mathrm{Re}\left(\frac{e^{-i\theta(D^*)}g_{21}}{2}\right)\\
&= \frac{1}{2}\mathrm{Re}(\overline{\lambda(D^*)}g_{21})\\
&= \frac{1}{4a_{12}f_2'(0)}\{(T_1+T_2)[a_{11}a_{12}f_2'(0)f_1'''(0) - a_{21}a_{22}f_1'(0)f_2'''(0)]\\
&\quad - (a_{11}a_{12}f_2'(0)f_1'''(0) + a_{21}a_{22}f_1'(0)f_2'''(0))(T_1-T_2)\\
&\quad - a_{12}^2 a_{21} f_2'(0)^2 f_1'''(0) + a_{12}a_{21}^2 f_1'(0)^2 f_2'''(0)\}\\
&= \frac{1}{4a_{12}f_2'(0)}\{2T_2 a_{11}a_{12}f_2'(0)f_1'''(0) - 2T_1 a_{21}a_{22}f_1'(0)f_2'''(0)\\
&\quad - a_{12}^2 a_{21} f_2'(0)^2 f_1'''(0) + a_{12}a_{21}^2 f_1'(0)^2 f_2'''(0)\}\\
&= \frac{1}{4a_{12}f_2'(0)}\{(\beta + a_{22}f_2'(0))a_{11}a_{12}f_2'(0)f_1'''(0)\\
&\quad - (\alpha + a_{11}f_1'(0))a_{21}a_{22}f_1'(0)f_2'''(0) - a_{12}^2 a_{21} f_2'(0)^2 f_1'''(0)\\
&\quad + a_{12}a_{21}^2 f_1'(0)^2 f_2'''(0)\}\\
&= \frac{1}{4a_{12}f_2'(0)}\left\{(\beta + a_{22}f_2'(0))a_{11}a_{12}f_2'(0)f_1'''(0)\right.\\
&\quad - (\alpha + a_{11}f_1'(0))a_{21}a_{22}f_1'(0)f_2'''(0)\\
&\quad \left. + \frac{1-4T_1T_2}{f_1'(0)f_2'(0)}(a_{12}f_2'(0)^2 f_1'''(0) - a_{21}f_1'(0)^2 f_2'''(0))\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4a_{12}f_1'(0)f_2'(0)^2}\{(\beta + a_{22}f_2'(0))a_{11}a_{12}f_1'(0)f_2'(0)^2 f_1'''(0)\\
&\quad - (\alpha + a_{11}f_1'(0))a_{21}a_{22}f_1'(0)^2 f_2'(0)f_2'''(0)\\
&\quad + [1 - (\alpha + a_{11}f_1'(0))(\beta + a_{22}f_2'(0))](a_{12}f_2'(0)^2 f_1'''(0) - a_{21}f_1'(0)^2 f_2'''(0))\}\\
&= \frac{1}{4a_{12}f_1'(0)f_2'(0)^2}\{\beta a_{11}a_{12}f_1'(0)f_2'(0)^2 f_1'''(0) + a_{11}a_{12}a_{22}f_1'(0)f_2'(0)^3 f_1'''(0)\\
&\quad - \alpha a_{21}a_{22}f_1'(0)^2 f_2'(0)f_2'''(0) - a_{11}a_{21}a_{22}f_1'(0)^3 f_2'(0)f_2'''(0)\\
&\quad + a_{12}f_2'(0)^2 f_1'''(0)(1 - \alpha\beta) - a_{21}f_1'(0)^2 f_2'''(0)(1 - \alpha\beta)\\
&\quad - \alpha a_{12}a_{22}f_2'(0)^3 f_1'''(0) + \alpha a_{21}a_{22}f_1'(0)^2 f_2'(0)f_2'''(0)\\
&\quad - \beta a_{11}a_{12}f_1'(0)f_2'(0)^2 f_1'''(0) + \beta a_{11}a_{21}f_1'(0)^3 f_2'''(0)\\
&\quad - a_{11}a_{12}a_{22}f_1'(0)f_2'(0)^3 f_1'''(0) + a_{11}a_{21}a_{22}f_1'(0)^3 f_2'(0)f_2'''(0)\}\\
&= \frac{a_{21}f_1'(0)f_2'''(0)}{4a_{12}f_2'(0)f_2'(0)}[-1 + \alpha\beta + \beta a_{11}f_1'(0)] + \frac{f_1'''(0)}{4f_1'(0)}[1 - \alpha\beta - \alpha a_{22}f_2'(0)].
\end{aligned}
\tag{3.10}
$$

Theorem 3. *Suppose that the condition* (H_2) *hold and* $T \in (-\frac{\alpha+\beta}{2}, 1 - \frac{\alpha+\beta}{2})$, *then the direction of the Neimark–Sacker bifurcation and stability of bifurcating periodic solution can be determined by the sign of* $a(D^*)$. *In fact, if* $a(D^*) < 0(>0)$, *then the Neimark–Sacker bifurcation is supercritical* (*subcritical*) *and the bifurcating periodic solution is asymptotically stable* (*unstable*), *where* D^* *is given by* (2.8).

This method is introduced by Kuznetsov in [11].

Remark 1. If the two-neuron network (1.1) is without self-connections, that is, $a_{11} = a_{22} = 0$, then system (1.1) has the following form:

$$
\begin{aligned}
x_1(n+1) &= \alpha x_1(n) + a_{12}f_2(x_2(n)),\\
x_2(n+1) &= \beta x_1(n) + a_{21}f_1(x_2(n)), \quad n = 0, 1, 2, \ldots
\end{aligned}
\tag{3.11}
$$

From (3.10), we can obtain

$$
a(D^*) = \frac{1-\alpha\beta}{4}\left\{\frac{f_1'''(0)}{f_1'(0)} - \frac{a_{21}f_1'(0)f_2'''(0)}{a_{12}f_2'(0)f_2'(0)}\right\}.
\tag{3.12}
$$

From the previous analysis, we know $a_{12}a_{21}f_1'(0)f_2'(0) < 0$. If $\mathrm{sgn}(f_1'(0)f_1'''(0)) = \mathrm{sgn}(f_2'(0)f_2'''(0))$, we have $\mathrm{sgn}(a(D^*)) = \mathrm{sgn}(f_1'(0)f_1'''(0)) = \mathrm{sgn}(f_2'(0)f_2'''(0))$ from (3.12). Thus we obtain the following result.

Corollary 1. *Suppose that* (H_2) *is satisfied and* $\mathrm{sgn}(f_1'(0)f_1'''(0)) = \mathrm{sgn}(f_2'(0)f_2'''(0))$. *Then the direction of the Neimark–Sacker bifurcation and stability of bifurcating periodic solution are determined by the sign of* $f_k'(0)f_k'''(0)$. *Indeed, if* $f_k'(0)f_k'''(0) < 0(> 0)$, *the Neimark–Sacker bifurcation is supercritical* (*subcritical*) *and the bifurcating periodic solution is asymptotically stable* (*unstable*).

Remark 2. If the two-neuron network with self-connections modelled by a discrete-time system of the form (1.1) with $\alpha = \beta$

$$
\begin{aligned}
x_1(n+1) &= \beta x_1(n) + a_{11}f_1(x_1(n)) + a_{12}f_2(x_2(n)),\\
x_2(n+1) &= \beta x_1(n) + a_{21}f_1(x_1(n)) + a_{22}f_2(x_2(n)),
\end{aligned}
\tag{3.13}
$$

which is the model studied by Yuan et al. [10]. From (3.10), we obtain

$$
a(D^*) = \frac{a_{21}f_1'(0)f_2'''(0)}{4a_{12}f_2'(0)f_2'(0)}[-1 + \beta^2 + \beta a_{11}f_1'(0)] + \frac{f_1'''(0)}{4f_1'(0)}[1 - \beta^2 - \beta a_{22}f_2'(0)].
\tag{3.14}
$$

It tallies with the result of Yuan et al. [10]. Here we find the two bifurcation parameters are different. In our paper, we assume $D = -a_{12}a_{21}f_1'(0)f_2'(0)$ as the bifurcation parameter. In [10] the bifurcation parameter is $D_0 = (a_{11}a_{22} - a_{12}a_{21})f_1'(0)f_2'(0)$. We obtain $D_0 = D + a_{11}a_{22}f_1'(0)f_2'(0)$. In fact it does not influence the sign of $a(D^*)$. The bifurcation parameter in our paper is simpler. System (1.1) includes the model of Yuan et al. [10]. Yuan et al. [10] only discussed a two-neuron model with the same internal decay of the neurons, however our model can deal with the system with different internal decay of the neurons, which is more general.

4. Numerical simulations

In this section, we give numerical simulations to support our theoretical analysis.

Example 1. Let $\alpha = \frac{1}{4}, \beta = \frac{3}{4}, a_{11} = 1, a_{12} = -1, a_{22} = -1$ and $f_1(u) = \sin(u), f_2(u) = \arctan(u/2)$ in the system (1.1). By the simple calculation, we obtain

$$\begin{aligned} &f_1'(0) = 1, \quad f_2'(0) = \frac{1}{2}, \quad f_1''(0) = f_2''(0) = 0, \quad f_1'''(0) = -1, \quad f_2'''(0) = -\frac{1}{4},\\ &T = \frac{1}{2}(a_{11}f_1'(0) + a_{22}f_2'(0)) = \frac{1}{4}, \quad -1 + 2(T_1 + T_2) - 4T_1T_2 = -\frac{3}{16},\\ &\quad -1 - 2(T_1 + T_2) - 4T_1T_2 = -\frac{45}{16}, \quad 1 - 4T_1T_2 = \frac{11}{16}. \end{aligned}$$

It follows from (2.8) that $D^* = \frac{11}{16}\left(a_{21} = \frac{11}{8}\right)$, the Neimark–Sacker bifurcation occurs when $D = \frac{11}{16}$. Choosing $a_{21} = \frac{5}{4}$, so that $D = \frac{5}{8} < D^*, (T_1, T_2, D) \in X_0$. By Theorem 1, we know the origin is asymptotically stable, the corresponding waveform and phase plots are shown in Fig. 1. Choosing $a_{21} = \frac{3}{2}$, then $D = \frac{3}{4} > D^*$. By Theorem 2, we know that a Neimark–Sacker bifurcation occurs when $D = D^* = \frac{11}{16}$. The origin loses its stability and a periodic solution bifurcates from the origin when $D = \frac{3}{4} > D^*$. From (3.10), we have $a(D^*) = -\frac{131}{512} < 0$. By Theorem 3, we know the periodic solution is asymptotically stable. The corresponding phase plots are shown in Figs. 2 and 3. In Fig. 2 its zero solution undergoes a Neimark–Sacker bifurcation at the origin and in Fig. 3 it is stable.

Example 2. Let $\alpha = \frac{1}{3}, \beta = \frac{1}{2}, a_{11} = 1, a_{12} = -1, a_{22} = -1$ and $f_1(u) = \sin(u), f_2(u) = e^u - \left(\frac{1}{2}u^2 + 1\right)$ in the system (1.1). By the simple calculation, we obtain

$$\begin{aligned} &f_1'(0) = 1, \quad f_2'(0) = 1, \quad f_1''(0) = f_2''(0) = 0, \quad f_1'''(0) = -1, \quad f_2'''(0) = 1,\\ &T = \frac{1}{2}(a_{11}f_1'(0) + a_{22}f_2'(0)) = 0, \quad -1 + 2(T_1 + T_2) - 4T_1T_2 = \frac{1}{2},\\ &\quad -1 - 2(T_1 + T_2) - 4T_1T_2 = -\frac{7}{6}, \quad 1 - 4T_1T_2 = \frac{5}{3}. \end{aligned}$$

It follows from (2.8) that $D^* = \frac{5}{3}$ $(a_{21} = \frac{5}{3})$, the Neimark–Sacker bifurcation occurs when $D = \frac{5}{3}$. Choosing $a_{21} = 1.6$, so that $D = 1.6 < D^*$, $(T_1, T_2, D) \in X_0$. By Theorem 1, we know the origin is asymptotically stable,

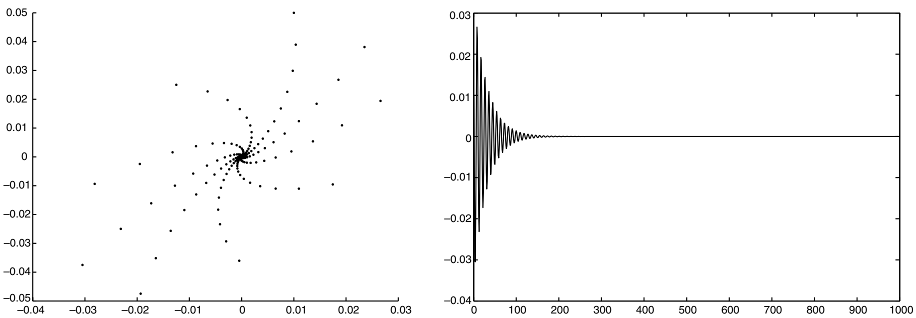

Fig. 1. Phase plot and waveform plot for system (1.1) with $a_{21} = \frac{5}{4}$, $D = \frac{5}{8} < D^*$.

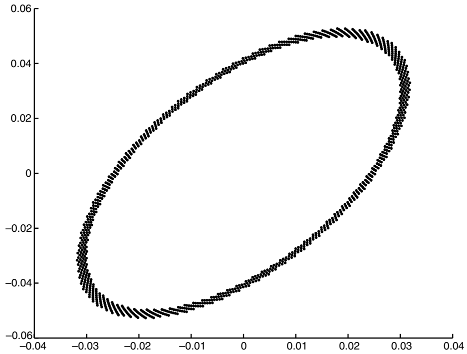

Fig. 2. $a_{21} = \frac{11}{8},\ D = \frac{11}{16} = D^*$.

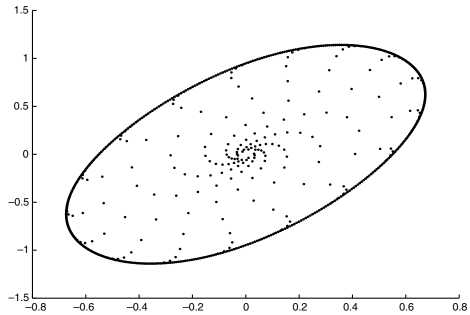

Fig. 3. $a_{21} = \frac{3}{2},\ D = \frac{3}{4} > D^*$.

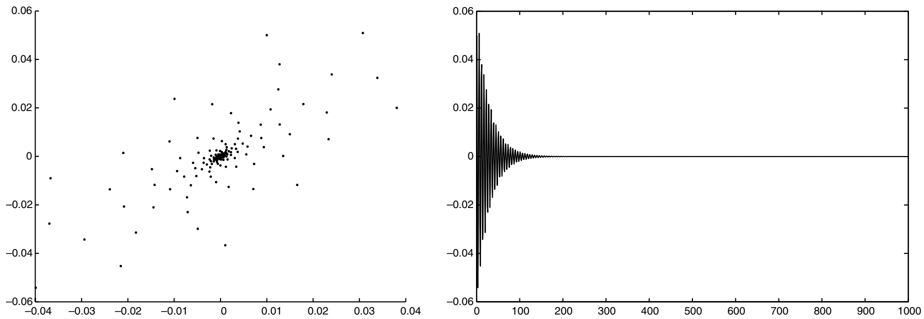

Fig. 4. Phase plot and waveform plot for system (1.1) with $a_{21} = 1.6$, $D = 1.6 < D^*$.

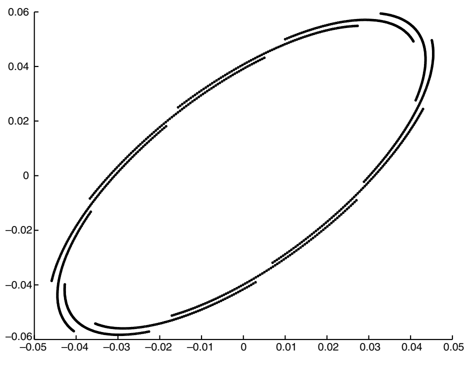

Fig. 5. $a_{21} = \frac{5}{3},\ D = \frac{5}{3} = D^*$.

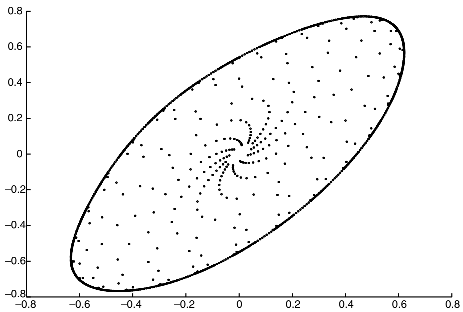

Fig. 6. $a_{21} = 1.7$, $D = 1.7 > D^*$.

the corresponding waveform and phase plots are shown in Fig. 4. Choosing $a_{21} = 1.7$, then $D = 1.7$. By Theorem 2, we know that a Neimark–Sacker bifurcation occurs when $D = D^* = \frac{5}{3}$. The origin loses its stability and a periodic solution bifurcates from the origin when $D = 1.7 > D^*$. From (3.10), we have $a(D^*) = -\frac{1}{8} < 0$. By Theorem 3, we know the periodic solution is asymptotically stable. The corresponding phase plots are shown in Figs. 5 and 6. In Fig. 5 its zero solution undergoes a Neimark–Sacker bifurcation at the origin and in Fig. 6 it is stable.

5. Conclusions

The discrete-time system of neural networks provides some dynamical behaviors which enriches the theory of continuous system and has potential applications in neural networks. Although the system discussed above is quite simple, it is potentially useful as the complexity found might be carried over to the model with delay.

By choosing a proper bifurcation parameter, we have shown that a Neimark–Sacker bifurcation occurs when this parameter passes through a critical value. The direction of Neimark–Sacker bifurcation and the stability of the bifurcating periodic solution are also discussed.

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