Nonlinear Systems and Control
Lecture # 6
Bifurcation

Bifurcation is a change in the equilibrium points or periodic orbits, or in their stability properties, as a parameter is varied

Example

$$\begin{aligned}\dot{x}_1 &= \mu - x_1^2\\ \dot{x}_2 &= -x_2\end{aligned}$$

Find the equilibrium points and their types for different values of μ

For $\mu > 0$ there are two equilibrium points at $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$

Linearization at $(\sqrt{\mu}, 0)$:

$$\begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

$(\sqrt{\mu}, 0)$ is a stable node

Linearization at $(-\sqrt{\mu}, 0)$:

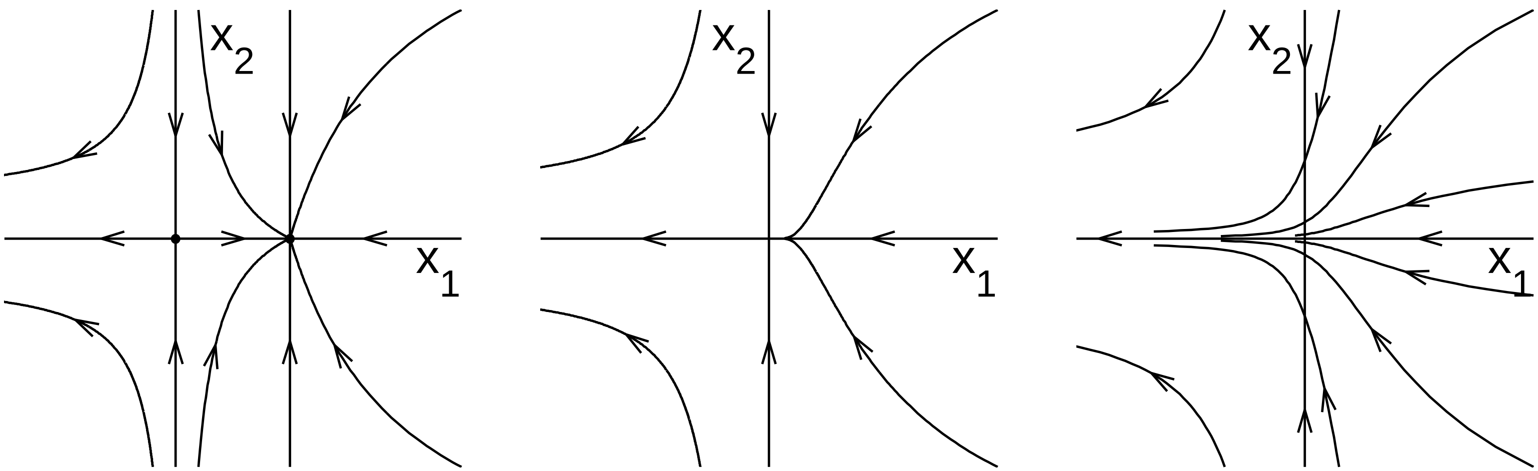
$$\begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

$(-\sqrt{\mu}, 0)$ is a saddle

$$\dot{x}_1 = \mu - x_1^2, \qquad \dot{x}_2 = -x_2$$

No equilibrium points when $\mu < 0$

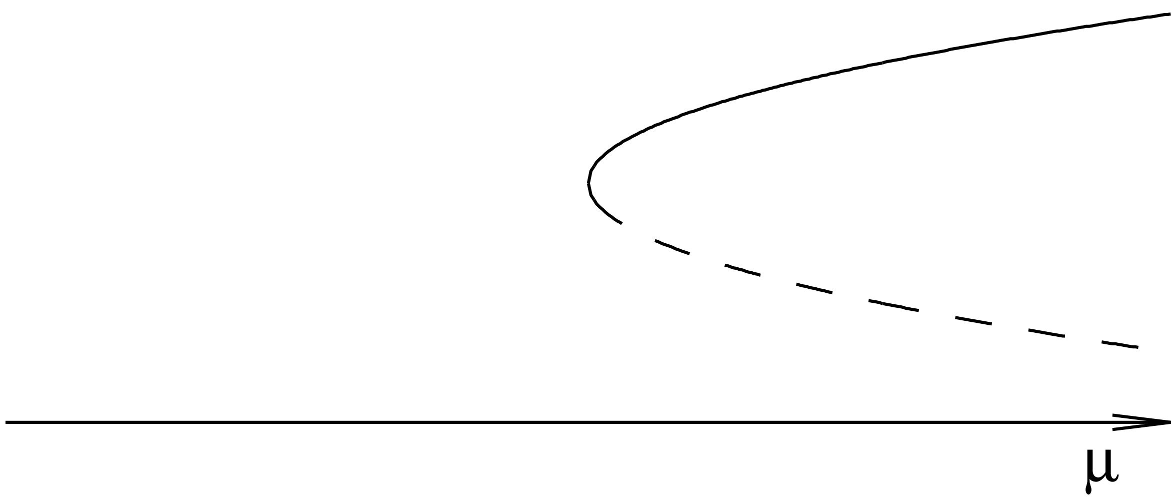
As μ decreases, the saddle and node approach each other, collide at $\mu = 0$, and disappear for $\mu < 0$



$\mu > 0$ $\qquad$ $\mu = 0$ $\qquad$ $\mu < 0$

μ is called the bifurcation parameter and $\mu = 0$ is the bifurcation point

Bifurcation Diagram



(a) Saddle−node bifurcation

Example

$$\dot{x}_1 = \mu x_1 - x_1^2, \qquad \dot{x}_2 = -x_2$$

Two equilibrium points at $(0,0)$ and $(\mu,0)$

The Jacobian at $(0,0)$ is $\begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}$

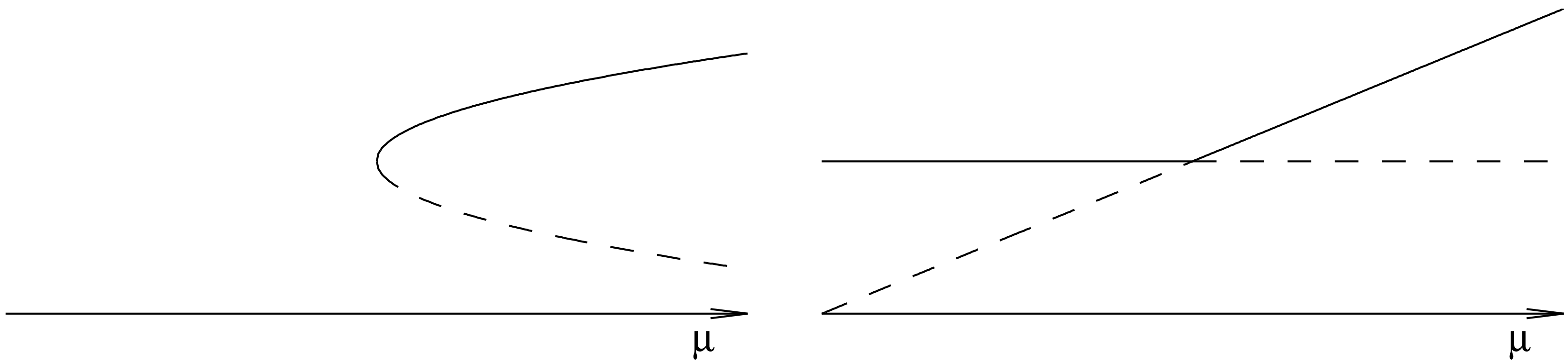
$(0,0)$ is a stable node for $\mu < 0$ and a saddle for $\mu > 0$

The Jacobian at $(\mu,0)$ is $\begin{bmatrix} -\mu & 0 \\ 0 & -1 \end{bmatrix}$

$(\mu,0)$ is a saddle for $\mu < 0$ and a stable node for $\mu > 0$

An eigenvalue crosses the origin as μ crosses zero

While the equilibrium points persist through the bifurcation point $\mu = 0$, $(0, 0)$ changes from a stable node to a saddle and $(\mu, 0)$ changes from a saddle to a stable node



(a) Saddle−node bifurcation (b) Transcritical bifurcation

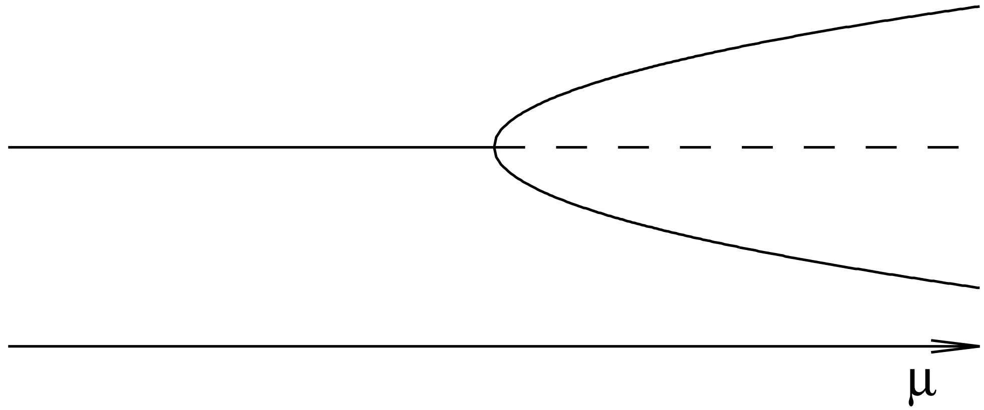
dangerous or hard safe or soft

Example

$$\dot{x}_1 = \mu x_1 - x_1^3, \qquad \dot{x}_2 = -x_2$$

For $\mu < 0$, there is a stable node at the origin

For $\mu > 0$, there are three equilibrium points: a saddle at $(0,0)$ and stable nodes at $(\sqrt{\mu}, 0)$, and $(-\sqrt{\mu}, 0)$



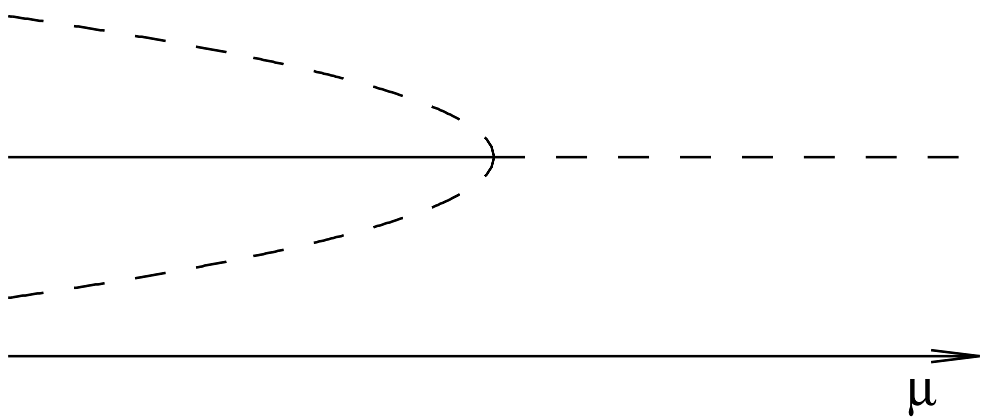
(c) Supercritical pitchfork bifurcation

Example

$$\dot{x}_1 = \mu x_1 + x_1^3, \qquad \dot{x}_2 = -x_2$$

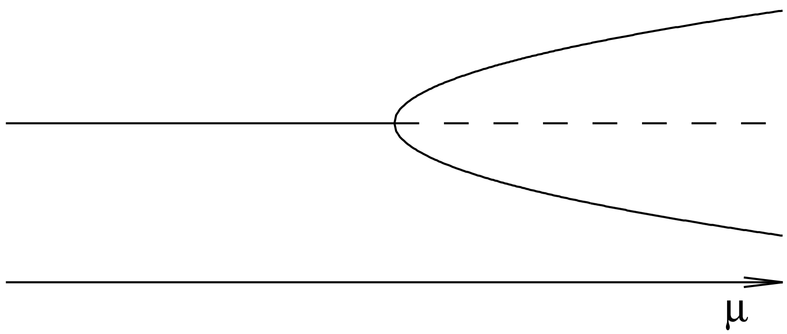
For $\mu < 0$, there are three equilibrium points: a stable node at $(0, 0)$ and two saddles at $(\pm\sqrt{-\mu}, 0)$

For $\mu > 0$, there is a saddle at $(0, 0)$

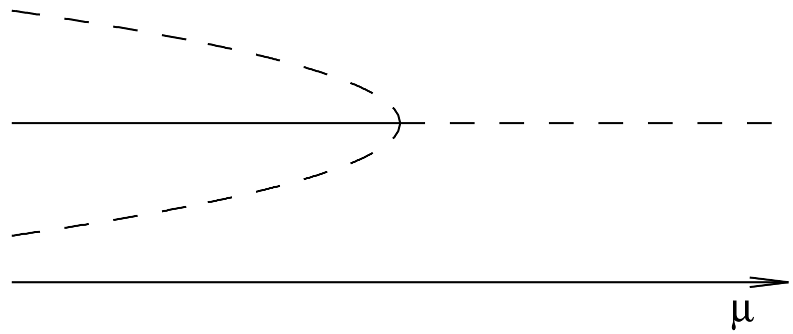


(d) Subcritical pitchfork bifurcation

Notice the difference between supercritical and subcritical pitchfork bifurcations



(c) Supercritical pitchfork bifurcation



(d) Subcritical pitchfork bifurcation

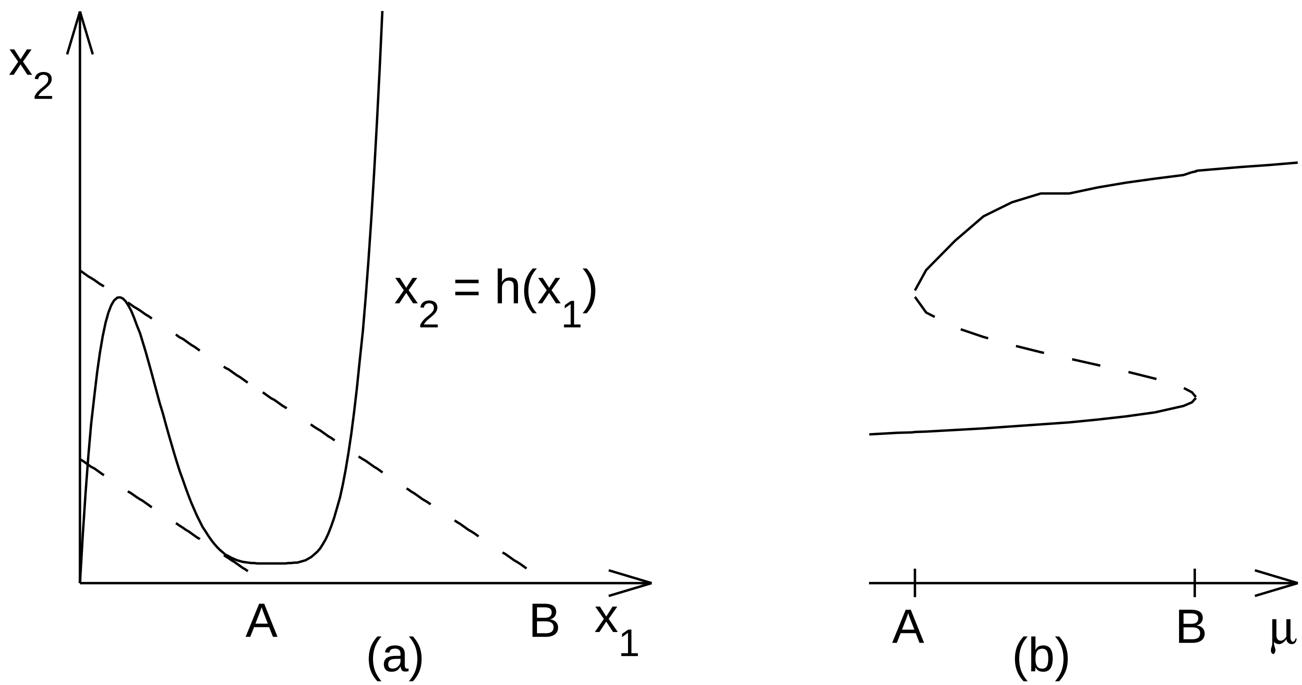
safe or soft

dangerous or hard

Example: Tunnel diode Circuit

$$\dot{x}_1 = \frac{1}{C}\left[-h(x_1) + x_2\right]$$

$$\dot{x}_2 = \frac{1}{L}\left[-x_1 - Rx_2 + \mu\right]$$



Example

$$\begin{aligned} \dot{x}_1 &= x_1(\mu - x_1^2 - x_2^2) - x_2 \\ \dot{x}_2 &= x_2(\mu - x_1^2 - x_2^2) + x_1 \end{aligned}$$

There is a unique equilibrium point at the origin

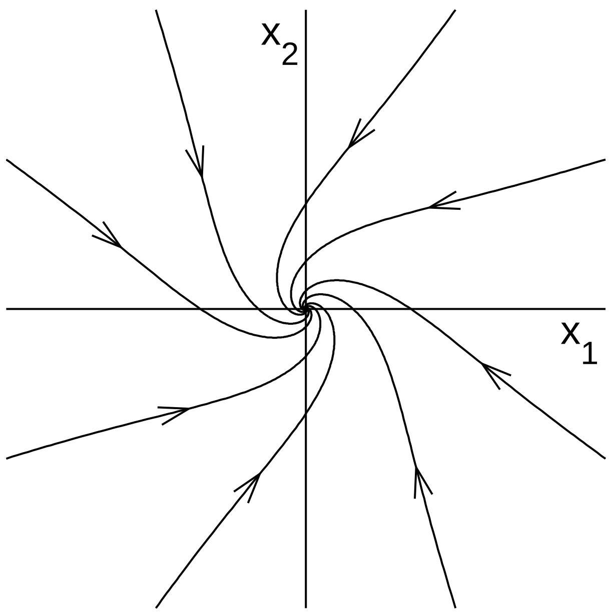
$$\text{Linearization:} \quad \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

Stable focus for $\mu < 0$, and unstable focus for $\mu > 0$

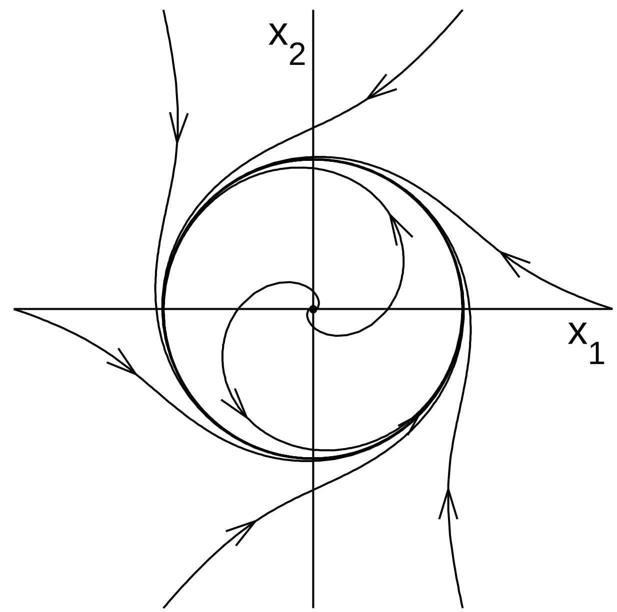
A pair of complex eigenvalues cross the imaginary axis as μ crosses zero

$$\dot{r} = \mu r - r^3 \quad \text{and} \quad \dot{\theta} = 1$$

For $\mu > 0$, there is a stable limit cycle at $r = \sqrt{\mu}$



$\mu < 0$



$\mu > 0$

Supercritical Hopf bifurcation

Example

$$\begin{aligned}
\dot{x}_1 &= x_1\left[\mu + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2\right] - x_2 \\
\dot{x}_2 &= x_2\left[\mu + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2\right] + x_1
\end{aligned}$$

There is a unique equilibrium point at the origin

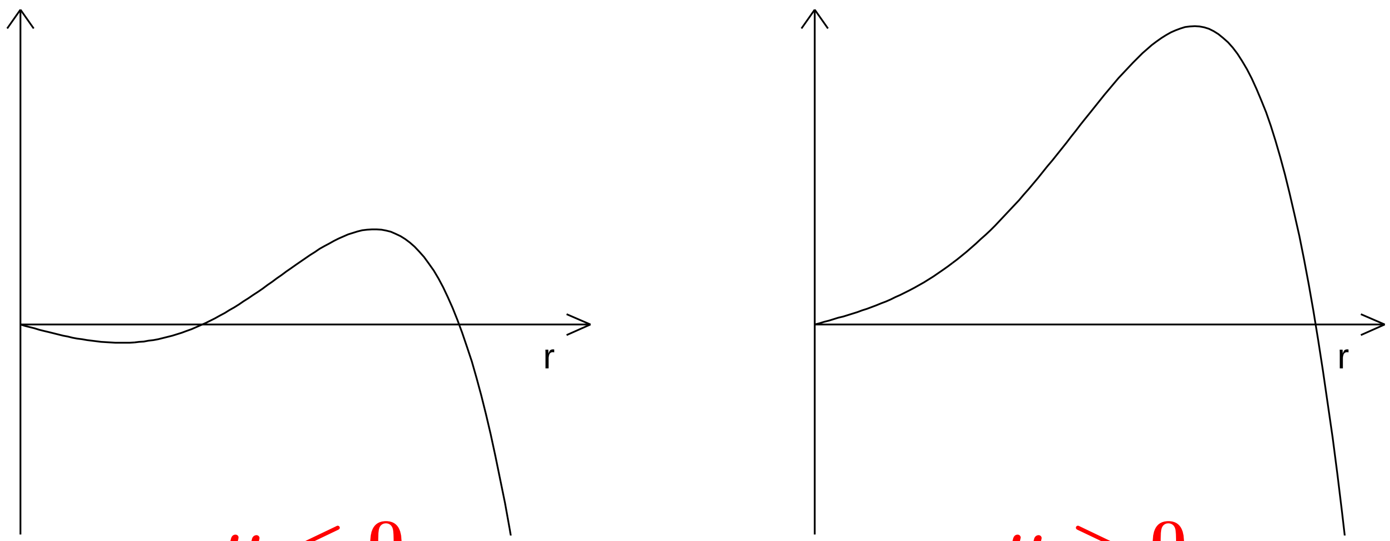
$$\text{Linearization:} \quad \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

Stable focus for $\mu < 0$, and unstable focus for $\mu > 0$

A pair of complex eigenvalues cross the imaginary axis as μ crosses zero

$$\dot{r} = \mu r + r^3 - r^5 \quad \text{and} \quad \dot{\theta} = 1$$

Sketch of $\mu r + r^3 - r^5$:

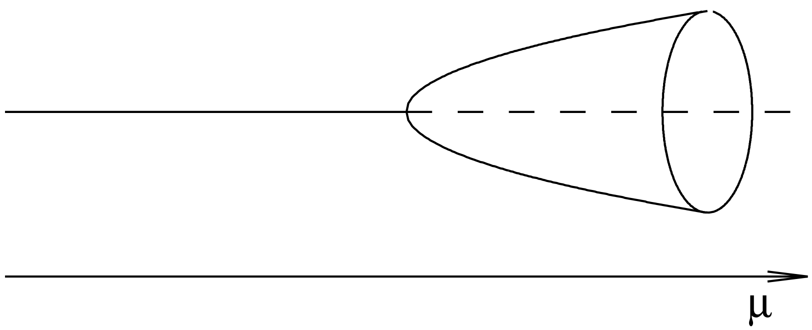


$\mu < 0$ $\mu > 0$

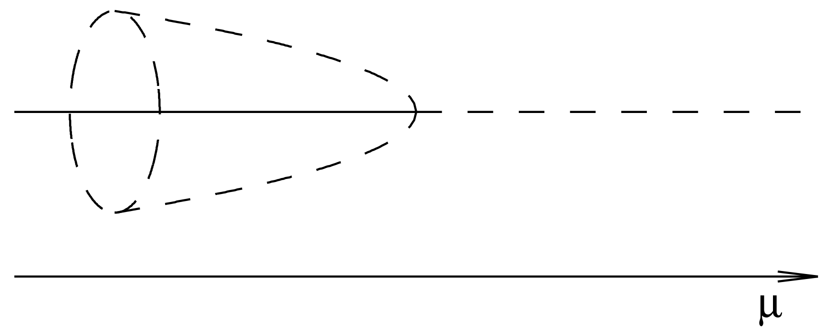
For small $|\mu|$, the stable limit cycles are approximated by $r = 1$, while the unstable limit cycle for $\mu < 0$ is approximated by $r = \sqrt{|\mu|}$

As μ increases from negative to positive values, the stable focus at the origin merges with the unstable limit cycle and bifurcates into unstable focus

Subcritical Hopf bifurcation



(e) Supercritical Hopf bifurcation



(f) Subcrtitical Hopf bifurcation

safe or soft

dangerous or hard

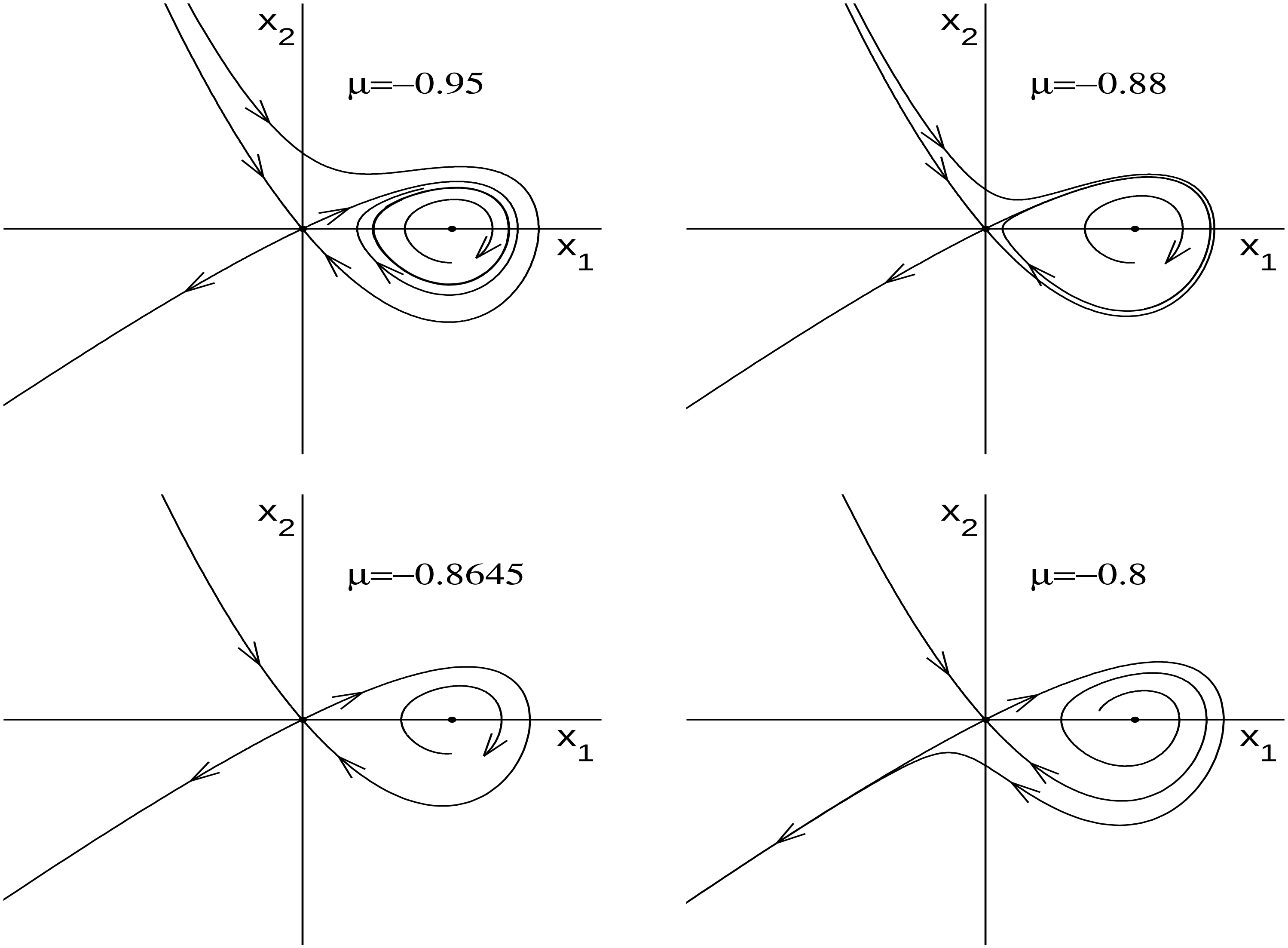
All six types of bifurcation occur in the vicinity of an equilibrium point. They are called local bifurcations

Example of Global Bifurcation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mu x_2 + x_1 - x_1^2 + x_1 x_2 \end{aligned}$$

There are two equilibrium points at $(0, 0)$ and $(1, 0)$. By linearization, we can see that $(0, 0)$ is always a saddle, while $(1, 0)$ is an unstable focus for $-1 < \mu < 1$

Limit analysis to the range $-1 < \mu < 1$



Saddle–connection (or homoclinic) bifurcation