Inductance

FIZ102E: Electricity & Magnetism

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Contents

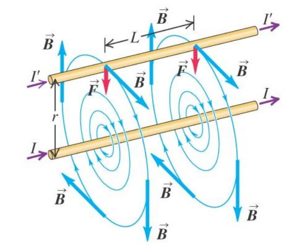
Learning outcomes

- How a time-varying current in one coil can induce an emf in a second, unconnected coil.
- How to relate the induced emf in a circuit to the rate of change of current in the same circuit.
- How to calculate the energy stored in a magnetic field.
- How to analyze circuits that include both a resistor and an inductor (coil).
- Why electrical oscillations occur in circuits that include both an inductor and a capacitor.
- Why oscillations decay in circuits with an inductor, a resistor, and a capacitor.

Introduction

- A changing current in a coil induces an emf in an adjacent coil. The coupling between the coils is described by their *mutual inductance*.
- A changing current in a coil also induces an emf in that same coil.
- Such a coil is called an *inductor*,
- and the relationship of current to emf is described by the inductance (also called *self-inductance*) of the coil.

Mutual inductance



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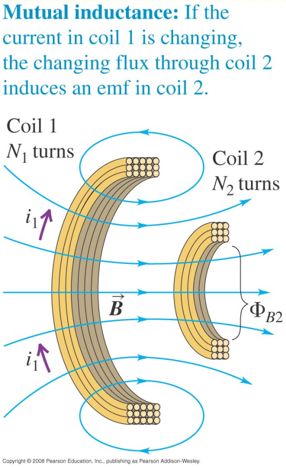
- We have considered the magnetic interaction between two wires carrying steady currents.
- The current in one wire causes a magnetic field, which exerts a force on the current in the second wire.
- An additional interaction arises between two circuits when there is a *changing* current in one of the circuits.

Mutual inductance



- We use lowercase letters to represent quantities that vary with time (e.g. i).
- i_1 produces B, according to Biot-Savarts law:

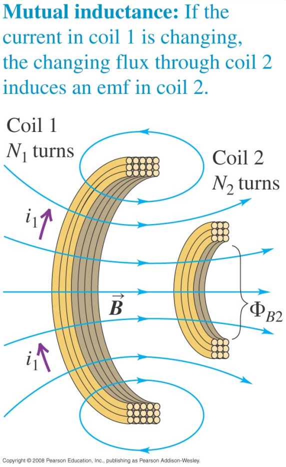
$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \int \frac{i_1 \mathrm{d}\vec{\ell} \times \hat{\mathbf{r}}}{r^2} \Rightarrow B \propto i_1$$

- Φ through the 2nd loop is proportional to B

$$\Phi_{B2} = \int \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{A}} \Rightarrow \Phi_{B2} \propto B$$

- Thus $\Phi_{B2} \propto i_1$.

Mutual inductance



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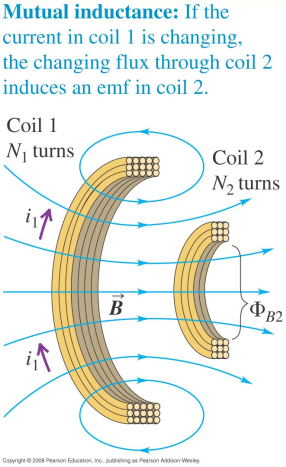
- We call the proportionality constant as the *mutual inductance*, M_{21}.
- If there are N_2 turns, the same flux passes through each loop, and

$$N_2 \Phi_{B2} = M_{21} i_1 \tag{1}$$

defines M_{21}.

Mutual inductance



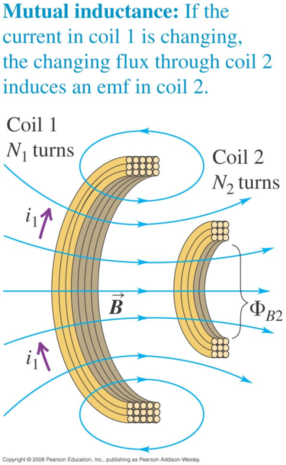
- When i_1 changes, Φ_{B2} changes; this changing flux induces an emf $\mathcal{E}_2$ in coil 2, given by

$$\mathcal{E}_2 = -N_2 \frac{d\Phi_{B2}}{dt} \quad (1)$$

Using $N_2\Phi_{B2} = M_{21}i_1$

$$\mathcal{E}_2 = -M_{21} \frac{di_1}{dt} \quad (2)$$

Mutual inductance



- We found $\Phi_{B2} \propto i_1$.
- Then the mutual inductance

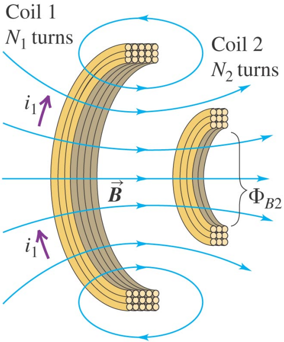
$$M_{21} = \frac{N_2 \Phi_{B2}}{i_1}$$

is a constant that depends only on the geometry of the two coils (the size, shape, number of turns, and orientation of each coil and the separation between the coils).

- This is always valid if the coils are in vacuum.

Mutual inductance

Mutual inductance: If the current in coil 1 is changing, the changing flux through coil 2 induces an emf in coil 2.



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- If a magnetic material is present, M_{21} also depends on the magnetic properties of the material.
- If the material has nonlinear magnetic properties-that is, if the relative permeability K_m is not constant and magnetization is not proportional to magnetic field-then Φ_{B2} is no longer directly proportional to i_1.

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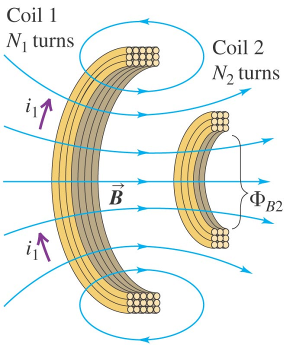

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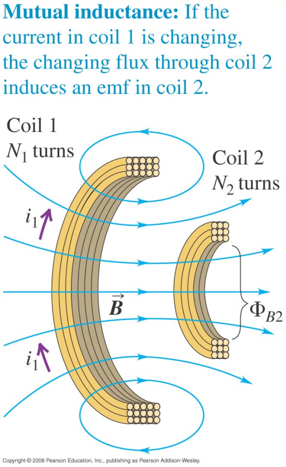
- A changing current i_2 in coil 2 causes a changing flux Φ_{B1} and an emf $\mathcal{E}_1$ in coil 1.
- The corresponding constant $M_{12} = M_{21}$ *always*, even though in general the two coils are not identical and Φ through them is not the same.

$$M = M_{21} = \frac{N_2 \Phi_{B2}}{i_1} = M_{12} = \frac{N_1 \Phi_{B1}}{i_2} \tag{1}$$

- Mutual inductance, M, characterizes completely the induced-emf interaction of two coils.

Mutual inductance



Mutually induced emfs are then

$$\mathcal{E}_2 = -M\frac{di_1}{dt}, \quad \mathcal{E}_1 = -M\frac{di_2}{dt} \qquad (1)$$

Unit of inductance

- The SI unit of mutual inductance is called the *henry* ($1\,\mathrm{H}$) [1]
- One henry is equal to one weber per ampere ($M = N_2\Phi_{B2}/i_1$)
- Other equivalent units

$$1\,\mathrm{H} = 1\,\mathrm{Wb/A} = 1\,\mathrm{V}\cdot\mathrm{s/A} = 1\,\Omega\cdot\mathrm{s} = 1\,\mathrm{J/A^2} \tag{2}$$

[1] In honor of the American physicist Joseph Henry (1797-1878), one of the discoverers of electromagnetic induction.

Drawbacks and Uses of Mutual Inductance

- Mutual inductance can be a nuisance in electric circuits, since variations in current in one circuit can induce unwanted emfs in other nearby circuits.
- To minimize these effects two coils would be placed far apart.
- Mutual inductance also has many useful applications:
- A transformer
- How do electric toothbrushes charge through plastic?

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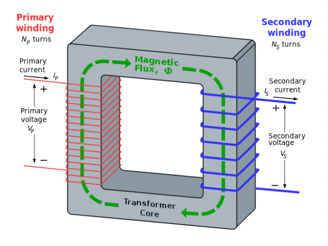

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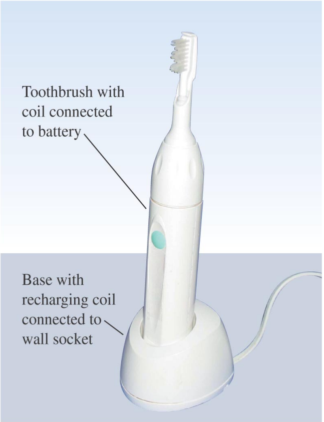

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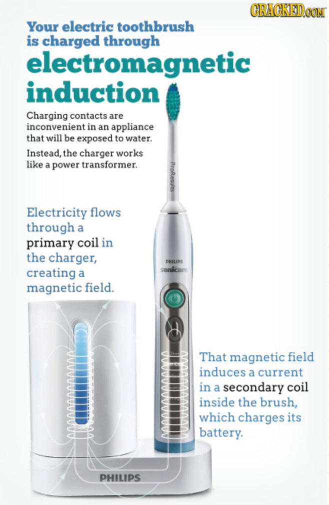

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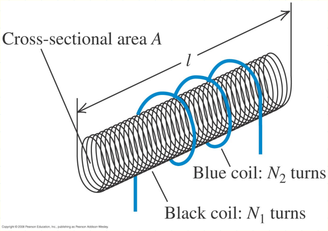
Ex: Calculating mutual inductance

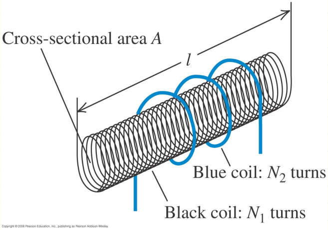


This is a form of Tesla coil (a high-voltage generator popular in science museums)

Question

A long solenoid with length l and cross-sectional area A is closely wound with N_1 turns of wire. A coil with N_2 turns surrounds it at its center. Find the mutual inductance M.

Ex: Calculating mutual inductance



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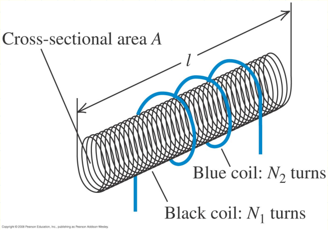
Solution

- B_1 at the center of the solenoid

$$B_1 = \mu_0 n_1 i_1 = \frac{\mu_0 N_1 i_1}{l}$$

  where $n_1 = N_1/l$.
- The flux through a cross section of the solenoid equals $B_1 A$.
- This also equals the flux Φ_{B2} through each turn of the outer coil, independent of its cross-sectional area as there is almost no mag- netic field outside a very long solenoid.

Ex: Calculating mutual inductance



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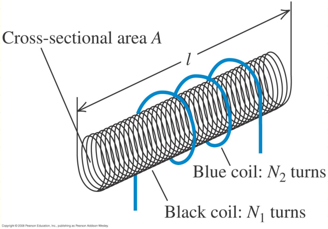
Solution

- The mutual inductance is then

$$M = \frac{N_2 \Phi_{B2}}{i_1} = \frac{N_2 B_1 A}{i_1}$$
$$= \frac{\mu_0 N_1 N_2 A i_1}{l i_1} = \frac{\mu_0 N_1 N_2 A}{l}$$

- The mutual inductance M of any two coils is proportional to the product $N_1 N_2$ of their numbers of turns.
- Notice that M depends only on the geometry of the two coils, not on the current.

Ex: Calculating mutual inductance



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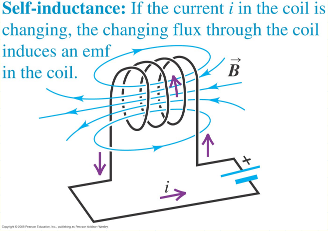
Numerical

Suppose $l = 0.50\,\mathrm{m}$,
$A = 10\,\mathrm{cm}^2 = 1.0 \times 10^{-3}\,\mathrm{m}^2$,
$N_1 = 1000\,\mathrm{turns}$, and $N_2 = 10\,\mathrm{turns}$.
The mutual inductance is then

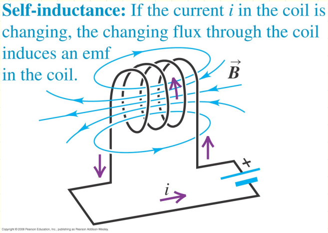
$$\begin{aligned} M &= \frac{\mu_0 N_1 N_2 A}{l} \\ &= \frac{4\pi \times 10^{-7}\mathrm{Wb/A\cdot m} \times 1000 \times 10 \times 1.0 \times 10^{-3}\,\mathrm{m}^2}{0.50\,\mathrm{m}} \\ &= 25 \times 10^{-6}\,\mathrm{H} \end{aligned}$$

Self-inductance



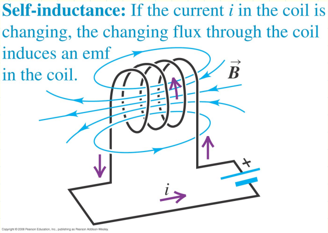
- A current in a circuit sets up a magnetic field that causes a magnetic flux through the *same* circuit;
- this flux changes when the current changes.
- Thus any circuit that carries a varying current has an emf induced in it by the variation in its own B. Such an emf is called a *self-induced emf*.

Self-inductance



- By *Lenz's law*, a self-induced emf opposes the change in i that caused the emf and so tends to make it more difficult for variations in i to occur.
- Self-induced emfs can occur in any circuit, since there is always some Φ_B through the closed loop of a current-carrying circuit.
- But the effect is greatly enhanced if the circuit includes a coil with N turns of wire.

Self-inductance



- The self-inductance of the circuit is defined as

$$L = \frac{N\Phi_B}{i} \tag{3}$$

From Faraday's law $\mathcal{E} = -N\mathrm{d}\Phi_B/\mathrm{d}t$

$$\mathcal{E} = -L\frac{\mathrm{d}i}{\mathrm{d}t} \tag{4}$$

- The $-$ sign is a reflection of Lenz's law: the self-induced emf in a circuit opposes any change in the current in that circuit.

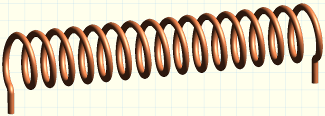
Inductors as circuit elements

- A circuit device that is designed to have a particular inductance is called an inductor, or a choke.
- The usual circuit symbol for an inductor is 
- Their purpose is to oppose any variations in the current through the circuit.

Application

- If lightning strikes part of an electrical power transmission system, it causes a sudden spike in voltage that can damage the components of the system as well as anything connected to that system.
- To minimize these effects, large inductors are incorporated into the transmission system.
- An inductor opposes and suppresses any rapid changes in the current.

Self-inductance of an ideal soleoid



- Prototype of inductors.
- $n = N/l$
- $B = \mu_0 ni$
- $\Phi_B = BA = \mu_0 niA$
- Then

$$L = \frac{N\Phi_B}{i} = \mu_0 nNA$$

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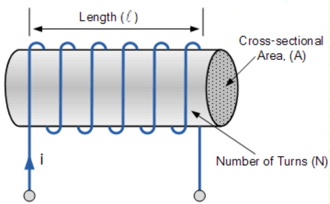

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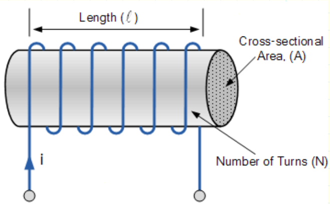

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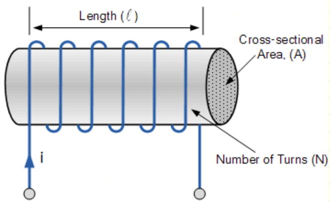

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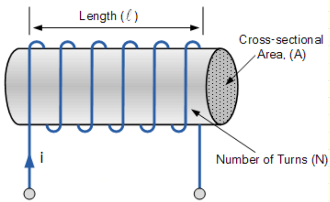

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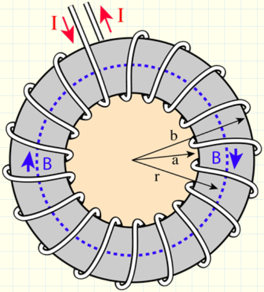
$$L = \frac{N\Phi_B}{i} = \mu_0 n N A$$

Self-inductance of a toroidal soleoid

- $B = \frac{\mu_0 N i}{2\pi r}$
- $\Phi_B = \int \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{A}} = \int_a^b B h \, \mathrm{d}r = \frac{\mu_0 N i h}{2\pi} \ln \frac{b}{a}$
- $L = \frac{N \Phi_B}{i} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}$

Self-inductance of a toroidal soleoid



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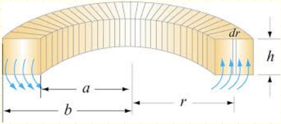

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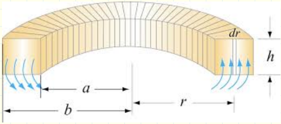

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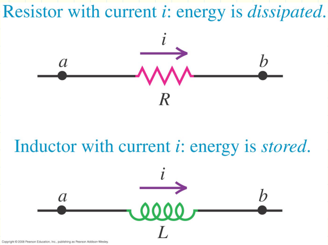
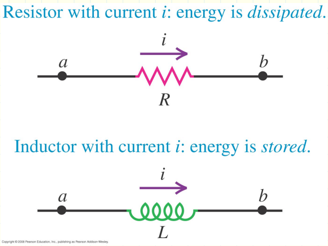
Magnetic field energy



- An inductor carrying a current has energy stored in it.
- Establishing a current in an inductor requires an input of energy.

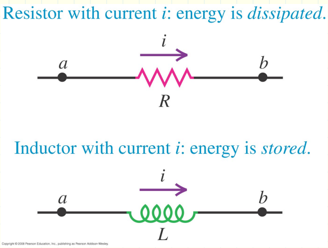
Energy Stored in an Inductor



- What is the total energy input U needed to establish a final current I in an inductor with inductance L if the initial current is zero.
- The rate P at which energy is being delivered to the inductor is

$$P = V_{ab}i = L\frac{\mathrm{d}i}{\mathrm{d}t}i$$

Energy Stored in an Inductor



- The energy $\mathrm{d}U$ supplied to the inductor during an infinitesimal time interval $\mathrm{d}t$ is $\mathrm{d}U = P\mathrm{d}t$, so

$$\mathrm{d}U = Li\,\mathrm{d}i$$

- The total energy U supplied while the current increases from zero to a final value I is

$$U = L\int_0^I i\,\mathrm{d}i = \frac{1}{2}LI^2$$

- Where is this energy stored?

Magnetic energy density

Recall

- The electric energy density was defined as

$$u_E = \frac{1}{2}\epsilon_0 E^2$$

- The energy given to a capacitor was stored in the electric field

- The magnetic energy density

$$u_B = \frac{B^2}{2\mu_0} \tag{5}$$

- Ex: The magnetic energy density in an ideal solenoid is then $u_B = \frac{(\mu_0 n i)^2}{2\mu_0} = \frac{1}{2}\mu_0 n^2 i^2$
- The energy in an ideal solenoid is then $U_B = u_B A l = \frac{1}{2}\mu_0 n^2 l A i^2$
- Recalling $L = \mu_0 n^2 l A$ we get $U_B = \frac{1}{2} L i^2 = U$.
- All given energy is stored in the magnetic field.

Magnetic energy density

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- All given energy is stored in the magnetic field.

Energy stored in a toroidal solenoid

- Recall $L = \frac{N\Phi}{i} = \frac{\mu_0 N^2 h}{2\pi} \ln\frac{b}{a}$ and so

$$U = \frac{1}{2}Li^2 = \frac{\mu_0 N^2 h}{4\pi} \ln\frac{b}{a} i^2$$

- Let us find this from the energy density.
- Recall $B = \mu_0 Ni/2\pi r$
- Energy density: $u_B = B^2/2\mu_0 = \mu_0 N^2 i^2/8\pi^2 r^2$ (not uniform)
- Consider a cylindrical shell of radius r, thickness $\mathrm{d}r$ and height h. It's volume is $\mathrm{d}V = 2\pi r\,\mathrm{d}r h$. The energy stored is $\mathrm{d}U = u_B \mathrm{d}V$

$$\begin{aligned} U_B &= \int_a^b u_b\,\mathrm{d}V = \int_a^b \frac{\mu_0 N^2 i^2}{8\pi^2 r^2} 2\pi r\,\mathrm{d}r\,h \\ &= \frac{\mu_0 N^2 h i^2}{4\pi} \int_a^b \frac{\mathrm{d}r}{r} \\ &= \frac{\mu_0 N^2 h i^2}{4\pi} \ln\frac{b}{a} = U \end{aligned}$$

Energy stored in a toroidal solenoid

- Recall $L = \frac{N\Phi}{i} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}$ and so

$$U = \frac{1}{2} L i^2 = \frac{\mu_0 N^2 h}{4\pi} \ln \frac{b}{a} i^2$$

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Energy stored in a toroidal solenoid

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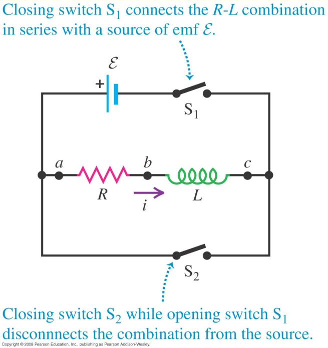
The $R - L$ circuit



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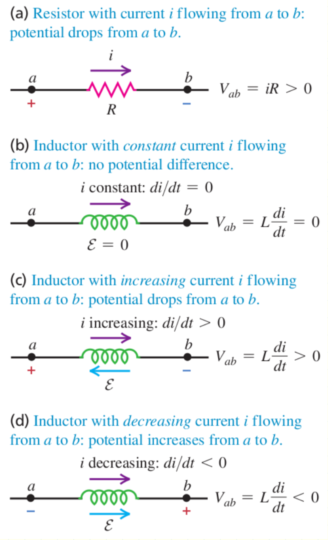
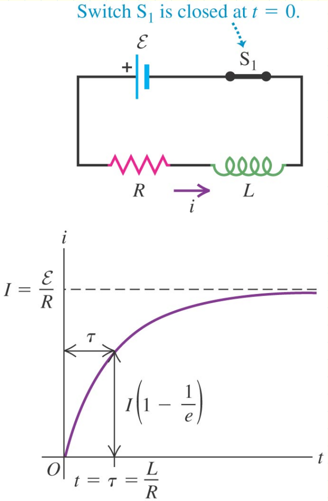
- How does an inductor behave in a circuit?
- An inductor in a circuit makes it difficult for rapid changes in current to occur, thanks to the effects of self-induced emf.
- According to $\mathcal{E} = -L\frac{\mathrm{d}i}{\mathrm{d}t}$ the greater the rate of change of current $\mathrm{d}i/\mathrm{d}t$, the greater the self-induced emf and the greater the potential difference between the inductor terminals.

The potential difference across an inductor



- The potential difference across a resistor depends on the current (a).
- whereas the potential difference across an inductor depends on the rate of change of the current (b), (c), (d) .

The $R - L$ circuit



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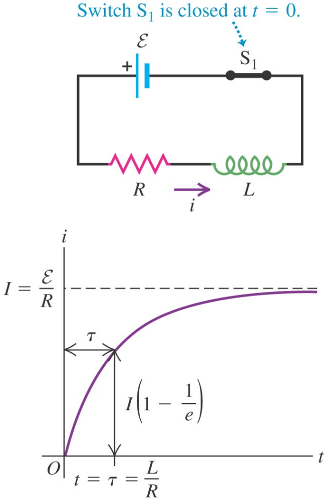
- Apply Kirchhoff's loop rule

$$\mathcal{E} - iR - L\frac{\mathrm{d}i}{\mathrm{d}t} = 0$$

- Arrange

$$\frac{\mathrm{d}i}{i - (\mathcal{E}/R)} = -\frac{R}{L}\mathrm{d}t$$

The $R-L$ circuit



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- Integrate

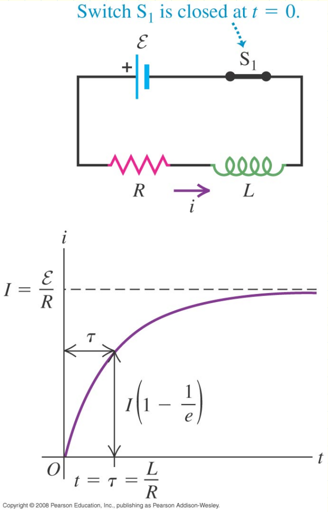
$$\int_0^i \frac{di'}{i' - (\mathcal{E}/R)} = -\int_0^t \frac{R}{L} dt'$$

$$\ln\left(\frac{i - (\mathcal{E}/R)}{-\mathcal{E}/R}\right) = -\frac{R}{L} t$$

- take exponentials of both sides and solve for i

$$i = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right), \qquad \tau = L/R$$

The $R - L$ circuit



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- Recall

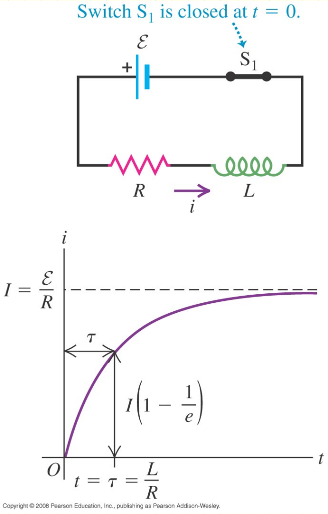
$$i = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right)$$

- Take derivative:

$$\frac{di}{dt} = \frac{\mathcal{E}}{L}e^{-t/\tau}$$

- At time $t = 0$, $i = 0$ and $di/dt = \mathcal{E}/L$. As $t \to \infty$, $i \to \mathcal{E}/R$ and $di/dt \to 0$

The $R-L$ circuit

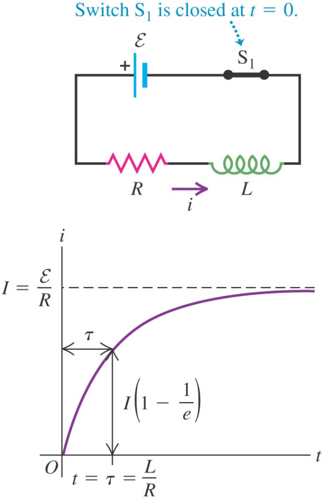


- Recall

$$i = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right)$$

- At $t = \tau$ the current has risen to $(1 - 1/e) \simeq 63\%$ of its final value.
- At $t = 5\tau$ the current has risen to 99.3% of its final value.

Energy considerations



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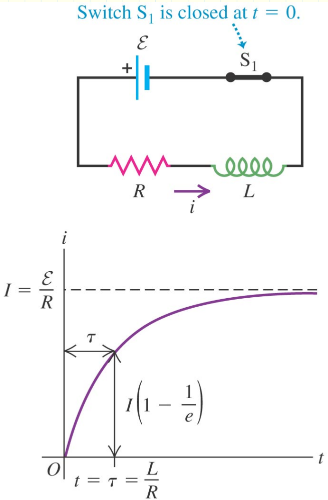
- The rate at which the source delivers energy to the circuit is $P_{\mathcal{E}} = \mathcal{E}i$.
- The rate at which energy is dissipated in the resistor is $P_R = i^2R$.
- The rate at which energy is stored in the inductor is $P_L = Li\mathrm{d}i/\mathrm{d}t$
- Multiply $\mathcal{E} - iR - L\frac{\mathrm{d}i}{\mathrm{d}t} = 0$ by i and arrange

$$\mathcal{E}i = i^2R + Li\frac{\mathrm{d}i}{\mathrm{d}t}$$

Of the power supplied by the source $\mathcal{E}i$, part i^2R is dissipated in R and part $Li\frac{\mathrm{d}i}{\mathrm{d}t}$ goes to store energy in L.

Energy delivered by the source



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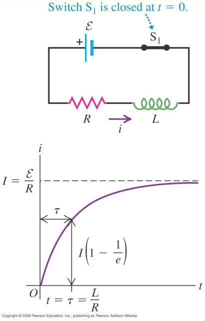
- The rate at which the source delivers energy to the circuit is $P_{\mathcal{E}} = \mathcal{E} i$.
- $dU_{\mathcal{E}} = P_{\mathcal{E}} \mathrm{d}t$ and $i = \frac{\mathcal{E}}{R}\left(1 - \mathrm{e}^{-t/\tau}\right)$
- $U_{\mathcal{E}} = \int_0^\infty \mathcal{E} i \mathrm{d}t$

$$U_{\mathcal{E}} = \frac{\mathcal{E}^2}{R} \int_0^\infty \left(1 - \mathrm{e}^{-t/\tau}\right) \mathrm{d}t$$
$$= \frac{\mathcal{E}^2}{R}\tau \int_0^\infty \left(1 - \mathrm{e}^{-x}\right) \mathrm{d}x$$

where $x \equiv t/\tau$. The integral diverges. If we integrate to some finite $x_f \gg 1$

$$U_{\mathcal{E}} = \frac{\mathcal{E}^2}{R}\tau \left(x + \mathrm{e}^{-x}\right)_0^{x_f} \simeq \frac{\mathcal{E}^2}{R}\tau (x_f - 1)$$

Energy dissipated on the resistor



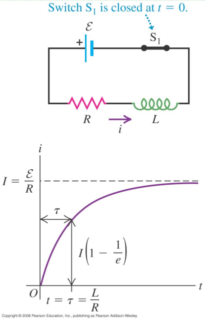
- The rate at which the resistor dissipates energy is $P_R = Ri^2$.
- $dU_R = P_R \mathrm{d}t$ and $i = \frac{\mathcal{E}}{R}\left(1 - \mathrm{e}^{-t/\tau}\right)$
- $U_R = R\int_0^\infty i^2 \mathrm{d}t$

$$
\begin{aligned}
U_R &= \frac{\mathcal{E}^2}{R}\int_0^\infty \left(1 - \mathrm{e}^{-t/\tau}\right)^2 \mathrm{d}t \\
&= \frac{\mathcal{E}^2}{R}\tau\int_0^\infty \left(1 - \mathrm{e}^{-x}\right)^2 \mathrm{d}x
\end{aligned}
$$

where $x \equiv t/\tau$. If we integrate to some finite $x_f \gg 1$

$$
U_R = \frac{\mathcal{E}^2}{R}\tau\left(x + 2\mathrm{e}^{-x} - \frac{1}{2}\mathrm{e}^{-2x}\right)_0^{x_f} \simeq \frac{\mathcal{E}^2}{R}\tau(x_f - \frac{3}{2})
$$

Energy stored on the inductor



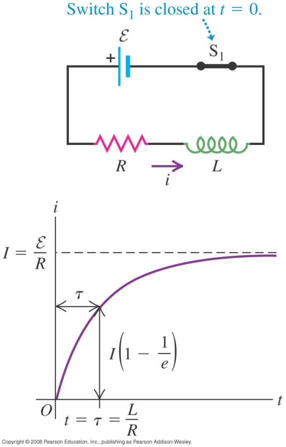
- The rate at which the inductor stores energy is $P_L = L\mathrm{d}i/\mathrm{d}t$.
- $dU_L = P_L\mathrm{d}t$ and $i = \frac{\mathcal{E}}{R}\left(1 - \mathrm{e}^{-t/\tau}\right)$, $\mathrm{d}i/\mathrm{d}t = \frac{\mathcal{E}}{R\tau}\mathrm{e}^{-t/\tau}$
- $U_L = L\int_0^\infty i(\mathrm{d}i/\mathrm{d}t)\mathrm{d}t$

$$U_L = L\frac{\mathcal{E}^2}{R^2\tau}\int_0^\infty \left(1 - \mathrm{e}^{-t/\tau}\right)\mathrm{e}^{-t/\tau}\mathrm{d}t$$
$$= \frac{\mathcal{E}^2}{R}\tau\int_0^\infty \left(1 - \mathrm{e}^{-x}\right)\mathrm{e}^{-x}\mathrm{d}x$$

where $x \equiv t/\tau$. If we integrate to some finite $x_f \gg 1$

$$U_L = \frac{\mathcal{E}^2}{R}\tau\frac{1}{2}\left((1 - \mathrm{e}^{-x})^2\right)_0^{x_f} \simeq \frac{\mathcal{E}^2}{2R}\tau = \frac{1}{2}LI^2$$

In summary



Since the current does not go to zero, there is always some dissipation of energy on the resistor which is supplied by the source.

- Energy delivered by the source to the circuit

$$U_{\mathcal{E}} \simeq \frac{\mathcal{E}^2}{R}\tau(x_f - 1)$$

- Energy dissipated on the resistor

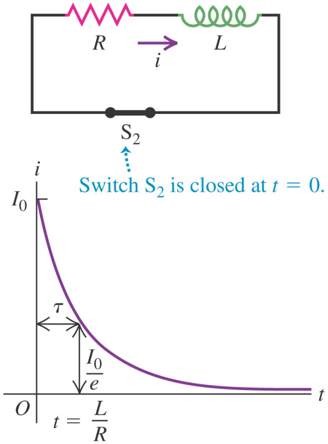
$$U_R \simeq \frac{\mathcal{E}^2}{R}\tau(x_f - \frac{3}{2})$$

- Energy stored on the inductor

$$U_L \simeq \frac{\mathcal{E}^2}{R}\tau\frac{1}{2}$$

- $U_{\mathcal{E}} = U_R + U_L$

Current decay in an R-L circuit



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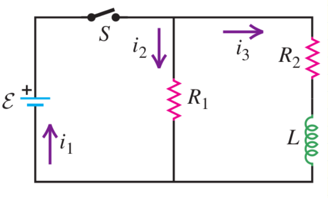
- Now suppose switch S_1 has been closed for a while and the current has reached the value I_0.
- We reset our stopwatch to redefine the initial time, we close switch S_2 at time $t = 0$, bypassing the battery.
- Kirchhoff's loop eqn: $-iR - L\frac{\mathrm{d}i}{\mathrm{d}t} = 0$ whose solution is

$$i = I_0 \mathrm{e}^{-t/\tau}, \qquad \tau = L/R \qquad (6)$$

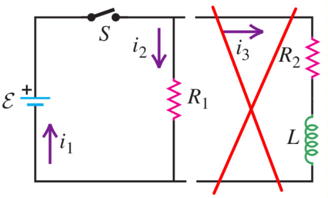
Ex: $R - L$ circuit



Question

An inductor with inductance $L = 0.300\,\mathrm{H}$ and negligible resistance is connected to a battery, a switch S, and two resistors, $R_1 = 12.0\,\Omega$ and $R_2 = 16.0\,\Omega$. The battery has emf $96.0\,\mathrm{V}$ and negligible internal resistance. S is closed at $t = 0$. (a) What are the currents i_1, i_2, and i_3 just after S is closed? (b) What are i_1, i_2, and i_3 after S has been closed a long time?

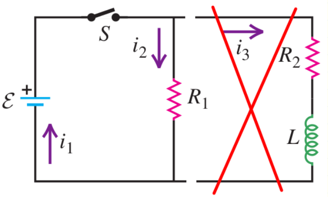
Ex: $R - L$ circuit



Solution a

- As soon as the S is closed there is no current on the L branch.
- $i_3 = 0$
- $i_1 = i_2 = \mathcal{E}/R_1$

Ex: $R - L$ circuit



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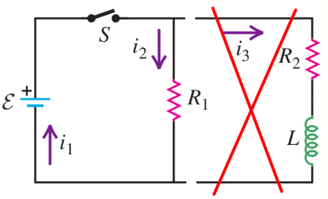

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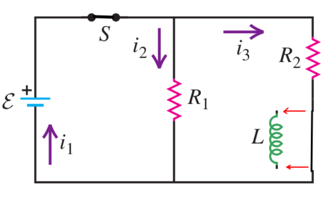
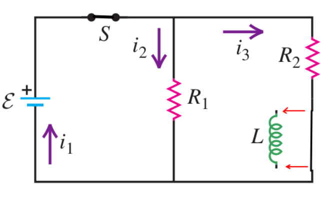


Solution b

- After S has been closed a long time L acts just like a wire (as $\mathrm{d}i/\mathrm{d}t = 0$).
- $i_2 = \mathcal{E}/R_1$
- $i_3 = \mathcal{E}/R_2$
- $i_1 = i_2 + i_3$

Ex: $R - L$ circuit



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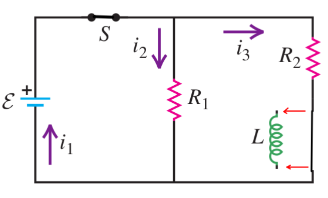

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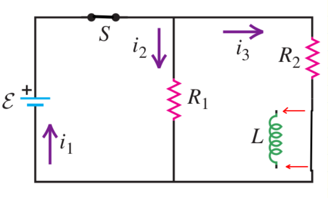
Ex: $R - L$ circuit



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