AA 284a
Advanced Rocket Propulsion

Lecture 2
Thrust Equation, Nozzles and Definitions

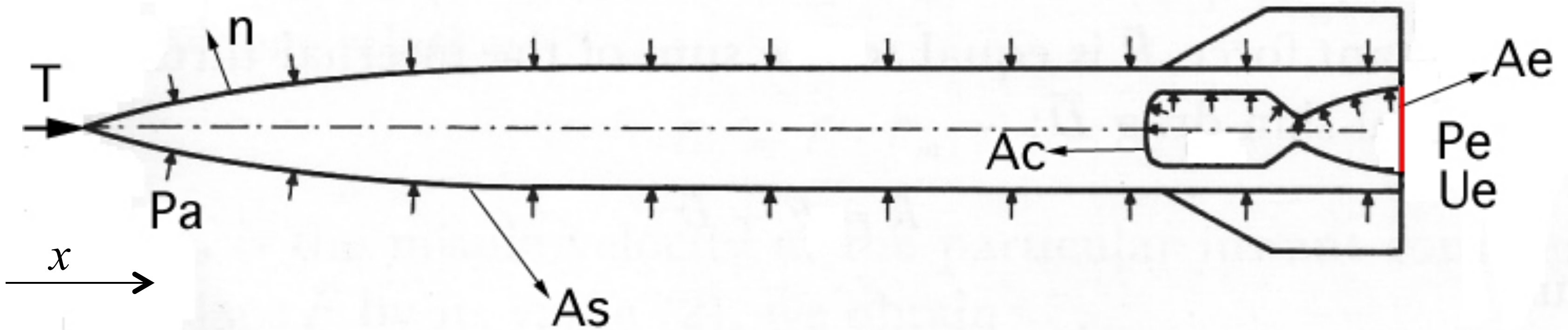
Prepared by
Arif Karabeyoglu

Department of Aeronautics and Astronautics
Stanford University
and
Mechanical Engineering
KOC University

September 25, 2015

Derivation of the Static Thrust Expression

As: Exterior Surface Ac: Interior Surface Ae: Exit Plane



- Force balance in the *x* direction

$$T + \int_{A_s} P_a \hat{n}\, dA\Big|_x + \int_{A_c} \left(P\bar{\bar{I}} - \bar{\bar{\tau}}\right)\cdot \hat{n}\, dA\Big|_x = 0$$

- *Assumption 1:* Static firing/ External gas is at rest
- *Assumption 2:* No body forces acting on the rocket

$\bar{\bar{I}}$: *Unity Tensor*

$\bar{\bar{\tau}}$: *Stress Tensor*

P : Pressure

P_a : *Ambient* Pressure

P_e : *Exit* Pressure

U_e : *Exit Velocity*

T : *Thrust Force*

Derivation of the Static Thrust Expression

- Combined to obtain the thrust force

$$\dot{m} : Mass\ Flow\ Rate$$

$$T = \rho_e u_e^2 A_e + \left(P_e - P_a\right) A_e$$

- Introduce the mass flow rate: $\dot{m} = \rho_e u_e A_e$

$$T = \dot{m}\ u_e + \left(P_e - P_a\right) A_e$$

- Two terms can be combined by introducing the effective exhaust velocity, V_e

$$T = \dot{m} V_e$$

- Maximum thrust for unit mass flow rate requires
  - High exit velocity
  - High exit pressure
- This cannot be realized. Compromise -> optimal expansion

Maximum Thrust Condition

- Thrust equation:

$$T = \dot{m}\, u_e + \left(P_e - P_a\right)A_e$$

- At fixed flow rate, chamber and atmospheric pressures, the variation in thrust can be written as

$$dT = \dot{m}\, du_e + \left(P_e - P_a\right)dA_e + A_e\, dP_e$$

- Momentum equation in 1D

$$A\rho u\, du + A\, dP = 0 \qquad \dot{m}\, du = -A\, dP$$

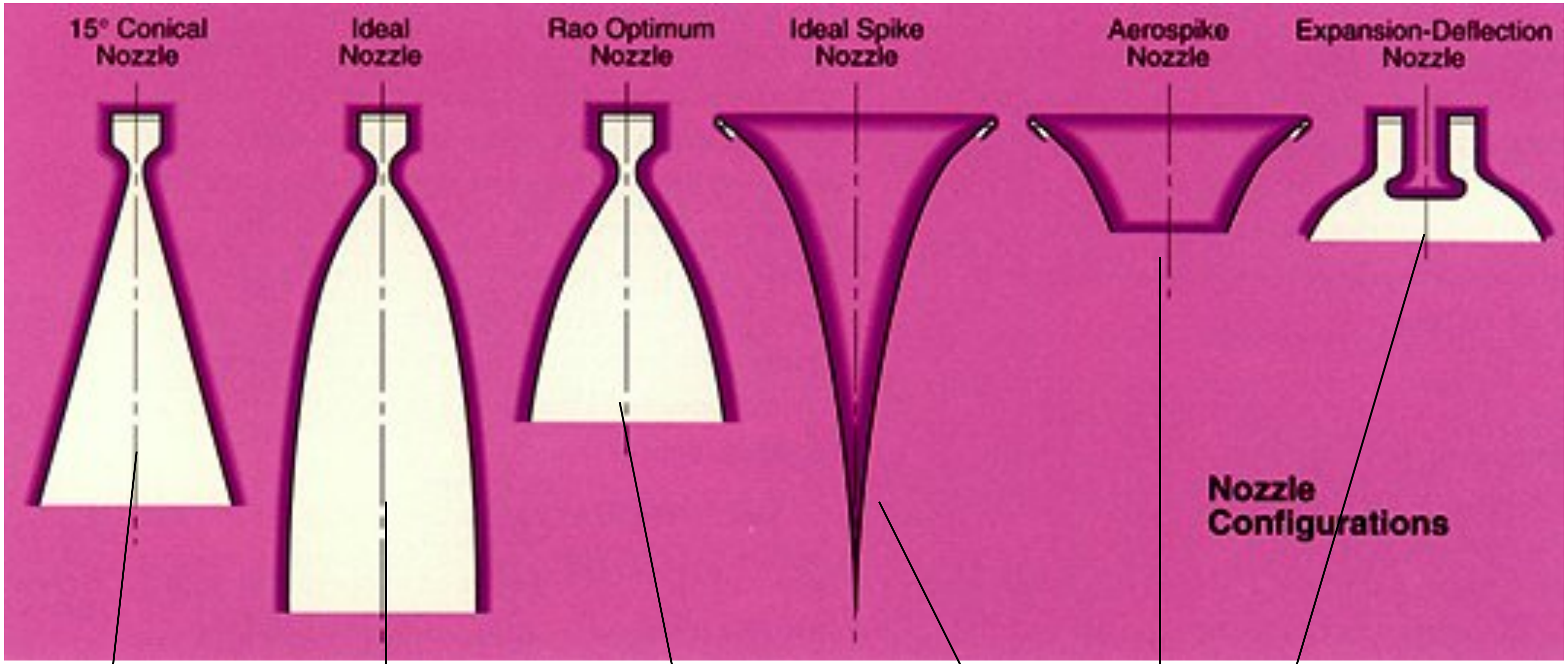
- Substitute in the differential expression for thrust

$$dT = \left(P_e - P_a\right)dA_e \qquad \frac{dT}{dA_e} = \left(P_e - P_a\right)$$

- Maximum thrust is obtained for a perfectly expanded nozzle

$$P_e = P_a$$

Nozzle Types



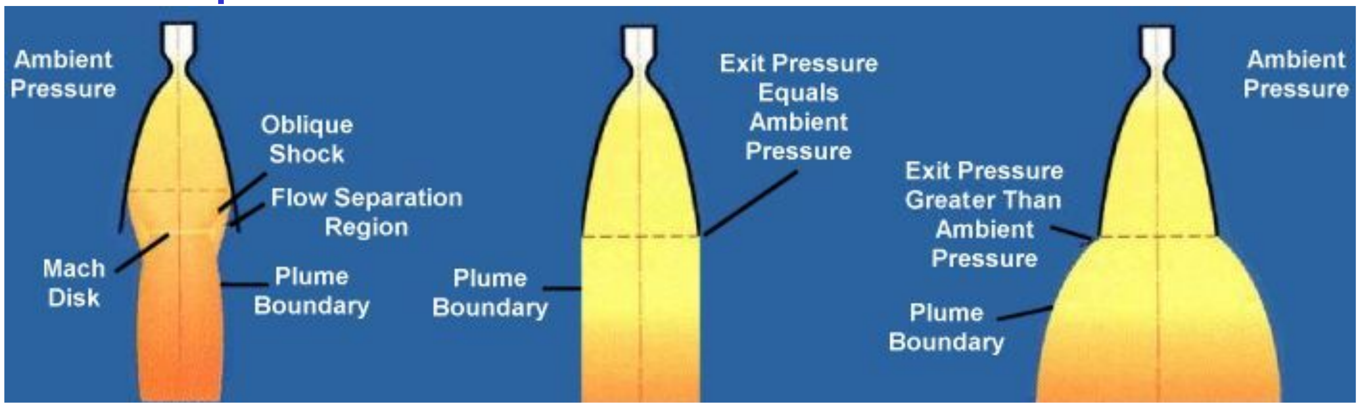
Used for small area ratios

Not used- too long

Used for large area ratios

Not used
In research phase

Bell Nozzle Operation

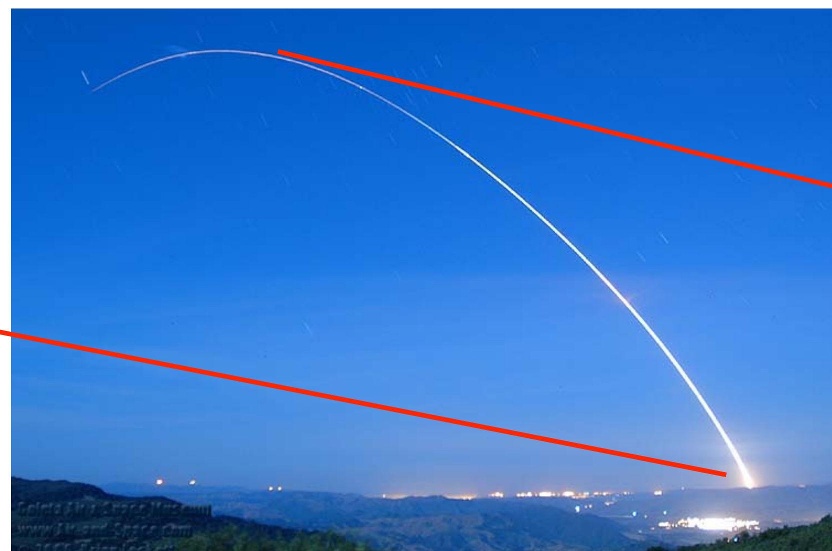


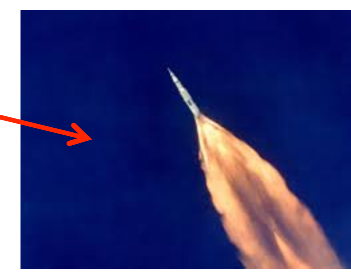
Pe<Pa

Pe=Pa

Pe>Pa

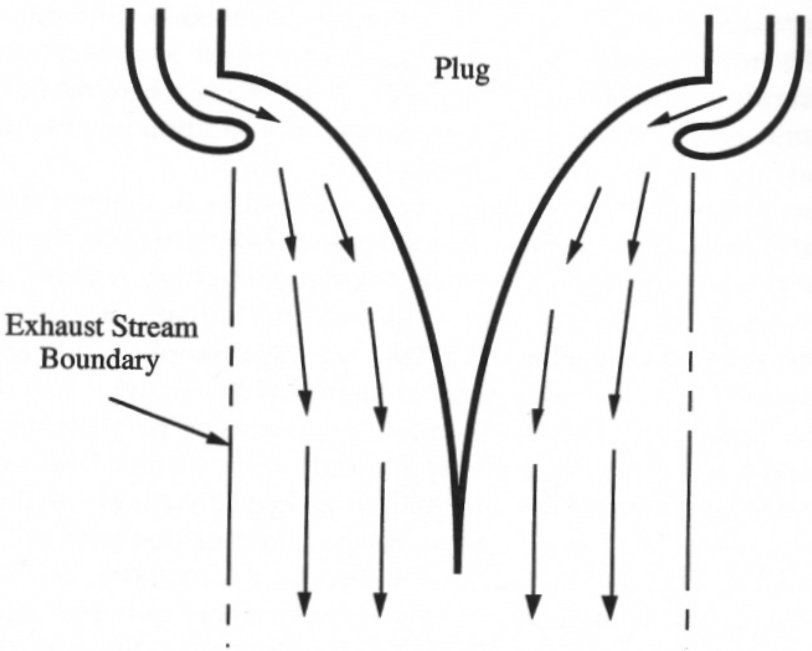
Low Altitude
Overexpanded
Nozzle

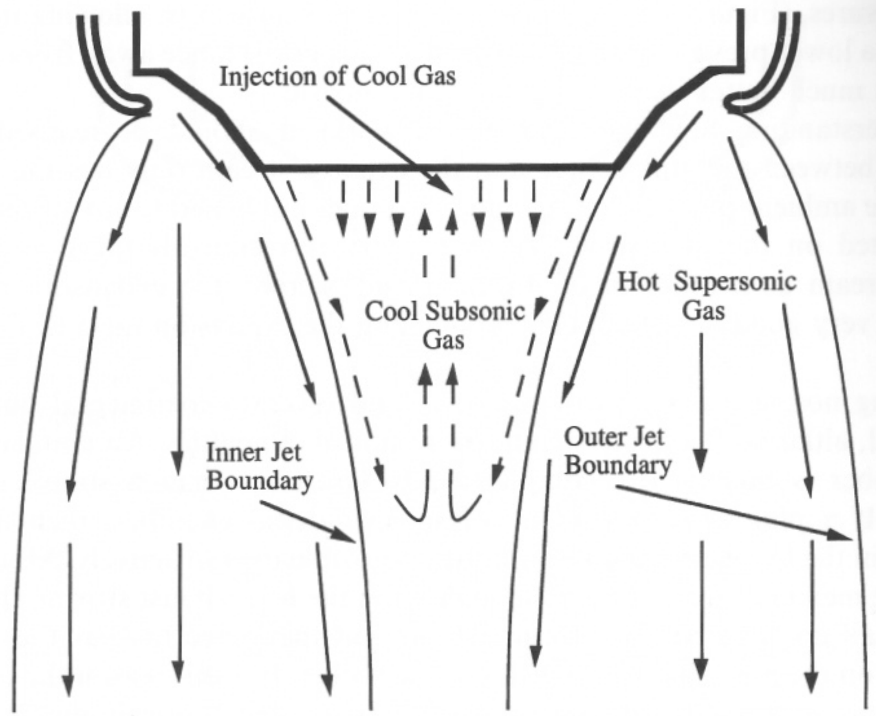




High Altitude
Underexpanded
Nozzle

Plug and Aerospike Nozzles





Definitions-Thrust Coefficient

- Thrust equation:

$$T = \dot{m}\, u_e + \left(P_e - P_a\right)A_e$$

- Thrust coefficient:

$$C_F \equiv \frac{T}{A_t P_{t2}} = \frac{\dot{m}\, u_e}{A_t P_{t2}} + \left(\frac{P_e}{P_{t2}} - \frac{P_a}{P_{t2}}\right)\frac{A_e}{A_t}$$

- For isentropic flow and calorically perfect gas in the nozzle the thrust coefficient can be written as

$$C_F = \left\{\left(\frac{2\gamma^2}{\gamma-1}\right)\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1-\left(\frac{P_e}{P_{t2}}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} + \left(\frac{P_e}{P_{t2}} - \frac{P_a}{P_{t2}}\right)\frac{A_e}{A_t}$$

Definitions - c* Equation

- Mass flow equation (choked and isentropic flow of a calorically perfect gas in the convergent section of the nozzle):

$$\dot{m} = \frac{A_t P_{t2}}{\sqrt{R T_{t2}}} \left[\gamma \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{\gamma-1}} \right]^{1/2}$$

- Definition of c*:

$$c^* \equiv \frac{P_{t2} A_t}{\dot{m}}$$

- c* can be expressed in terms of the operational parameters as

$$c^* = \left[\frac{1}{\gamma} \left(\frac{\gamma + 1}{2} \right)^{\frac{\gamma+1}{\gamma-1}} \frac{R_u T_{t2}}{M_w} \right]^{1/2}$$

Definitions – Specific Impulse and Impulse density

- Combine the definitions of the thrust coefficient and c* to express the thrust

$$T = \dot{m}\, c^* C_F$$

Energy of the gases
In the chamber

Flow in the nozzle

- Think of nozzle as a thrust amplifier and C_F as the gain
- Specific Impulse: Thrust per unit mass expelled

$$I_{sp} \equiv \frac{T}{\dot{m}\, g_o} = \frac{c^* C_F}{g_o}$$

g_o : *Gravitational Cons. on Earth*
ρ_p : *Propellant Density*

- Impulse Density: Thrust per unit volume of propellant expelled

$$\delta \equiv \frac{T}{\dot{V}_p} = \frac{T}{\dot{m}\, g_o} \frac{\dot{m}\, g_o}{\dot{V}_p} = I_{sp} \rho_p$$

$$\dot{V}_p \equiv \dot{m} / \rho_p$$

Definitions – Total Impulse, Average Thrust, Delivered Isp

- The total impulse is defined as

$$I_{tot} \equiv \int_0^{t_b} T dt = \int_0^{t_b} \dot{m}\, I_{sp} g_o dt$$

- Average thrust

$$\overline{T} \equiv \frac{1}{t_b} \int_0^{t_b} T dt = \frac{I_{tot}}{t_b}$$

t_b : *Total Burn Time*

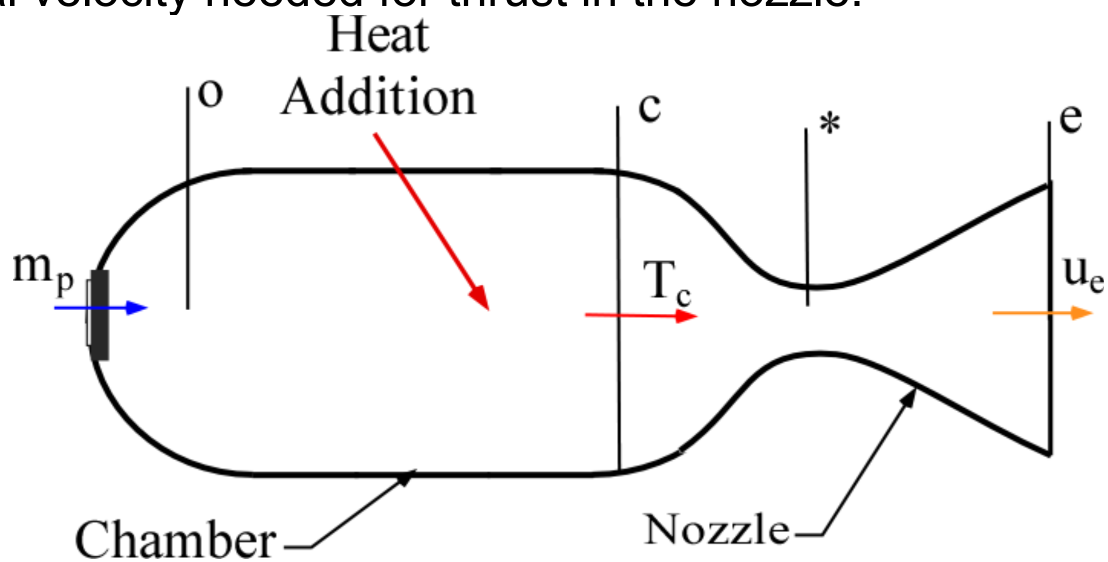
M_p : *Total Propellant Mass*

- Delivered I_{sp}

$$\left(I_{sp}\right)_{del} = \frac{I_{tot}}{M_p g_o}$$

Thermal Rocket – General Concept

- In a thermal rocket the propellant molecules are thermalized by addition of heat in a chamber.
- This thermal energy (random motion of the molecules) is converted to the useful directional velocity needed for thrust in the nozzle.



- The heat source varies
  - Nuclear energy: Thermonuclear rockets
  - Chemical bond energy: Chemical rockets
  - Electric energy: Resistojets and Arcjets
  - Thermal energy of the stored propellant: Cold gas thrusters