

# Lecture 9

## Laminar Diffusion Flame Configurations

# Different Flame Geometries and Single Droplet Burning

Solutions for the velocities and the mixture fraction fields for some typical **laminar flame configurations**.

Based on the assumption of **fast chemistry** we will then be able to calculate the flame contour defined by the condition  $Z(\mathbf{x}, t) = Z_{st}$ .

We will for simplicity always assume  $Le = 1$  and  $c_p = \text{const.}$

1<sup>st</sup> example: the flame stagnation point boundary layer (similar to the counterflow flow of the previous lecture but with different boundary conditions).

2<sup>nd</sup> example : vertical laminar 2D jet diffusion flame without/with buoyancy

3<sup>rd</sup> example : combustion of a single droplet surrounded by a diffusion flame.

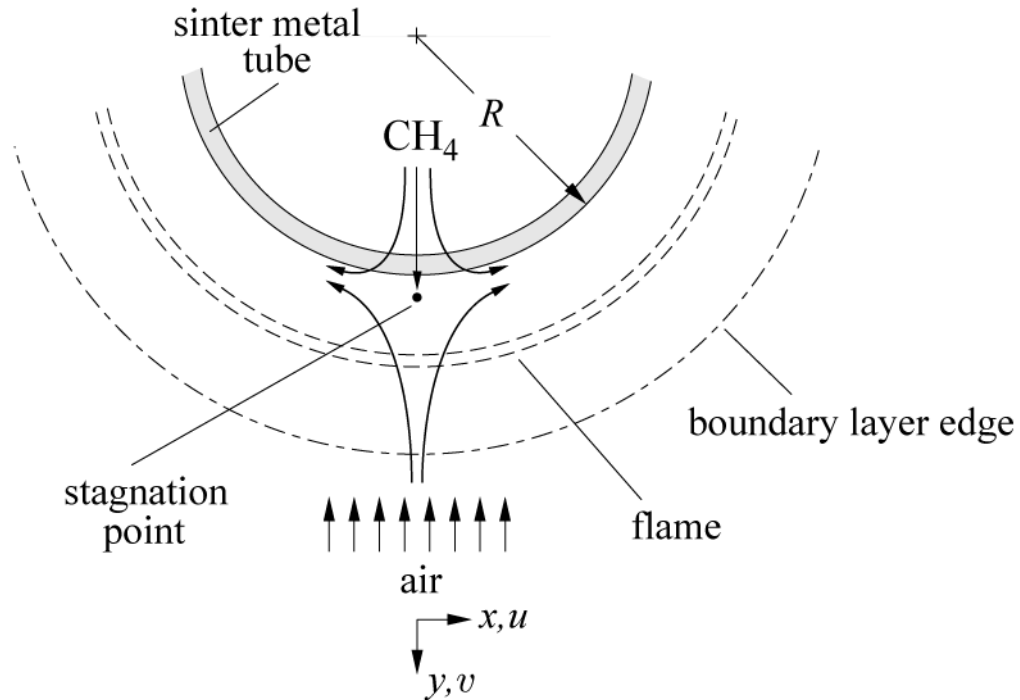
# Diffusion Flames in a Stagnation Point Boundary Layer: The Tsuji Flame

Gaseous fuel from a sinter metal tube is injected into the surrounding air which flows vertically upwards.

Below the tube a **stagnation point** is formed.

This burner is known as the **Tsuji burner**

If the Reynolds number based on the cylinder radius and the free stream velocity is large, the flow field may be split into an **inviscid outer flow** and a **boundary layer** close to the surface.



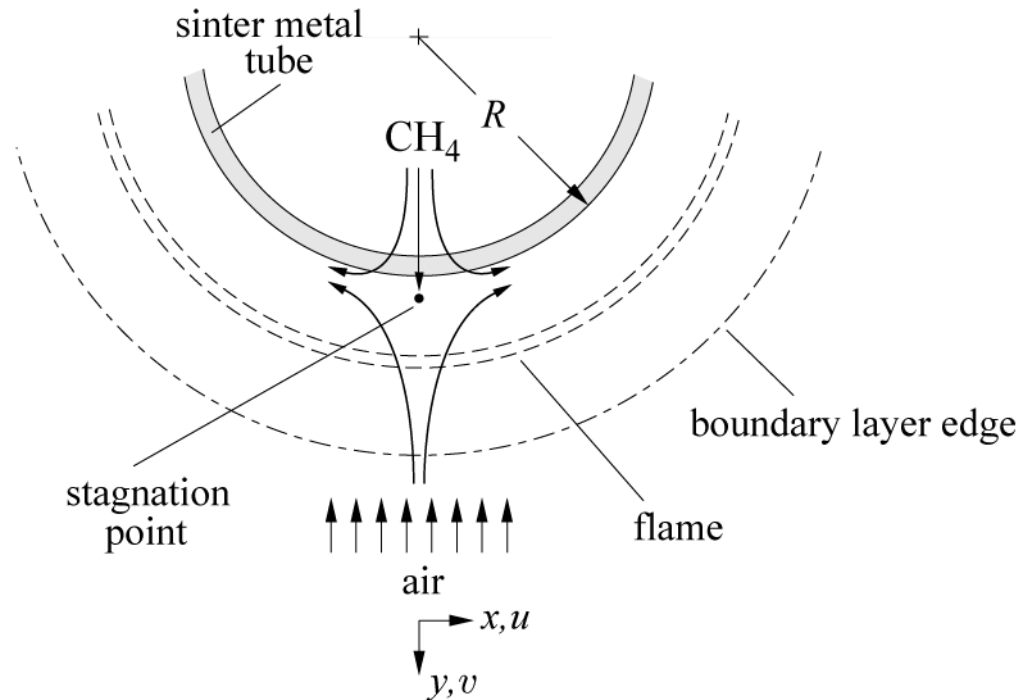
The potential flow solution for the flow around a cylinder then yields the velocity gradient at the stagnation point

$$a = \frac{2v_{\infty}}{R}$$

where  $v_{\infty}$  is the velocity very far from the cylinder.

The free-stream velocities at the edge of the boundary layer are

$$u_e = ax, \quad v_e = -ay.$$

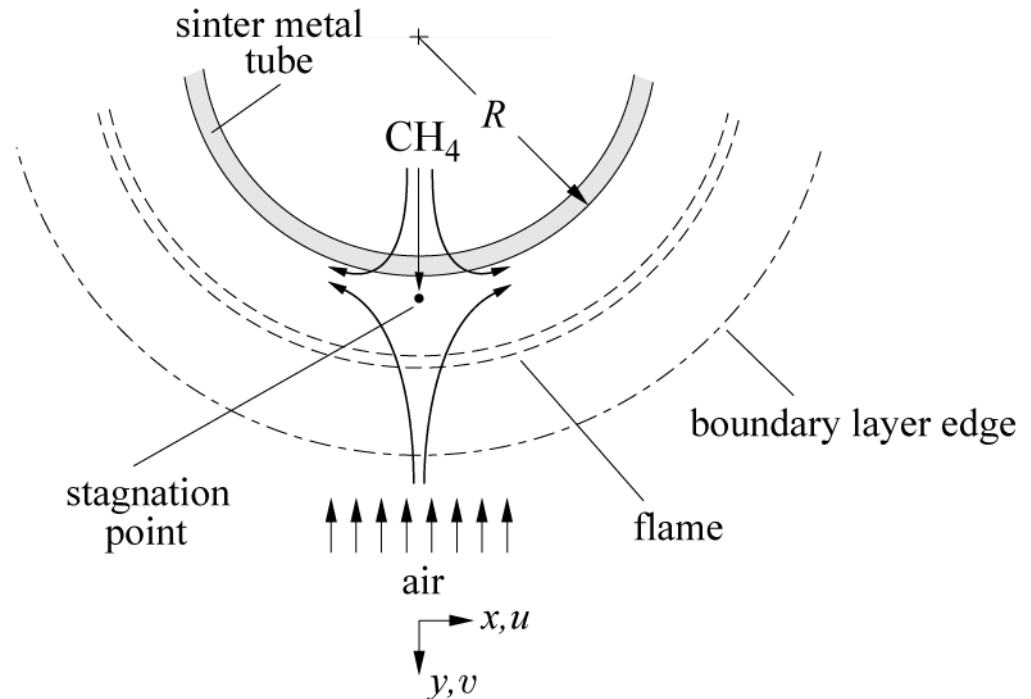


If the kinematic viscosity is small, the boundary layer thickness  $\delta$  is proportional to the viscous length

$$l_\nu = \sqrt{\frac{\nu_e}{a}}$$

where  $\nu_e$  is the kinematic viscosity at the edge of the boundary layer.

In case the boundary layer thickness is thin compared to the cylinder radii the curvature of the cylinder surface may be neglected and the boundary may be treated as two-dimensional allowing the usage of a Cartesian coordinate system.



# Governing equations for the boundary layer flow

continuity

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

momentum

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \left. \frac{dp}{dx} \right|_e + \frac{\partial}{\partial y} \left( \rho \nu \frac{\partial u}{\partial y} \right)$$

mixture fraction

$$\rho u \frac{\partial Z}{\partial x} + \rho v \frac{\partial Z}{\partial y} = \frac{\partial}{\partial y} \left( \rho D \frac{\partial Z}{\partial y} \right)$$

Bernoulli's equation for the pressure gradient at the boundary layer edge

$$\rho_e u_e \frac{du_e}{dx} = - \left. \frac{dp}{dx} \right|_e = \rho_e a^2 x$$

## Boundary conditions

$$u = 0, \quad \dot{m}_w = (\rho v)_w, \quad (\rho D)_w \frac{\partial Z}{\partial y} \Big|_w = m_w(Z_w - 1) \quad \text{at} \quad y = 0$$

and

$$u = ax, \quad Z = 0 \quad \text{at} \quad y \rightarrow \infty.$$

Here  $u = 0$  is the symmetry condition at the surface.

The mass flow rate  $\dot{m}_w$  of fuel issuing through the porous metal into the boundary layer is imposed.

The boundary condition for the mixture fraction equation represents the integrated mixture fraction balance at the surface of the porous metal by assuming that the mixture fraction gradient within the metal is zero.

The continuity equation is satisfied by introducing the stream function  $\psi$  such that

$$\rho u = \frac{\partial \psi}{\partial y}, \quad \rho v = -\frac{\partial \psi}{\partial x}$$

Introducing the similarity variable

$$\eta = \left(\frac{a}{\nu_e}\right)^{1/2} \int_0^y \frac{\rho}{\rho_e} dy$$

a non-dimensional stream function  $f(\eta)$  is defined as

$$f(\eta) = \frac{\psi}{\rho_e x \sqrt{a \nu_e}}$$



The velocities are

$$u = ax \frac{\partial f}{\partial \eta}, \quad v = -\frac{\rho_e}{\rho} (a\nu_e)^{1/2} f(\eta)$$

One obtains the similarity equations

$$-f \frac{d^2 f}{d\eta^2} = \left[ \frac{\rho_e}{\rho} - \left( \frac{df}{d\eta} \right)^2 \right] \frac{d}{d\eta} \left( C \frac{d^2 f}{d\eta^2} \right)$$

$$-f \frac{dZ}{d\eta} = \frac{d}{d\eta} \left( \frac{C}{Sc} \frac{dZ}{d\eta} \right)$$

Here  $Sc = \nu / D$  is the **Schmidt number** and  $C$  is the **Chapman-Rubesin parameter**

$$C = \frac{\rho^2 \nu}{\rho_e^2 \nu_e}$$

Since  $\nu$  changes with temperature as  $T^{1.7}$  and  $\rho$  as  $T^1$ , this quantity changes less than the viscosity itself in a flow with strong heat release.

The boundary conditions for the similar solution are

$$f_w = -\frac{(\rho\nu)_w}{(\rho_e^2 a \nu_e)^{1/2}}, \quad f' = 0, \quad \left. \frac{C}{Sc} \frac{dZ}{d\eta} \right|_w = f_w(1-Z) \quad \text{at} \quad \eta = 0$$

$$f'(\infty) = 1 \quad \text{for} \quad \eta \rightarrow \infty$$

The mixture fraction equation may be transformed into

$$\frac{d}{d\eta} \left[ \ln \left( \frac{C}{Sc} \frac{dZ}{d\eta} \right) \right] = -\frac{fSc}{C}$$

which can formally be solved as

$$Z = f_w \frac{I(\eta) - I(\infty)}{1 - f_w I(\infty)}$$

where

$$I(\eta) = \int_0^\eta \frac{Sc}{C} \exp \left( - \int_0^\eta \frac{fSc}{C} d\eta \right) d\eta$$

The mixture fraction at the surface is given by

$$Z_w = \frac{-f_w I(\infty)}{1 - f_w I(\infty)}$$

This indicates that the mixture fraction varies between  $Z = 0$  and  $Z = Z_w$  rather than between 0 and 1.

The boundary condition for the fuel and oxidizer fractions satisfy the Burke-Schumann solution at  $Z_w$ , as may easily be shown.

The boundary condition for the temperature at the surface is to be imposed at  $Z = Z_w$ .

If the mass flow rate at the surface is increased and  $f_w$  takes large negative values, the mixture fraction at the surface tends towards unity.

This is the limit of a counter-flow diffusion flame detached from the surface.

The equations

$$-f \frac{d^2 f}{d\eta^2} = \left[ \frac{\rho_e}{\rho} - \left( \frac{df}{d\eta} \right)^2 \right] \frac{d}{d\eta} \left( C \frac{d^2 f}{d\eta^2} \right)$$

$$-f \frac{dZ}{d\eta} = \frac{d}{d\eta} \left( \frac{C}{Sc} \frac{dZ}{d\eta} \right)$$

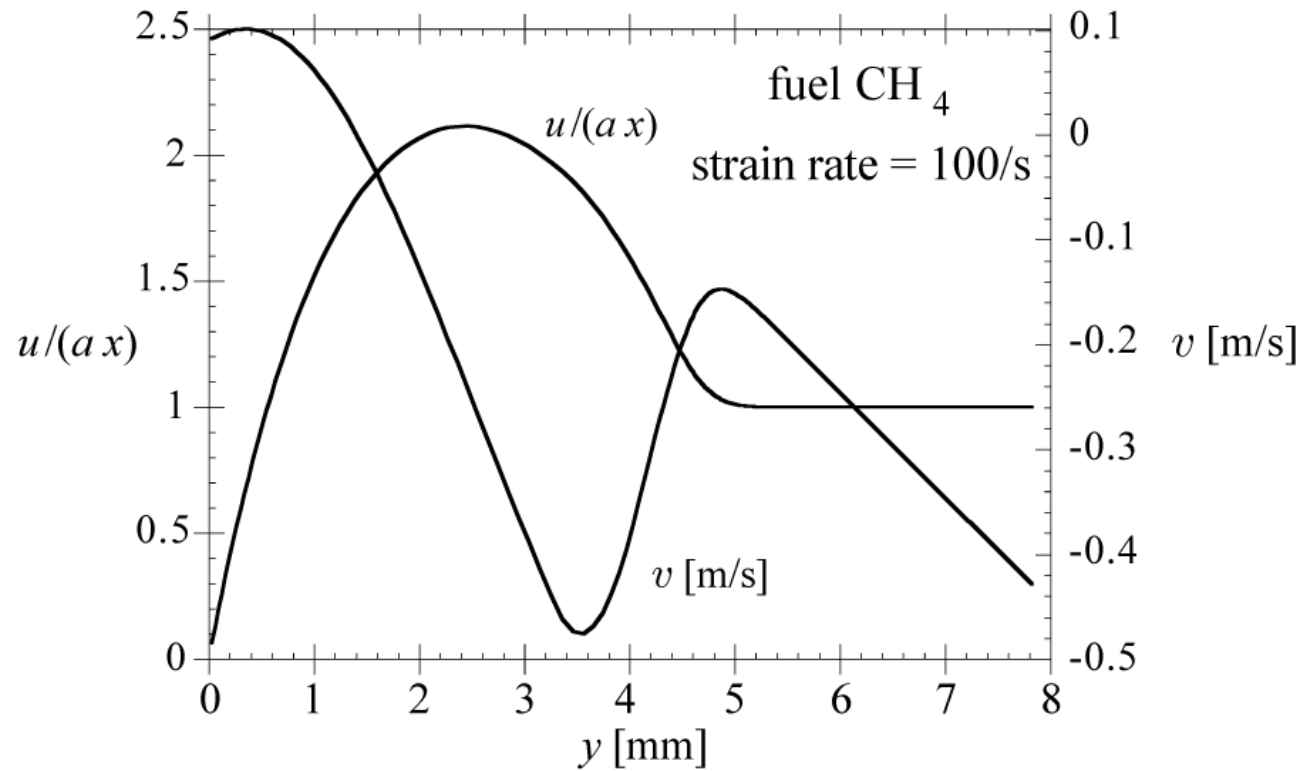
have been solved numerically using the Burke-Schumann solution for combustion of methane in air

$$T_b(Z) = T_u(Z) + \frac{Q_{\text{ref}} Y_{F,1}}{c_p \nu'_F W_F} Z, \quad Z \leq Z_{st},$$

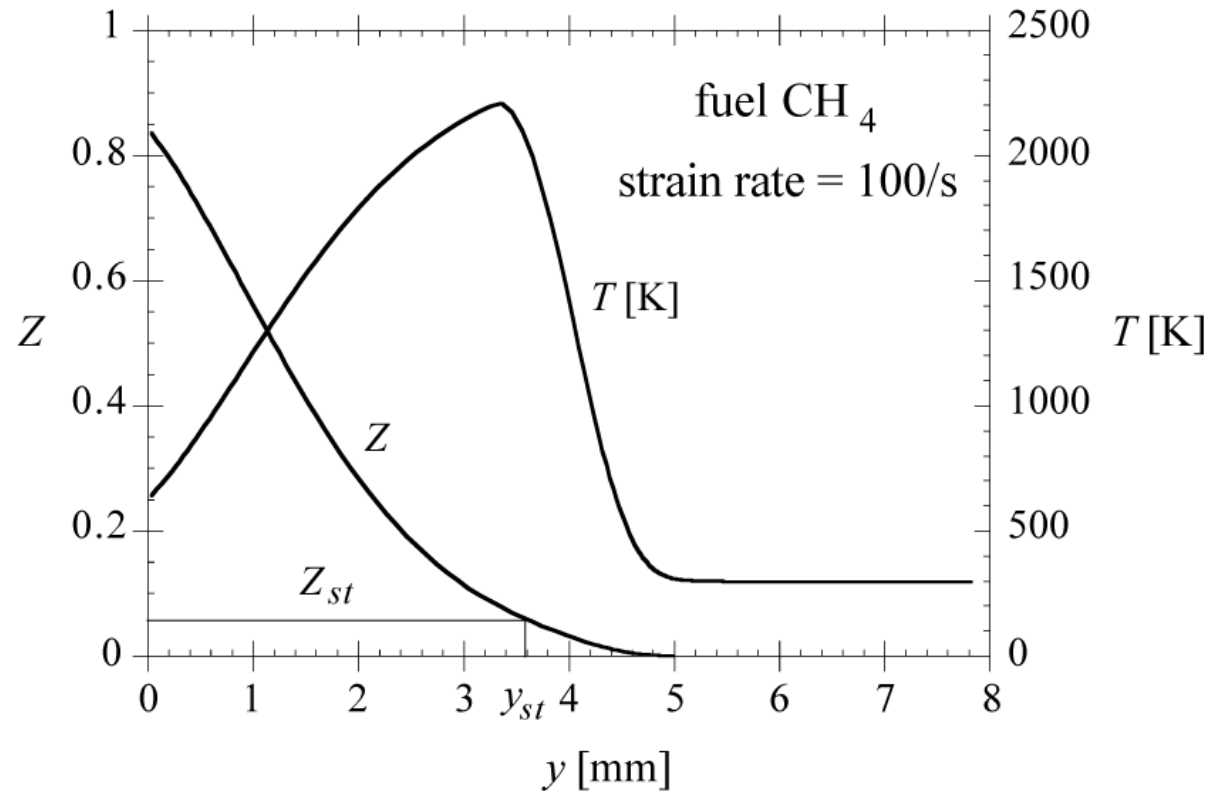
$$T_b(Z) = T_u(Z) + \frac{Q_{\text{ref}} Y_{O_2,2}}{c_p \nu'_{O_2} W_{O_2}} (1 - Z) \quad Z \geq Z_{st}$$

with  $Z_{st} = 0.055$ ,  $T_u = 300$  K and  $T_{st} = T_b(Z_{st}) = 2263$  K.

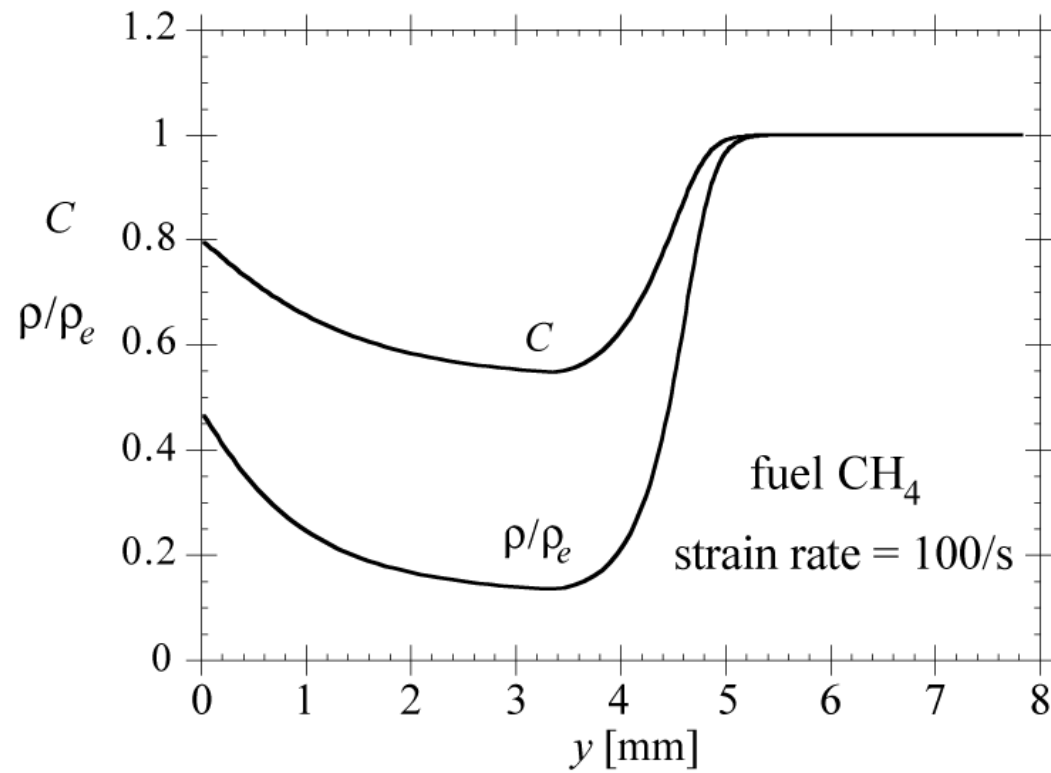
## Profiles of $u=ax$ , $v$



## Profiles of $Z$ , $T$



## Profiles of $\rho = \rho_e$ and $C$



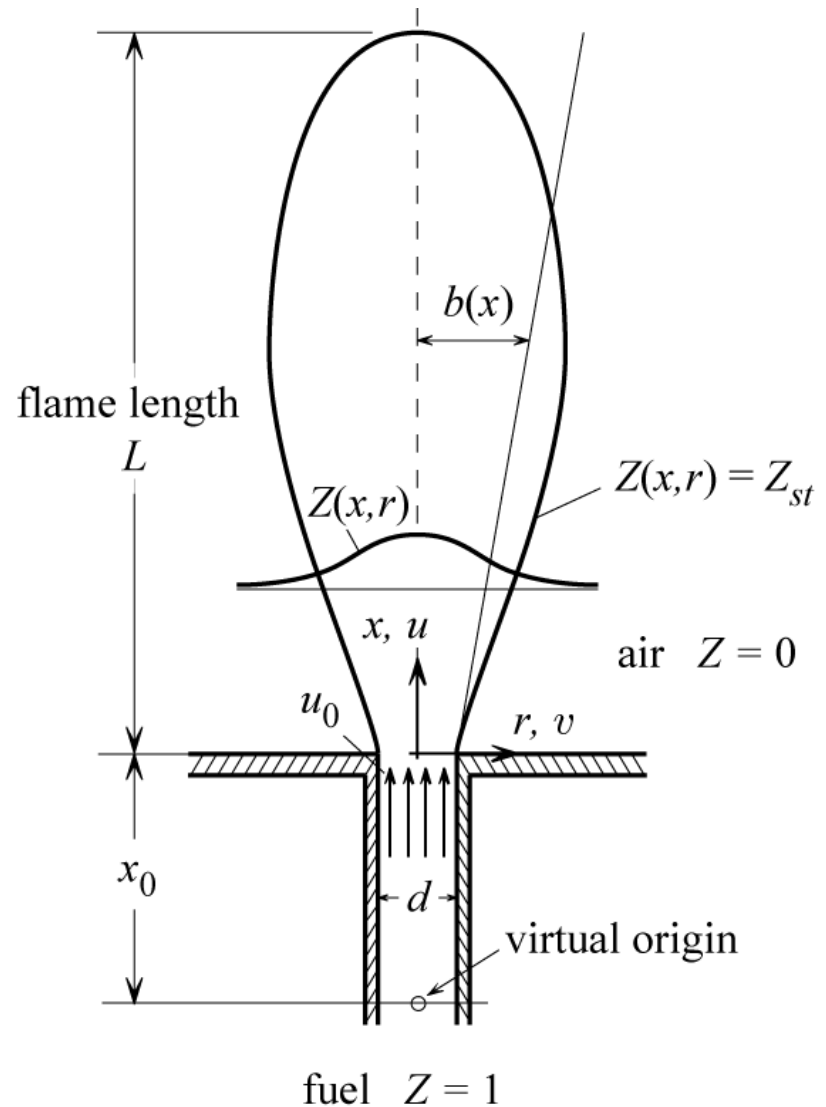


## The Round Laminar Diffusion Flame

Fuel enters into the combustion chamber as a round jet, either laminar or turbulent.

To provide an understanding of the basic properties of jet diffusion flames, we will consider here the easiest case, the **axisymmetric jet flame without buoyancy**.

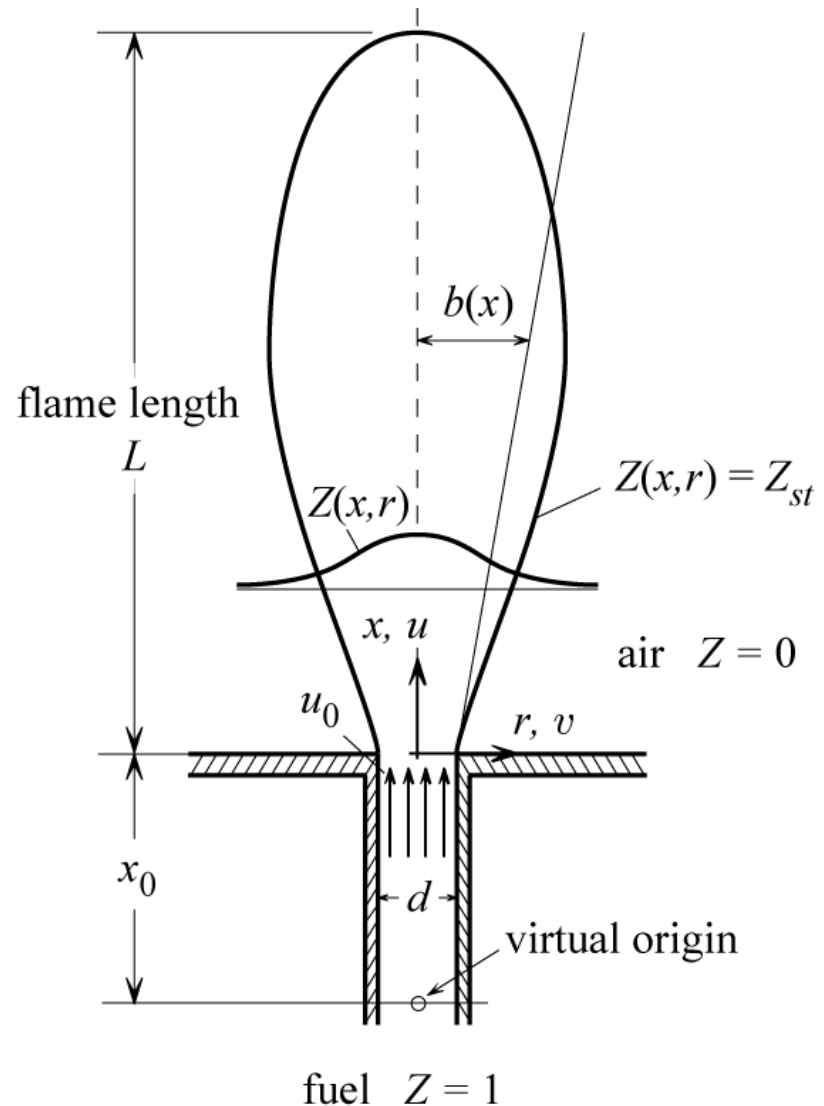
This will enable us to determine the **flame length**.



The flame length is defined as the distance from the nozzle to the point on the centerline of the flame where for the mixture fraction  $Z = Z_{st}$ .

We consider a fuel jet issuing from a nozzle with diameter  $d$  and exit velocity  $u_0$  into quiescent air.

The indices 0 and  $\infty$  denote conditions at the nozzle and in the ambient air.



Using the boundary layer assumption with constant pressure we obtain a system of two-dimensional axisymmetric equations, in terms of the axial coordinate  $x$  and the radial coordinate  $r$ :

continuity

$$\frac{\partial(\rho ur)}{\partial x} + \frac{\partial(\rho vr)}{\partial r} = 0$$

momentum in  $x$  direction

$$\rho ur \frac{\partial u}{\partial x} + \rho vr \frac{\partial u}{\partial r} = \frac{\partial}{\partial r} \left( \mu r \frac{\partial u}{\partial r} \right)$$

mixture fraction

$$\rho ur \frac{\partial Z}{\partial x} + \rho vr \frac{\partial Z}{\partial r} = \frac{\partial}{\partial r} \left( \frac{\mu}{Sc} r \frac{\partial Z}{\partial r} \right)$$

Schmidt number is defined as  $Sc = \nu/D$ .

The dimensionality of the problem may be reduced by introducing the similarity transformation

$$\eta = \frac{\bar{r}}{\xi}, \quad \bar{r}^2 = 2 \int_0^r \frac{\rho}{\rho_\infty} r \, dr, \quad \xi = x + x_0,$$

which contains a density transformation defining the density weighted radial coordinate.

The new axial coordinate  $\xi$  starts from the virtual origin of the jet located at  $x = -x_0$ .

Introducing a stream function by

$$\rho u r = \frac{\partial \psi}{\partial r}, \quad \rho v r = -\frac{\partial \psi}{\partial x}$$

we can satisfy the continuity equation.

The convective terms in the momentum equation and in the equation for the mixture fraction may be expressed using the transformation rules

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial r} = \frac{\partial \eta}{\partial r} \frac{\partial}{\partial \eta}$$

which leads to

$$\rho_{ur} \frac{\partial}{\partial x} + \rho_{vr} \frac{\partial}{\partial r} = \frac{\partial \eta}{\partial r} \left( \frac{\partial \psi}{\partial \eta} \frac{\partial}{\partial \xi} - \frac{\partial \psi}{\partial \xi} \frac{\partial}{\partial \eta} \right)$$

For the diffusive terms one obtains

$$\frac{\partial}{\partial r} \left( \mu r \frac{\partial}{\partial r} \right) = \mu_{\infty} \frac{\partial \eta}{\partial r} \frac{\partial}{\partial \eta} \left( C \eta \frac{\partial}{\partial \eta} \right)$$

Chapman-Rubesin-parameter  $C = \frac{\rho \mu r^2}{2 \mu_{\infty} \int_0^r \rho r dr}$

For constant density with  $\mu = \mu_{\infty}$  one obtains  $C = 1$ .

The axial and radial velocity components may now be expressed in terms of the nondimensional stream function  $F(\xi, \eta)$  defined by

$$\psi = \mu_{\infty} \xi F(\xi, \eta)$$

as

$$u = \frac{\frac{\partial F}{\partial \eta} \mu_{\infty}}{\eta \rho_{\infty} \xi}, \quad \rho v r = -\mu_{\infty} \left( \xi \frac{\partial F}{\partial \xi} + F - \frac{\partial F}{\partial \eta} \eta \right)$$

For the mixture fraction the ansatz

$$Z = Z_{CL}(\xi) \omega(\eta)$$

is introduced, where  $Z_{CL}$  stands for the mixture fraction on the centerline.

For a jet into still air a similarity solution exists if the nondimensional stream function  $F$  and Chapman Rubesin parameter  $C$  are no function of  $\xi$ .

Then one obtains the ordinary differential equations, valid in the similarity region of the jet:

$$-\frac{d}{d\eta}\left(\frac{F}{\eta}\frac{dF}{d\eta}\right) = \frac{d}{d\eta}\left(C\eta\frac{d}{d\eta}\left(\frac{1}{\eta}\frac{dF}{d\eta}\right)\right)$$

$$-\frac{d}{d\eta}(F\omega) = \frac{d}{d\eta}\left(\frac{C}{Sc}\eta\frac{d\omega}{d\eta}\right).$$

To derive an analytical solution we must assume that  $C$  is a constant in the entire jet. The solution is:

$$F(\eta) = \frac{C(\gamma\eta)^2}{1 + (\gamma\eta/2)^2}, \quad \omega(\eta) = \left(\frac{1}{1 + (\gamma\eta/2)^2}\right)^{2Sc}$$

The axial velocity profile then is obtained from

$$u = \frac{\frac{dF}{d\eta}}{\eta} \frac{\mu_{\infty}}{\rho_{\infty} \xi}$$

as

$$u = \frac{2C\gamma^2\nu}{\xi} \left( \frac{1}{1 + (\gamma\eta/2)^2} \right)^2$$

with the jet spreading parameter

$$\gamma^2 = \frac{3 \cdot \text{Re}^2}{64} \frac{\rho_0}{\rho_{\infty} C^2}$$



The jet spreading parameter

$$\gamma^2 = \frac{3 \cdot \text{Re}^2}{64} \frac{\rho_0}{\rho_\infty C^2}$$

is obtained from the requirement of integral momentum conservation along the axial direction:

$$\int_0^\infty \rho u^2 r dr = \rho_0 u_0^2 d^2 / 8$$

Here  $\rho_0$  is the density of the fuel and  $\text{Re} = \rho_\infty u_0 d / \mu_\infty$  is the [Reynolds number](#).

Similarly, conservation of the mixture fraction integral across the jet yields the mixture fraction on the centerline

$$Z_{CL} = \frac{\text{Re}(1 + 2\text{Sc})}{32} \frac{\rho_0}{\rho_\infty C} \frac{d}{\xi}$$

such that the mixture fraction profile is given by

$$Z = \frac{(1 + 2\text{Sc})d\text{Re}}{\xi} \frac{\rho_0}{32 \rho_\infty C} \left( \frac{1}{1 + (\gamma\eta/2)^2} \right)^{2\text{Sc}}$$

From this equation the flame length  $L$  can be calculated by setting  $Z = Z_{st}$  at  $x = L$ ,  $r = 0$ :

$$\frac{L + x_0}{d} = \frac{(1 + 2\text{Sc})}{32 \cdot Z_{st}} \frac{\rho_0}{\rho_\infty C} \frac{u_0 d}{\nu}$$

This shows that the flame length of a laminar round jet increases linearly with increasing exit velocity  $u_0$ .

# Single Droplet Combustion

In many practical applications liquid fuel is injected into the combustion chamber resulting in a fuel spray.

By the combined action of aerodynamical shear, strain, and surface tension the liquid spray will decompose into a large number of single droplets of different diameters.

The fuel will then evaporate and a non-homogeneous fuel air mixture will be formed in the flow field surrounding the droplets.

When the spray is ignited, the droplets will burn either as a cloud surrounded by a enveloping flame or as single droplets, each being surrounded by its own diffusion flame.

The former will be the case if the fuel air mixture between different droplets is fuel rich such that the surface of stoichiometric mixture will surround the droplet cloud.

We will consider here the latter case, where the surface of stoichiometric mixture surrounds the single droplet.

We will furthermore consider very small droplets which follow the flow very closely and assume that the velocity difference between the droplet and the surrounding fuel is zero.

Therefore we may consider the case of a spherically symmetric droplet in a quiescent surrounding.

We assume the evaporation and combustion process as quasi-steady and can therefore use the steady state

continuity

$$\frac{d}{dr}(r^2 \rho v) = 0$$

mixture fraction

$$\rho v \frac{dZ}{dr} = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \rho D \frac{dZ}{dr} \right)$$

temperature

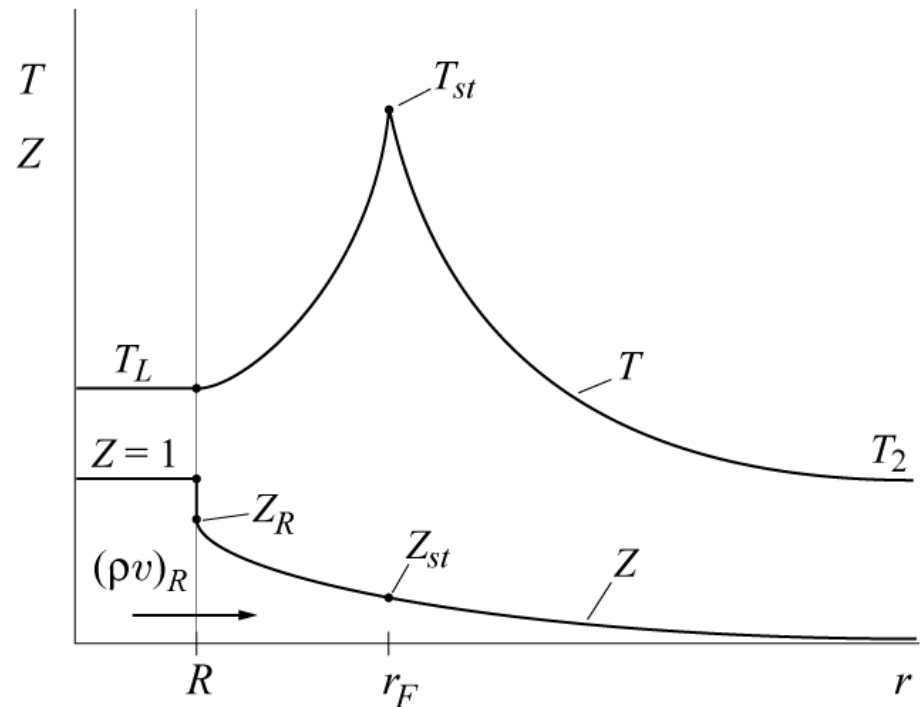
$$\rho v \frac{dT}{dr} = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{\lambda}{c_p} \frac{dT}{dr} \right) + \frac{Q}{c_p} \omega$$

In these equations  $r$  is the radial coordinate, and  $v$  is the flow velocity in radial direction.

Here again  $Le = 1$  leading to  $\lambda = \rho c_p D$  and a one step reaction with fast chemistry will be assumed.

The reaction rate  $\omega$  is then a  $\delta$ -function at the flame surface located at  $Z = Z_{st}$ .

Expected temperature and mixture fraction profiles



The boundary conditions at the droplet surface  $r = R$  are obtained by integrating the balance equations once in radial direction.

Since temperature and concentration gradients within the droplet are assumed negligible, the convective flux through the surface equals the diffusive flux in the gas phase at the droplet surface.

The convective heat flux through the boundary involves a change of enthalpy, namely the enthalpy of evaporation  $h_L$ .

Therefore

$$r = R : \quad \lambda \frac{dT}{dr} \Big|_R = (\rho v)_R h_L$$

Here  $(\rho v)_R$  is the convective mass flux through the surface.

The mixture fraction of the convective flux involves the difference between the mixture fraction within the droplet, which is unity by definition, and that in the gas phase at the droplet surface, where  $Z = Z_R$ .

This leads to

$$r = R : \quad \rho D \frac{dZ}{dr} \Big|_R = (\rho v)_R (Z_R - 1)$$

The boundary conditions in the surrounding air are

$$r \rightarrow \infty : \quad T = T_2, \quad Z = 0$$



In addition, we assume that the temperature  $T$  at the droplet surface is equal to the boiling temperature of the liquid  $T = T_L$ .

Then the temperature equation must satisfy three boundary conditions.

This leads to an **eigenvalue** problem for the mass burning rate

$$\dot{m} = 4\pi R^2(\rho v)_R$$

of the droplet which thereby can be determined.

Integration of the continuity equation leads to

$$\rho v r^2 = R^2(\rho v)_R$$

We will now introduce the nondimensional coordinate

$$\eta = \int_r^\infty \frac{v}{D} \exp(-\zeta) dr$$

where

$$\zeta = \int_r^\infty \frac{v}{D} dr = \frac{\dot{m}}{4\pi} \int_r^\infty (\rho D r^2)^{-1} dr$$

Between  $\eta$  and  $\zeta$  there is the relation

$$\frac{d\eta}{d\zeta} = \frac{d\eta/dr}{d\zeta/dr} = \exp(-\zeta)$$

$$\frac{d\eta}{d\zeta} = \frac{d\eta/dr}{d\zeta/dr} = \exp(-\zeta)$$

may be integrated with the boundary conditions at  $r \rightarrow \infty$ :  $\zeta = 0$ ,  $\eta = 0$  to yield

$$\eta = 1 - \exp(-\zeta)$$

and at  $r = R$

$$\eta_R = 1 - \exp(-\zeta_R).$$

## Transformation of

$$\rho v \frac{dZ}{dr} = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \rho D \frac{dZ}{dr} \right), \quad \rho v \frac{dT}{dr} = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{\lambda}{c_p} \frac{dT}{dr} \right) + \frac{Q}{c_p} \omega$$

with their boundary conditions leads to

$$\frac{d^2 Z}{d\eta^2} = 0$$

$$\eta = \eta_R : \quad (\eta_R - 1) \frac{dZ}{d\eta} = Z_R - 1$$

$$\eta \rightarrow \infty : \quad Z = 0, \quad \rho D \left( \frac{d\eta}{dr} \right)^2 \frac{d^2 T}{d\eta^2} = -\frac{Q}{c_p} \omega$$

$$\eta = \eta_R : \quad (\eta - 1) \frac{dT}{d\eta} = \frac{h_L}{c_p}, \quad T_R = T_L$$

$$\eta \rightarrow 0 : \quad T = T_2$$

The solution of the mixture fraction equation with its boundary condition is readily seen to be

$$Z = \eta$$

This and the scalar dissipation rate defined here as

$$\chi = 2D \left( \frac{dZ}{dr} \right)^2 = 2D \left( \frac{d\eta}{dr} \right)^2$$

is introduced into the temperature equation.

$$\rho \frac{\chi d^2 T}{2 dZ^2} = -\frac{Q}{c_p} \omega$$

This equation

$$\rho \frac{\chi}{2} \frac{d^2 T}{dZ^2} = -\frac{Q}{c_p} \omega$$

compares to the flamelet equation derived in the previous lecture

$$\rho \frac{\partial T}{\partial \tau} - \frac{\chi}{2} \frac{\partial^2 T}{\partial Z^2} \approx \sum_{i=1}^k \dot{m}_i \frac{h_i}{c_p} + \frac{\dot{q}_R}{c_p} + \frac{1}{c_p} \frac{\partial p}{\partial t}$$

if steady state, negligible heat loss, and one-step chemistry are assumed.

→ The one-dimensional droplet combustion problem satisfies the laminar flamelet assumptions exactly.

Here we want again consider the Burke-Schumann-solution

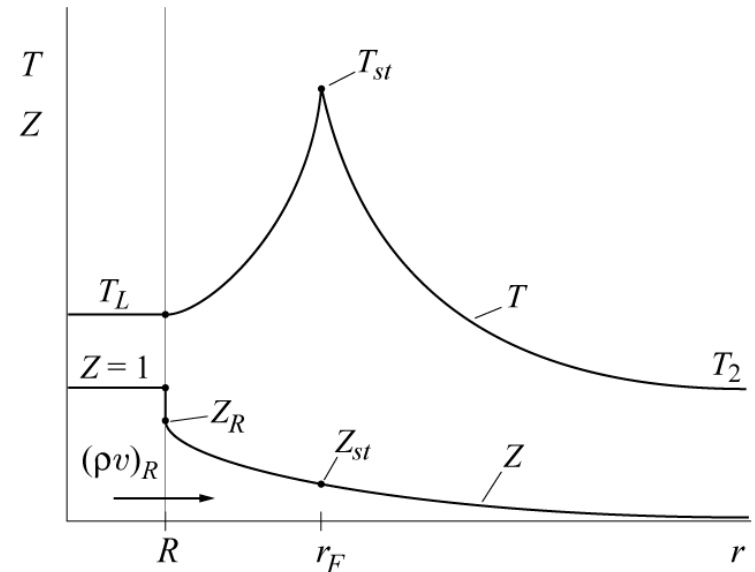
$$T_b(Z) = T_u(Z) + \frac{Q_{\text{ref}} Y_{F,1}}{c_p \nu'_F W_F} Z, \quad Z \leq Z_{st},$$

$$T_b(Z) = T_u(Z) + \frac{Q_{\text{ref}} Y_{O_2,2}}{c_p \nu'_{O_2} W_{O_2}} (1 - Z) \quad Z \geq Z_{st}$$

Then, in the fuel rich region between  
 $r = R$  and  $r = r_F$  we have

$$T(Z) = T_2 + Z(T_1 - T_2) + \frac{Y_{O_2,2} Q}{c_p \nu'_{O_2} W_{O_2}} (1 - Z).$$

Here  $T_1$  is by definition the temperature at  $Z = 1$ ,  
 problem.



We know, however, from the boundary conditions

$$\eta = \eta_R : \quad (\eta - 1) \frac{dT}{d\eta} = \frac{h_L}{c_p}, \quad T_R = T_L$$

the slope and the value at  $Z = Z_R$  where

$$T_L = T_2 + Z_R(T_1 - T_2) + \frac{Y_{O_2,2} Q}{c_p \nu'_{O_2} W_{O_2}} (1 - Z_R)$$

Introducing this and

$$T(Z) = T_2 + Z(T_1 - T_2) + \frac{Y_{O_2,2} Q}{c_p \nu'_{O_2} W_{O_2}} (1 - Z)$$

into the boundary condition above one obtains

$$T_1 = T_L - \frac{h_L}{c_p}$$



$T_1$  is a hypothetical temperature corresponding to the fuel if one considers the droplet as a point source of gaseous fuel.

The heat of vaporization then decreases the temperature of the liquid fuel by the amount  $h_L/c_p$ .

It should be used in flamelet calculations if one wishes to calculate flamelet profiles in the range  $0 < Z < 1$  rather than  $0 < Z < Z_R$ .

The boundary condition

$$\eta = \eta_R : \quad (\eta - 1) \frac{dT}{d\eta} = \frac{h_L}{c_p}, \quad T_R = T_L$$

may also be used with

$$T(Z) = T_2 + Z(T_1 - T_2) + \frac{Y_{O_2,2} Q}{c_p \nu'_{O_2} W_{O_2}} (1 - Z).$$

and

$$\eta_R = 1 - \exp(-\zeta_R)$$

to calculate the non-dimensional mass burning rate

$$\zeta_R = \ln \left( 1 + \frac{c_p(T_2 - T_L) + Y_{O_2,2} Q / \nu'_{O_2} W_{O_2}}{h_L} \right)$$

From this, the mass burning rate may be determined using

$$\zeta = \int_r^\infty \frac{v}{D} dr = \frac{\dot{m}}{4\pi} \int_r^\infty (\rho D r^2)^{-1} dr.$$

We will introduce radially averaged properties

$$(\overline{\rho D})^{-1} = R \int_R^\infty \frac{dr}{\rho D r^2}$$

to obtain

$$\dot{m} = 4\pi \overline{\rho D} R \zeta_R$$

Now it is possible to determine the time needed to burn a droplet with initial radius  $R_0$  at time  $t = 0$ .

The droplet mass is

$$m = 4\pi\rho_L R^3/3$$

where  $\rho_L$  is the density of the liquid.

Its negative time rate of change equals the mass loss due to the mass burning rate

$$\frac{dm}{dt} = 4\pi\rho_L R^2 \frac{dR}{dt} = -\dot{m}$$

Introducing  $\dot{m} = 4\pi\overline{\rho D} R \zeta_R$  obtains with constant mean properties by separation of variables

$$dt = -\frac{\rho_L}{\zeta_R \overline{\rho D}} R dR, \quad t \cong R_0^2 - R^2.$$

Integrating of

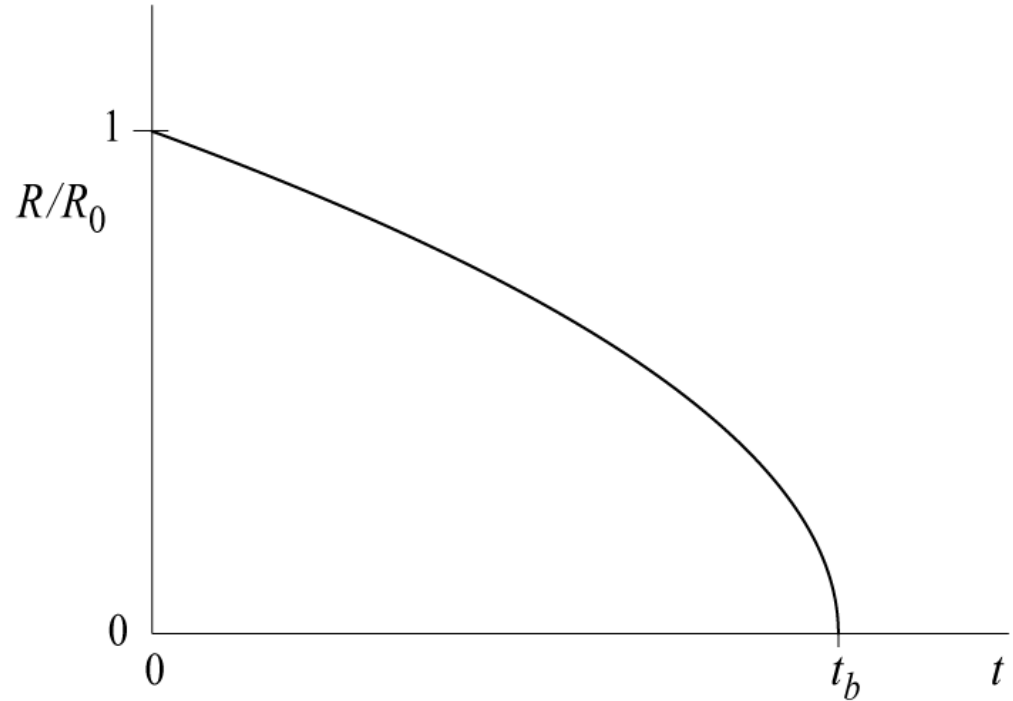
$$dt = -\frac{\rho_L}{\zeta_R \rho D} R dR, \quad t \cong R_0^2 - R^2.$$

from  $R = R_0$  to  $R = 0$  one obtains the  
burnout time

$$t_b = \frac{\rho_L}{8\zeta_R \rho D} d^2$$

where  $d = 2 R_0$  is the  
initial droplet diameter.

This is called the  $d^2$ -law of  
droplet combustion.



It represents a very good first approximation for the droplet combustion time and  
has often be confirmed by experiments.

Finally, we want to calculate the radial position of the surrounding flame.

Evaluating

$$\eta = 1 - \exp(-\zeta)$$

For  $Z = Z_{st} = \eta_{st}$  one obtains  $1 - Z_{st} = \exp(-\zeta_{st})$ ,

where with  $\rho v r^2 = R^2(\rho v)_R$  and  $\zeta = \int_r^\infty \frac{v}{D} dr = \frac{\dot{m}}{4\pi} \int_r^\infty (\rho D r^2)^{-1} dr$ .

$$\zeta_{st} = \frac{\dot{m}}{4\pi \overline{\rho D} r_{st}}$$

Here

$$(\overline{\rho D})^{-1} = R \int_{r_{st}}^\infty \frac{dr}{\rho D r^2}$$

If  $(\overline{\rho D})^{-1} = R \int_R^{\infty} \frac{dr}{\rho D r^2}$  and  $(\overline{\rho D})^{-1} = R \int_{r_{st}}^{\infty} \frac{dr}{\rho D r^2}$

are assumed equal one may use

$$\dot{m} = 4\pi \overline{\rho D} R \zeta_R$$

to determine the flame radius as

$$\frac{r_{st}}{R} = \frac{\zeta_R}{-\ln(1 - Z_{st})}$$

For sufficiently small values of  $Z_{st}$  the denominator may be approximated by  $Z_{st}$  itself showing that ratio  $r_{st} / R$  may take quite large values.

## Exercise

Determine the non-dimensional mass burning rate and  $r_{st} / R$  for a Diesel fuel where

$$h_L/c_p = 160 \text{ K}, T_L = 560 \text{ K}, T_2 = 800 \text{ K}$$

and

$$T_{st} - T_u(Z_{st}) = 2000 \text{ K}, Z_{st} = 0.036$$



## Solution

Using the Burke-Schumann solution the non-dimensional mass burning rate may be written as

$$\zeta_R = \ln \left[ 1 + \frac{T_2 - T_L + (T_{st} - T_u(Z_{st}))(1 - Z_{st})}{h_L/c_p} \right] = 2.74$$

The ratio of the flame radius to the droplet radius is then

$$\frac{r_{st}}{R} \approx 75$$

## Exercise

Compare the evaporation rate for the same droplet as in the previous problem to that of the mass burning rate.

## Solution

The non-dimensional evaporation rate of a droplet may be obtained in a similar way as

$$\zeta_R = \ln \left( 1 + \frac{c_p(T_2 - T_L) + Y_{O_2,2}Q/\nu'_{O_2}W_{O_2}}{h_L} \right)$$

It follows from

$$\zeta_R = \ln \left( 1 + \frac{c_p(T_2 - T_L) + Y_{O_2,2}Q/\nu'_{O_2}W_{O_2}}{h_L} \right)$$

in the limit  $Q = 0$ , therefore

$$\zeta_R = \ln \left( 1 + \frac{T_2 - T_L}{h_L/c_p} \right) = 0.916$$

The combustion rate is approximately three times faster than the evaporation rate.