Discrete Mathematics

Counting

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Topics

Combinatorics

- combinatorics: study of arrangements of objects
- enumeration: counting of objects with certain properties
- solve a complicated problem:
  - break it down into smaller problems
  - piece together solutions to these smaller problems

Sum Rule

- $task_1$ can be performed in n_1 distinct ways
- $task_2$ can be performed in n_2 distinct ways
- $task_1$ and $task_2$ cannot be performed simultaneously
- perform either $task_1$ or $task_2$:
  $n_1 + n_2$ ways

Sum Rule Example

- 40 books on sociology and 50 books on anthropology
- learn about sociology or anthropology:
  choose from $40 + 50 = 90$ books

Sum Rule Example

- one friend has 3 books on "Discrete Mathematics"
- another friend has 5
- n: maximum number of different books that can be borrowed
- $5 \leq n \leq 8$

Product Rule

- a procedure that can be broken down into two stages
- n_1 possible outcomes for the first stage
- for each outcome, n_2 possible outcomes for the second stage
- procedure can be carried out in
  $n_1 \cdot n_2$ ways

Product Rule Example

- drama club is holding tryouts for a play
- 6 men and 8 women auditioning for the leading roles
- director can cast leading couple in $6 \cdot 8 = 48$ ways

Product Rule Example

- license plates with 2 letters, followed by 4 digits
- how many possible plates?
- no letter or digit can be repeated:
  $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 3,276,000$
- repetitions allowed:
  $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6,760,000$
- repetitions allowed, only vowels and even digits:
  $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 15,625$

Product Rule Example

- a byte consists of 8 bits
- a bit has two possible values: 0 or 1
- number of possible values for a byte:
  $2 \cdot 2 \cdots 2 = 2^8 = 256$

Counting Example

- pastry shop menu:
  6 kinds of muffins, 8 kinds of sandwiches
  hot coffee, hot tea, icea tea, cola, orange juice
- buy either a muffin and a hot beverage,
  or a sandwich and a cold beverage
- how many possible purchases?
- muffin and hot beverage: $6 \cdot 2 = 12$
- sandwich and cold beverage: $8 \cdot 3 = 24$
- total: $12 + 24 = 36$

Permutation

- permutation: a linear arrangement of distinct objects
- order is significant

Permutation Example

- a class has 10 students: $A, B, C, \ldots, I, J$
- 4 students to be seated in a row:
  $BCEF, CEFI, ABCF, \ldots$
- how many such arrangements?
- filling of a position: a stage
  $10 \cdot 9 \cdot 8 \cdot 7 = 5,040$

Permutation Example

$$
\begin{aligned}
10 \cdot 9 \cdot 8 \cdot 7 &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
&= \frac{10!}{6!}
\end{aligned}
$$

Permutations

- n distinct objects
- number of permutations of size r (where $1 \leq r \leq n$):

$$
\begin{aligned}
P(n, r) &= n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \\
&= \frac{n!}{(n-r)!}
\end{aligned}
$$

- if repetitions are allowed: n^r

Permutations Example

- if size equals number of objects: $r = n$

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

example

- number of permutations of the letters in "COMPUTER": 8!

Arrangements Example

- number of arrangements of the letters in "BALL"
- two L's are indistinguishable

A	B	L	L	L	A	B	L
A	L	B	L	L	A	L	B
A	L	L	B	L	B	A	L
B	A	L	L	L	B	L	A
B	L	A	L	L	L	A	B
B	L	L	A	L	L	B	A

- number of arrangements: $\frac{4!}{2} = 12$

Arrangements Example

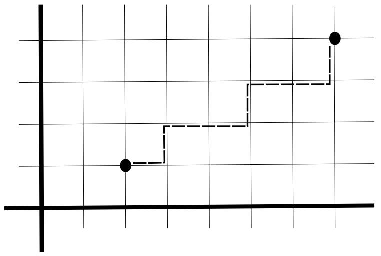
- arrangements of all letters in "DATABASES"
- for each arrangement where A's are **indistinguishable**, $3! = 6$ arrangements where A's are **distinguishable**:
  $DA_1TA_2BA_3SES$, $DA_1TA_3BA_2SES$, $DA_2TA_1BA_3SES$, $DA_2TA_3BA_1SES$, $DA_3TA_1BA_2SES$, $DA_3TA_2BA_1SES$
- for each of these, 2 arrangements where S's are distinguishable:
  $DA_1TA_2BA_3S_1ES_2$, $DA_1TA_2BA_3S_2ES_1$
- number of arrangements: $\frac{9!}{2! \cdot 3!} = 30,240$

Generalized Rule

- n objects
- n_1 indistinguishable objects of $type_1$
  n_2 indistinguishable objects of $type_2$
  ...
  n_r indistinguishable objects of $type_r$
- $n_1 + n_2 + ... + n_r = n$
- number of linear arrangements:

$$\frac{n!}{n_1! \cdot n_2! \cdots n_r!}$$

Arrangements Example



- go from $(2, 1)$ to $(7, 4)$
- each step one unit to the right (R) or one unit upwards (U)
- *RURRURRU*, *URRRUURR*
- how many such paths?

- each path consists of 5 R's and 3 U's
- number of paths: $\frac{8!}{5! \cdot 3!} = 56$

Circular Arrangements

- 6 people seated around a round table: A, B, C, D, E, F
- arrangements considered to be the same when one can be obtained from the other by rotation: $ABEFCD, DABEFC, CDABEF, FCDABE, EFCDAB, BEFCDA$
- how many different circular arrangements?

- each circular arrangement corresponds to 6 linear arrangements
- number of circular arrangements: $\frac{6!}{6} = 120$

Combination

- combination: choosing from distinct objects
- order is not significant

Combination Example

- a deck of 52 playing cards
- 4 suits: clubs, diamonds, hearts, spades
- 13 ranks in each suit: Ace, 2, 3, ..., 10, Jack, Queen, King

- draw 3 cards in succession, without replacement
- how many possible draws?

$$52 \cdot 51 \cdot 50 = \frac{52!}{49!} = P(52, 3) = 132,600$$

Combination Example

- one such draw:
  AH (ace of hearts), $9C$ (9 of clubs), KD (king of diamonds)
- if order insignificant:
  6 permutations correspond to one selection

$$\frac{52!}{3! \cdot 49!} = 22,100$$

Number of Combinations

- n distinct objects
- each combination of r objects: $r!$ permutations of size r
- number of combinations of size r (where $0 \leq r \leq n$):

$$C(n, r) = \binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{r! \cdot (n-r)!}$$

Number of Combinations

- number of combinations:

$$C(n, r) = \frac{n!}{r! \cdot (n-r)!}$$

- note that:

$$\begin{aligned} C(n, 0) &= 1 = & C(n, n) \\ C(n, 1) &= n = & C(n, n-1) \end{aligned}$$

Number of Combinations Example

- Lynn and Patti buy a powerball ticket
- match five numbers selected from 1 to 49
- and then match powerball, 1 to 42
- how many possible tickets?
- Lynn selects five numbers from 1 to 49: $C(49, 5)$
- Patti selects the powerball from 1 to 42: $C(42, 1)$
- possible tickets: $\binom{49}{5}\binom{42}{1} = 80,089,128$

Number of Combinations Examples

- for a volleyball team, gym teacher must select nine girls from junior and senior classes
- 28 junior and 25 senior candidates
- how many different ways?
- if no restrictions: $\binom{53}{9} = 4,431,613,550$
- if two juniors and one senior are best spikers and must be on the team: $\binom{50}{6} = 15,890,700$
- if there has to be four juniors and five seniors: $\binom{28}{4}\binom{25}{5} = 1,087,836,750$

Binomial Theorem

Theorem
if x and y are variables and n is a positive integer, then:

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots \\ &\quad + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 \\ &= \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}\end{aligned}$$

- $\binom{n}{k}$: binomial coefficient

Binomial Theorem Examples

- in the expansion of $(x+y)^7$, coefficient of x^5y^2: $\binom{7}{5} = \binom{7}{2} = 21$

Multinomial Theorem

Theorem
For positive integers n, t, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots n_t!}$$

where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$, and $n_1 + n_2 + n_3 + \ldots + n_t = n$.

Multinomial Theorem Examples

- in the expansion of $(x + y + z)^7$, coefficient of $x^2y^2z^3$:

$$\binom{7}{2,2,3} = \frac{7!}{2! \cdot 2! \cdot 3!} = 210$$

Combinations with Repetition

- 7 students visit a restaurant
- each of them orders one of the following:
  cheeseburger (c), hot dog (h), taco (t), fish sandwich (f)
- how many different purchases are possible?

Combinations with Repetition

c c h h t t f	x x | x x | x x | x
c c c c h t f	x x x x | x | x | x
c c c c c c f	x x x x x x | | | x
h t t f f f f	| x | x x | x x x x
t t t t t t t	| | x x x x x x x |
f f f f f f f	| | | x x x x x x x

- enumerate all arrangements of 10 symbols
  consisting of seven x's and three |'s
- number of different purchases: $\frac{10!}{7! \cdot 3!} = \binom{10}{7} = 120$

Number of Combinations with Repetition

- select, with repetition, r of n distinct objects
- considering all arrangements of r x's and $n - 1$ |'s

$$\frac{(n + r - 1)!}{r! \cdot (n - 1)!} = \binom{n + r - 1}{r}$$

Number of Combinations with Repetition Example

- distribute 7 bananas and 6 oranges among 4 children
- each child receives at least one banana
- how many ways?

- step 1: give each child a banana
- step 2: distribute 3 bananas to 4 children

1 1 1 0	b | b | b |
1 0 2 0	b | | b b |
0 0 1 2	| | b | b b
0 0 0 3	| | | b b b

- $C(6,3) = 20$ ways

Number of Combinations with Repetition Example

- step 3: distribute 6 oranges to 4 children

1 2 2 1	o | o o | o o | o
1 2 0 3	o | o o | | o o o
0 3 3 0	| o o o | o o o |
0 0 0 6	| | | o o o o o o

- $C(9,6) = 84$ ways

- step 4: by the rule of product: $20 \cdot 84 = 1,680$ ways

References

Required reading: Grimaldi

- Chapter 1: Fundamental Principles of Counting
  - 1.1. The Rules of Sum and Product
  - 1.2. Permutations
  - 1.3. Combinations
  - 1.4. Combinations with Repetition