# Discrete Mathematics

## Propositions

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2001-2023



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## Topics





## Proposition

**Definition**

proposition: declarative sentence that is either true or false

- law of the excluded middle:
  a proposition cannot be partially true or partially false
- law of contradiction:
  a proposition cannot be both true and false

## Proposition Examples

propositions

- The Moon revolves around the Earth.
- Elephants can fly.
- $3 + 8 = 11$

not propositions

- What time is it?
- Exterminate!
- $x < 43$



## Propositional Variable

- propositional variable:
  a name that represents the proposition

examples

- $p_1$: The Moon revolves around the Earth. ($T$)
- $p_2$: Elephants can fly. ($F$)
- $p_3$: $3 + 8 = 11$ ($T$)



## Logical Operators

- compound propositions obtained by applying logical operators
- truth table: a table that lists truth values
  for all possible values of variables



## Negation (NOT)

$\neg p$

| $p$ | $\neg p$ |
|---|---|
| $T$ | $F$ |
| $F$ | $T$ |

examples

- $\neg p_1$: The Moon does not revolve around the Earth.
  $\neg T : F$
- $\neg p_2$: Elephants cannot fly.
  $\neg F : T$

## Conjunction (AND)

$p \wedge q$

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

examples

- $p_1 \wedge p_2$: The Moon revolves around the Earth and elephants can fly.
  $T \wedge F : F$
- $p_1 \wedge p_3$: The Moon revolves around the Earth and $3 + 8 = 11$.
  $T \wedge T : T$



## Disjunction (OR)

$p \vee q$

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

example

- $p_1 \vee p_2$: The Moon revolves around the Earth or elephants can fly.
  $T \vee F : T$



## Exclusive Disjunction (XOR)

$p \veebar q$

| $p$ | $q$ | $p \veebar q$ |
|---|---|---|
| $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

examples

- $p_1 \veebar p_2$: Either the Moon revolves around the Earth or elephants can fly.
  $T \veebar F : T$
- $p_1 \veebar p_3$: Either the Moon revolves around the Earth or $3 + 8 = 11$.
  $T \veebar T : F$



## Implication (IF)

$p \rightarrow q$

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

- also called conditional
- if $p$ then $q$
- $p$: hypothesis
- $q$: conclusion

## Implication Examples

- $p_4$: $3 < 8$, $p_5$: $3 < 14$, $p_6$: $3 < 2$, $p_7$: $8 < 6$

- $p_4 \rightarrow p_5$:
  if $3 < 8$, then $3 < 14$
  $T \rightarrow T : T$
- $p_4 \rightarrow p_6$:
  if $3 < 8$, then $3 < 2$
  $T \rightarrow F : F$
- $p_6 \rightarrow p_4$:
  if $3 < 2$, then $3 < 8$
  $F \rightarrow T : T$
- $p_6 \rightarrow p_7$:
  if $3 < 2$, then $8 < 6$
  $F \rightarrow F : T$



## Implication Example

- "If I weigh over 70 kg, then I will exercise."

- $p$: I weigh over 70 kg.
- $q$: I exercise.
- when is this claim false?

$p \rightarrow q$

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |



## Biconditional (IFF)

$p \leftrightarrow q$

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ |

- $p$ if and only if $q$



## Example

- mother tells child:
  "If you do your homework, you can play computer games."
- $h$: The child does her homework.
- $p$: The child plays computer games.
- what does the mother mean?
- $h \rightarrow p$
- $\neg h \rightarrow \neg p$
- $h \leftrightarrow p$

## Well-Formed Formula

- syntax
  - rules to form compound propositions
  - formula that obeys rules: well-formed formula (WFF)
- semantics
  - *interpretation*: calculating value of compound proposition by assigning values to variables
  - truth table: all interpretations



## Formula Examples

not well-formed

- $\vee p$
- $p \wedge \neg$
- $p \neg \wedge q$



## Operator Precedence

1. $\neg$
2. $\wedge$
3. $\vee$
4. $\rightarrow$
5. $\leftrightarrow$

- parentheses are used to change the order of calculation
- implication associates from the right:
  $p \rightarrow q \rightarrow r$ means $p \rightarrow (q \rightarrow r)$



## Precedence Examples

- $s$: Phyllis goes out for a walk.
- $t$: The Moon is out.
- $u$: It is snowing.
- what do the following WFFs mean?

- $t \wedge \neg u \rightarrow s$
- $t \rightarrow (\neg u \rightarrow s)$
- $\neg(s \leftrightarrow (u \vee t))$
- $\neg s \leftrightarrow u \vee t$

## Metalanguage

- target language: language being worked on
- metalanguage: language used when talking *about* the properties of the target language



## Metalanguage Examples

- native Turkish speaker learning English:
  - target language: English
  - metalanguage: Turkish
- student learning programming
  - target language: C, Python, Java, ...
  - metalanguage: English, Turkish, ...



## Formula Properties

- WFF is true for all interpretations: tautology
- WFF is false for all interpretations: contradiction
- *concepts of the metalanguage*



## Tautology Example

$$p \wedge (p \rightarrow q) \rightarrow q$$

| $p$ | $q$ | $p \rightarrow q$ $(A)$ | $p \wedge A$ $(B)$ | $B \rightarrow q$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ |

## Contradiction Example

$p \wedge (\neg p \wedge q)$

| $p$ | $q$ | $\neg p$ | $\neg p \wedge q$ ($A$) | $p \wedge A$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $F$ |



## Logical Implication and Equivalence

- if $P \rightarrow Q$ is a tautology, then $P$ logically implies $Q$: $P \Rightarrow Q$
- if $P \leftrightarrow Q$ is a tautology, then $P$ and $Q$ are logically equivalent: $P \Leftrightarrow Q$



## Logical Implication Example

$p \wedge (p \rightarrow q) \Rightarrow q$

$p \wedge (p \rightarrow q) \rightarrow q$

| $p$ | $q$ | $p \rightarrow q$ ($A$) | $A \wedge p$ ($B$) | $B \rightarrow q$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ |



## Logical Equivalence Example

$\neg p \Leftrightarrow p \rightarrow F$

$\neg p \leftrightarrow p \rightarrow F$

| $p$ | $\neg p$ | $p \rightarrow F$ ($A$) | $\neg p \leftrightarrow A$ |
|---|---|---|---|
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ |

## Logical Equivalence Example

$p \rightarrow q \Leftrightarrow \neg p \vee q$

$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$

| $p$ | $q$ | $p \rightarrow q$ ($A$) | $\neg p$ | $\neg p \vee q$ ($B$) | $A \leftrightarrow B$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |



## Logical Equivalence Example

- implication: $p \rightarrow q$
- *contrapositive*: $\neg q \rightarrow \neg p$
- *converse*: $q \rightarrow p$
- *inverse*: $\neg p \rightarrow \neg q$

$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

| $p$ | $q$ | $p \rightarrow q$ (A) | $\neg q$ | $\neg p$ | $\neg q \rightarrow \neg p$ (B) | $A \leftrightarrow B$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |



## Metalogic

- $P_1, P_2, \ldots, P_n \vdash Q$
  There is a proof which infers the conclusion $Q$ from the assumptions $P_1, P_2, \ldots, P_n$.
- $P_1, P_2, \ldots, P_n \vDash Q$
  $Q$ must be true if $P_1, P_2, \ldots, P_n$ are all true.



## Formal Systems

- a formal system is consistent if for all WFFs $P$ and $Q$:
  if $P \vdash Q$ then $P \vDash Q$
- if every provable proposition is actually true
- a formal system is complete if for all WFFs $P$ and $Q$:
  if $P \vDash Q$ then $P \vdash Q$
- if every true proposition can be proven

## Gödel's Theorem

- propositional logic is consistent and complete

Theorem (Gödel's Theorem)

*Any logical system that is powerful enough to express arithmetic must be either inconsistent or incomplete.*



## Propositional Calculus

1. semantic approach: *truth tables*
   too complicated when the number of primitive statements grow
2. syntactic approach: *rules of inference*
   obtain new propositions from known propositions
   using logical implications
3. axiomatic approach: *Boolean algebra*
   substitute logically equivalent formulas for one another



## Laws of Logic

Double Negation (DN)

$\neg(\neg p) \Leftrightarrow p$

Commutativity (Co)

$p \wedge q \Leftrightarrow q \wedge p$ $\quad$ $p \vee q \Leftrightarrow q \vee p$

Associativity (As)

$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ $\quad$ $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

Idempotence (Ip)

$p \wedge p \Leftrightarrow p$ $\quad$ $p \vee p \Leftrightarrow p$

Inverse (In)

$p \wedge \neg p \Leftrightarrow F$ $\quad$ $p \vee \neg p \Leftrightarrow T$



## Laws of Logic

Identity (Id)

$p \wedge T \Leftrightarrow p$ $\quad$ $p \vee F \Leftrightarrow p$

Domination (Do)

$p \wedge F \Leftrightarrow F$ $\quad$ $p \vee T \Leftrightarrow T$

Distributivity (Di)

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $\quad$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Absorption (Ab)

$p \wedge (p \vee q) \Leftrightarrow p$ $\quad$ $p \vee (p \wedge q) \Leftrightarrow p$

DeMorgan's Laws (DM)

$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\quad$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

## Equivalence Example

$$\begin{array}{lll} & p \to q & \\ \Leftrightarrow & \neg p \vee q & \\ \Leftrightarrow & q \vee \neg p & Co \\ \Leftrightarrow & \neg\neg q \vee \neg p & DN \\ \Leftrightarrow & \neg q \to \neg p & \end{array}$$



## Equivalence Example

$$\begin{array}{lll} & \neg(\neg((p \vee q) \wedge r) \vee \neg q) & \\ \Leftrightarrow & \neg\neg((p \vee q) \wedge r) \wedge \neg\neg q & DM \\ \Leftrightarrow & ((p \vee q) \wedge r) \wedge q & DN \\ \Leftrightarrow & (p \vee q) \wedge (r \wedge q) & As \\ \Leftrightarrow & (p \vee q) \wedge (q \wedge r) & Co \\ \Leftrightarrow & ((p \vee q) \wedge q) \wedge r & As \\ \Leftrightarrow & q \wedge r & Ab \end{array}$$



## Duality

- dual of $s$: $s^d$
  replace: $\wedge$ with $\vee$, $\vee$ with $\wedge$, $T$ with $F$, $F$ with $T$
- principle of duality: if $s \Leftrightarrow t$ then $s^d \Leftrightarrow t^d$

example

$$\begin{array}{rl} s: & (p \wedge \neg q) \vee (r \wedge T) \\ s^d: & (p \vee \neg q) \wedge (r \vee F) \end{array}$$



## Inference

- establish the validity of an argument
- starting from a set of propositions
- which are assumed or proven to be true

notation

$$\begin{array}{c} p_1 \\ p_2 \\ \dots \\ p_n \\ \hline \therefore q \end{array} \qquad\qquad p_1 \wedge p_2 \wedge \cdots \wedge p_n \Rightarrow q$$

## Trivial Rules

### Identity (ID)

$$\frac{p}{\therefore p}$$

### Contradiction (CTR)

$$\frac{F}{\therefore p}$$



## Basic Rules

### OR Introduction (OrI)

$$\frac{p}{\therefore p \vee q}$$

### AND Elimination (AndE)

$$\frac{p \wedge q}{\therefore p}$$

### AND Introduction (AndI)

$$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$$



## Modus Ponens

### Implication Elimination (ImpE)

$$\begin{array}{c} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

example

- If Lydia wins the lottery, she will buy a car.
- Lydia has won the lottery.
- Therefore, Lydia will buy a car.



## Modus Tollens

### Modus Tollens (ImpER)

$$\begin{array}{c} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

example

- If Lydia wins the lottery, she will buy a car.
- Lydia did not buy a car.
- Therefore, Lydia did not win the lottery.

## Modus Tollens

$$\frac{p \to q \qquad \neg q}{\therefore \neg p}$$

| | | |
|---|---|---|
| 1. | $p \to q$ | $A$ |
| 2. | $\neg q \to \neg p$ | $EQV : 1$ |
| 3. | $\neg q$ | $A$ |
| 4. | $\neg p$ | $ImpE : 2, 3$ |



## Fallacies

$$\frac{p \to q \qquad q}{\therefore p}$$

$(p \to q) \wedge q \not\Rightarrow p$

- $p : F,\ q : T$
  $(F \to T) \wedge T \to F : F$

example

- If Lydia wins the lottery, she will buy a car.
- Lydia has bought a car.
- Therefore, Lydia has won the lottery.



## Fallacies

$$\frac{p \to q \qquad \neg p}{\therefore \neg q}$$

$(p \to q) \wedge \neg p \not\Rightarrow \neg q$

- $p : F,\ q : T$
  $(F \to T) \wedge T \to F : F$

example

- If Lydia wins the lottery, she will buy a car.
- Lydia has not won the lottery.
- Therefore, Lydia will not buy a car.



## Implication Introduction

### Implication Introduction (ImpI)

$$\frac{p \vdash q}{\therefore \ \vdash p \to q}$$

- if it can be shown that $q$ is true assuming $p$ is true
- then $p \to q$ is true *without assuming $p$ is true*
- $p$ is a provisional assumption (PA)
- provisional assumptions have to be discharged

## Implication Introduction Example

$$\begin{array}{c} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

| | | |
|---|---|---|
| 1. | $p$ | $PA$ |
| 2. | $p \rightarrow q$ | $A$ |
| 3. | $q$ | $ImpE : 2, 1$ |
| 4. | $\neg q$ | $A$ |
| 5. | $q \rightarrow F$ | $EQV : 4$ |
| 6. | $F$ | $ImpE : 5, 3$ |
| 7. | $p \rightarrow F$ | $ImpI : 1, 6$ |
| 8. | $\neg p$ | $EQV : 7$ |



## OR Elimination

### OR Elimination (OrE)

$$\begin{array}{c} p \vee q \\ p \vdash r \\ q \vdash r \\ \hline \therefore \ \vdash r \end{array}$$

- $p$ and $q$ are provisional assumptions



## Disjunctive Syllogism

### Disjunctive Syllogism (DisSyl)

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

example

- Bart's wallet is either in his pocket or on his desk.
- Bart's wallet is not in his pocket.
- Therefore, Bart's wallet is on his desk.



## Disjunctive Syllogism

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

applying OrE:

$$\begin{array}{c} p \vee q \\ p \vdash q \\ q \vdash q \\ \hline \therefore q \end{array}$$

| | | |
|---|---|---|
| 1. | $p \vee q$ | $A$ |
| 2. | $\neg p$ | $A$ |
| 3. | $p \rightarrow F$ | $EQV : 2$ |
| $4a1$. | $p$ | $PA$ |
| $4a2$. | $F$ | $ImpE : 3, 4a1$ |
| $4a$. | $q$ | $CTR : 4a2$ |
| $4b1$. | $q$ | $PA$ |
| $4b$. | $q$ | $ID : 4b1$ |
| 5. | $q$ | $OrE : 1, 4a, 4b$ |

## Hypothetical Syllogism

### Hypothetical Syllogism (HypSyl)

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

| | | |
|---|---|---|
| 1. | $p$ | $PA$ |
| 2. | $p \rightarrow q$ | $A$ |
| 3. | $q$ | $ImpE : 2, 1$ |
| 4. | $q \rightarrow r$ | $A$ |
| 5. | $r$ | $ImpE : 4, 3$ |
| 6. | $p \rightarrow r$ | $ImpI : 1, 5$ |



## Constructive Dilemma

### Constructive Dilemma

$$\begin{array}{c} p \rightarrow q \\ r \rightarrow s \\ p \vee r \\ \hline \therefore q \vee s \end{array}$$

### Destructive Dilemma

$$\begin{array}{c} p \rightarrow q \\ r \rightarrow s \\ \neg q \vee \neg s \\ \hline \therefore \neg p \vee \neg r \end{array}$$



## Inference Examples

$$\begin{array}{c} p \rightarrow r \\ r \rightarrow s \\ x \vee \neg s \\ u \vee \neg x \\ \neg u \\ \hline \therefore \neg p \end{array}$$

| | | |
|---|---|---|
| 1. | $\neg u$ | $A$ |
| 2. | $u \vee \neg x$ | $A$ |
| 3. | $\neg x$ | $DisSyl : 2, 1$ |
| 4. | $x \vee \neg s$ | $A$ |
| 5. | $\neg s$ | $DisSyl : 4, 3$ |
| 6. | $r \rightarrow s$ | $A$ |
| 7. | $\neg r$ | $ImpER : 6, 5$ |
| 8. | $p \rightarrow r$ | $A$ |
| 9. | $\neg p$ | $ImpER : 8, 7$ |



## Inference Examples

$$\begin{array}{c} (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ r \rightarrow x \\ \neg x \\ \hline \therefore p \end{array}$$

| | | |
|---|---|---|
| 1. | $\neg x$ | $A$ |
| 2. | $r \rightarrow x$ | $A$ |
| 3. | $\neg r$ | $ImpER : 2, 1$ |
| 4. | $\neg r \vee \neg s$ | $OrI : 3$ |
| 5. | $\neg(r \wedge s)$ | $EQV : 4$ |
| 6. | $(\neg p \vee \neg q) \rightarrow (r \wedge s)$ | $A$ |
| 7. | $\neg(\neg p \vee \neg q)$ | $ImpER : 6, 5$ |
| 8. | $p \wedge q$ | $EQV : 7$ |
| 9. | $p$ | $AndE : 8$ |

## Inference Examples

$$\begin{array}{c} p \to (q \lor r) \\ s \to \neg r \\ q \to \neg p \\ p \\ s \\ \hline \therefore F \end{array}$$

| | | |
|---|---|---|
| 1. | $p$ | $A$ |
| 2. | $q \to \neg p$ | $A$ |
| 3. | $\neg q$ | $ImpER : 2, 1$ |
| 4. | $q \to F$ | $EQV : 3$ |
| 5. | $s$ | $A$ |
| 6. | $s \to \neg r$ | $A$ |
| 7. | $\neg r$ | $ImpE : 6, 5$ |
| 8. | $p \to (q \lor r)$ | $A$ |
| 9. | $q \lor r$ | $ImpE : 8, 1$ |
| 10. | $q$ | $DisSyl : 9, 7$ |
| 11. | $F$ | $ImpE : 4, 10$ |



## Inference Examples

If there is a chance of rain or her red headband is missing, then Lois will not mow her lawn. Whenever the temperature is over 20℃, there is no chance for rain. Today the temperature is 22℃ and Lois is wearing her red headband. Therefore, Lois will mow her lawn.

- $p$: There is a chance of rain.
- $q$: Lois' red headband is lost.
- $r$: Lois mows her lawn.
- $s$: The temperature is over 20℃.



## Inference Examples

$$\begin{array}{c} (p \lor q) \to \neg r \\ s \to \neg p \\ s \land \neg q \\ \hline \therefore r \end{array}$$

| | | |
|---|---|---|
| 1. | $s \land \neg q$ | $A$ |
| 2. | $s$ | $AndE : 1$ |
| 3. | $s \to \neg p$ | $A$ |
| 4. | $\neg p$ | $ImpE : 3, 2$ |
| 5. | $\neg q$ | $AndE : 1$ |
| 6. | $\neg p \land \neg q$ | $AndI : 4, 5$ |
| 7. | $\neg(p \lor q)$ | $EQV : 6$ |
| 8. | $(p \lor q) \to \neg r$ | $A$ |
| 9. | ? | $7, 8$ |



## References

Required reading: Grimaldi

- Chapter 2: Fundamentals of Logic
  - 2.1. Basic Connectives and Truth Tables
  - 2.2. Logical Equivalence: The Laws of Logic
  - 2.3. Logical Implication: Rules of Inference

Supplementary reading: O'Donnell, Hall, Page

- Chapter 6: Propositional Logic