Simplification (Minimization) of Logic Functions

A logic function has many algebraic expressions (see canonical forms and simplified expressions).

The purpose of simplification is to choose the most appropriate expression (with the minimum cost) from the set of all possible expressions according to a cost criterion.

The cost criterion may change and depend on the application.

For example, the design criteria may require the expression to have a minimum number of products (or sums), a minimum number of literals (variables) in each product, the use of only one type of gate (such as NAND), or the use of only the gates that are at our disposal.

Objectives of simplification:

- Decreasing the size of the circuit
- Decreasing power consumption (battery, cooling problem)
- Decreasing the delay (increasing the speed) (See 6.2: Propagation Delay)
- Decreasing the cost

Simplification Related Definitions: Implicant and Prime implicant

Implicant (in Sum-of-Products (SOP) form):

An **Implicant** of a function F (in SOP form) is a **product** P that is covered by this function, i.e., $P \le F$ (See Order relation on slide 2.12).

Reminder: Each minterm (product) of the 1st canonical form corresponds to a single 1-generating ("true") point.

Therefore, the minterms are implicants of the function ($m \le F$).

Example:

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$F(A, B, C)= \Sigma m(1,3,5,6,7)$: 1st canonical form

$= A'B'C + A'BC + AB'C + ABC' + ABC$

These products can be simplified into products with fewer literals, which still cover all the 1s of the function.

This function was simplified previously (slide 2.35):

$F= AB + C$

While the minterms in the canonical form cover only a single 1, the AB product covers two 1s, and C covers four 1s.

Note that products with fewer literals cover more 1s.

Like minterms, the product terms AB and C are also implicants of the function F because $AB \le F$ and $C \le F$.

Prime implicant (in SOP form):

A **prime implicant** is a product (implicant) of a function (in SOP form) that cannot be simplified (i.e., it cannot be combined with another term to eliminate a literal).

A **prime implicant** (PI) of F is an implicant that is minimal - that is, the removal of any literal from PI results in a non-implicant for F (*Willard Van Orman Quine*).

Example:

F(A, B, C)= Σm(1,3,5,6,7) : 1st canonical form
= A'B'C + A'BC + AB'C + ABC' + ABC
= AB + C

- For the given function above, the minterms are implicants but not prime implicants.

  For example, ABC' and ABC are not prime implicants because they can be combined to form AB, which includes fewer literals and covers more 1s.

  If we remove C from ABC, the new product AB is still an implicant of F (AB ≤ F).
- AB is a **prime implicant** because it cannot be simplified as A and B because the function does not have 1s in all the places A and B would require ($A \nleq F$, $B \nleq F$).

  If we remove A or B from AB, the new expression (A or B) is not an implicant of F.
- C is also a **prime implicant** of the function F.

Simplification process of a Boolean function:

The simplification procedure consists of two steps.

1. Finding the complete **set of all prime implicants.**

   We will find all products that are covered by the function and cannot be simplified (prime implicants).

   To find prime implicants, we will use two different methods:

   a) Karnaugh maps

   b) Quine-McCluskey (Tabular) method

2. Selection of the "most appropriate" subset of the prime implicants that covers all the 1s of the function.

   All prime implicants may not be necessary to cover all 1s of the function.

   We will calculate the cost of each prime implicant using the given cost criteria.

   Using the prime implicant chart, we will select a subset of prime implicants with minimum cost that covers the function.

   The sum of the selected prime implicants will be the cheapest expression of the function in POS form (minimal covering sum).

Finding Prime Implicants:

Using Boolean algebra, we can combine minterms to obtain products that have fewer variables and cover more 1s.

It is hard to perform these simplifications manually, especially for complicated functions (with many variables). Therefore, a computer program can be used.

A practical procedure (without using the logical expression of the function):

- Investigate the 1- generating input combinations (**output = 1**) in the truth table,
- Combine 1- generating input combinations with one or more constant variables (Hamming distance = 1).
- Retain the constant variables and remove the rest (variables with changing values).

Example:

Algebraic combining: F = A'B'+AB' = (A'+A)B' = B'

These input combinations are adjacent. Hamming distance =1

A	B	F
0	0	1
0	1	0
1	0	1
1	1	0

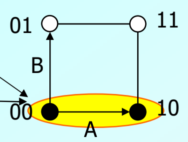
B is constant. For both lines, B=0.
Hence, B will be retained in the new product.

The value of A is changing.
Hence, it will be removed from the new product.

Since B=0, the new product will be B' .

Visualization of the process on the Boolean cube

A	B	F
0	0	1
0	1	0
1	0	1
1	1	0



Two points (having 0 dimensions) are combined to obtain a line (having 1 dimension).

This line represents B=0 (B is constant at zero, and A changes) which is the complement of B, namely, B'.

Remember algebraic combining:

F = A'B'+AB'= (A'+A)B' = B'

Visualization of the process on the Karnaugh map

F: A \ B	0	1
0	1	0
1	1	0

Karnaugh maps allow easier grouping of terms.

Neighboring 1s can be grouped together using the adjacency property.

In the grouped column above, B=0 is fixed, and A is changing.

This column represents the complement of B, namely, B'.

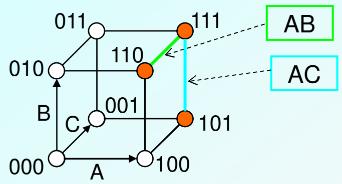
- **If more than one variable is fixed**, each one appears in the product that is the result of their grouping.

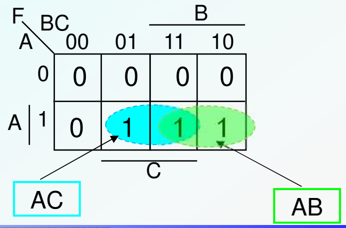
For example:

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

A=1 , C=1 are constant. B is changing.
The product AC is formed as the result of this grouping.
Algebraically: AB'C + ABC = A(B'+B)C = AC

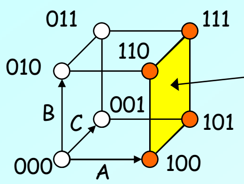
A=1 , B=1 are constant. C is changing.
The product AB is formed as the result of this grouping.
Algebraically: ABC' + ABC = AB(C'+C) = AB





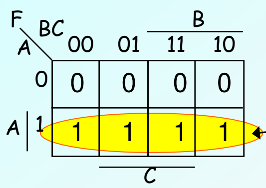
- **More than 2 points can also be combined** to establish new groups.

For example: $F(A,B,C) = \cup_1(4,5,6,7)$



A=1 is constant. B and C are changing.
This face of the cube (a plane) represents A.
Algebraically: AB'C' + AB'C + ABC' + ABC
= AB'+AB
= A

Using a Karnaugh map:



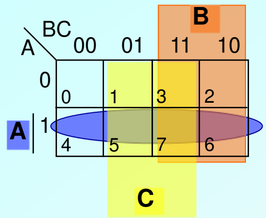
A=1 is constant. B and C are changing.

Finding Prime Implicants Using Karnaugh Maps (Diagrams):

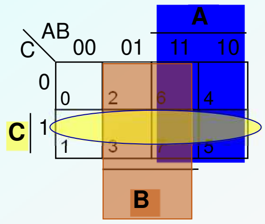
In Karnaugh maps, only a single variable changes between two neighboring cells; the rest remain constant.

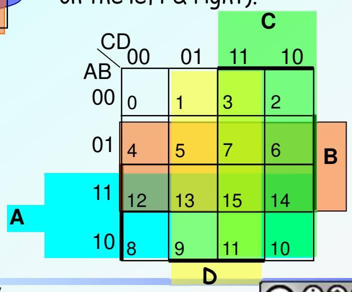
True points (1s) in the neighboring squares (cells) can be clustered into groups that contain 2, 4, 8, ..., 2^n cells.

Below, areas where the variable values stay constant are shown on the Karnaugh maps for functions with 3 and 4 variables.



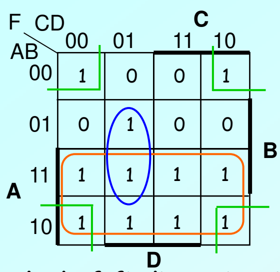
Karnaugh maps for the same function can be drawn in different ways by changing the order of variables (compare the diagrams on the left & right).





Example: Find the prime implicants of the following function.

$F(A,B,C,D) = \cup_1(0,2,5,8,9,10,11,12,13,14,15)$



Prime implicants: A , B'D' , BC'D

Method of finding prime implicants (grouping 1s):

- The groups must be rectangular and must have an area that is a power of two (i.e., 1, 2, 4, 8...).
- "True" points (1s) are placed into the largest groups possible.
- Two points within a larger group (or groups) cannot be combined into a smaller subgroup.

  For example, two points within different 4-point groups cannot be combined to form another 2-point group. It is possible to create a new 4-point group.
- However, if a point is not currently in any groups (such as 0101 above), that point can be grouped with an already grouped point.

The Set of All Prime Implicants and The Minimal Cover:

In logic circuit design, the simplification process starts by finding the set of all prime implicants.

In the second step of simplification, the most appropriate prime implicants are selected from the set of all prime implicants.

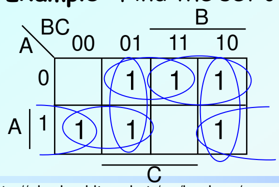
The minimum set of prime implicants that covers all true points (1s) of a function is called the **minimal cover**.

The sum (OR) of all prime implicants in the minimal cover is called the **minimal covering sum**.

If any of the prime implicants in the minimal covering sum is removed, some of the true points of the function will not be covered.

> Simplification of a function in SOP form is the selection of the most appropriate (having minimum cost) **minimal covering sum**.

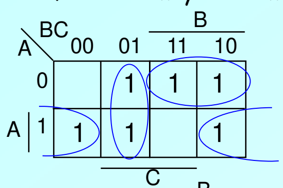
Example: Find the set of all prime implicants of the function.



Prime implicants:

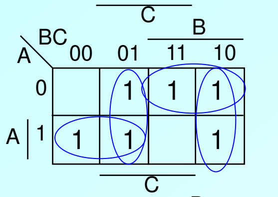
BC' , A'B , A'C , AB' , B'C , AC'

A function may have many minimal covering sums.

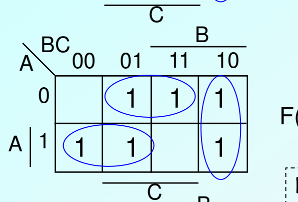


F(A,B,C)= A'B + B'C + AC'

> 1. The minimal covering sum covers all 1s.
> 2. If any of the prime implicants is removed from the minimal covering sum, some of the 1s will not be covered.

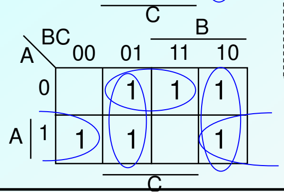


F(A,B,C)= A'B + BC' + B'C + AB'



F(A,B,C)= BC' + A'C + AB'

Note that all these expressions construct the same truth table. They generate same output values given the same input combinations.



F(A,B,C)= BC' + A'C + B'C + AC'

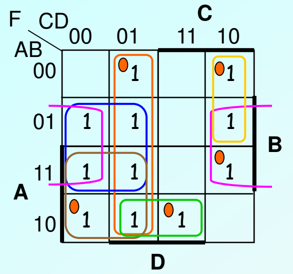
Distinguished Points and Essential Prime Implicants:

Some true points (1s) of a function may be covered only by a single prime implicant. These 1s are called **distinguished points**.

The prime implicant that covers a distinguished point is an **essential prime implicant**.

Essential prime implicants must be included in the minimal covering sum. Otherwise, covering all true points of a function is impossible.

Example:



Set of all prime implicants:

C'D , BC' , AC' , BD' ,A'CD' , AB'D

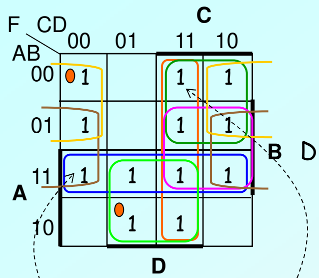
Distinguished points	Essential prime implicants
0001	C'D
0010	A'CD'
1000	AC'
1110	BD'
1011	AB'D

Here, essential prime implicants cover all true points of the function.

This is a special case. The minimum cost solution is:

$$F = \bar{C}D + \bar{A}C\bar{D} + A\bar{C} + B\bar{D} + A\bar{B}D$$

Example: Finding the set of all prime implicants, distinguished points and essential prime implicants.



Set of all prime implicants:

CD , AB , A'C , BC , AD , A'D' , BD'

Distinguished points	Essential prime implicants
0000	A'D'
1001	AD

Here, essential prime implicants do not cover all "true " points (1s) of the function.

We must decide which prime implicants to include to cover the remaining 1s.

For example, to cover 0011, we can include A'C or CD.

To cover 1100, we can include AB or BD'.

For this function, there are multiple possible minimal covering sums.

Only two of them are given below (alternative solutions exist):

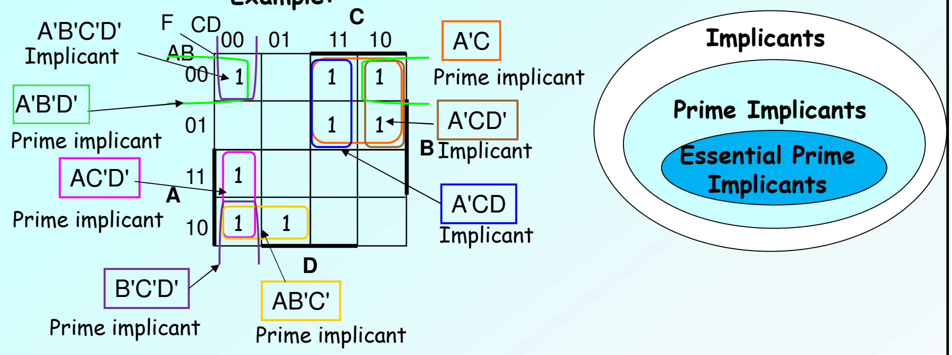
$$F = \bar{A}\bar{D} + AD + CD + AB$$

$$F = \bar{A}\bar{D} + AD + \bar{A}C + B\bar{D}$$

Summary:

Relationship between Implicants, Prime Implicants, and Essential Prime Implicants

Example:



An **Implicant** of a function in SOP form is a product that is covered by the function.
For example, A'B'C'D', A'B'D', A'C, A'CD are implicants of the given function.

A **Prime implicant** is an implicant that cannot be simplified.
For example, A'B'D', A'C, AC'D' are prime implicants, but A'CD is not.

An **Essential prime implicant** covers at least one true point of the function that is not covered by any other prime implicant.
For example, AC'D' is an essential prime implicant, but A'B'D' is not.

Simplification: Selection of the Most Appropriate Prime Implicants

Reminder: The simplification process has two steps:

1. Finding the set of all prime implicants
2. Selection of a subset of prime implicants with minimum cost that covers the function (minimal covering sum).

Prime implicant charts are used to select the minimal cover with the minimum cost.

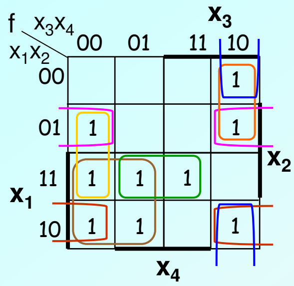
Prime Implicant Chart:

- Simple symbols (such as A, B, C, …) are assigned to each prime implicant.
- Using the given cost criterion, the cost of each prime implicant is calculated.
- The prime implicant chart is organized as a matrix:
  - The symbols of prime implicants are listed down the side of the chart (rows).
  - The numbers corresponding to the true points of the function are listed across the top of the chart (columns).
  - The cost of each prime implicant is placed in the last column.
  - If a prime implicant covers a given true point, an 'X' is placed at the intersection of the corresponding row and column.

Example: Find the set of all prime implicants and form the prime implicant chart for the following function.

$f(x_1, x_2, x_3, x_4)=\Sigma m(2, 4, 6, 8, 9, 10, 12, 13, 15)$

Cost criteria: 2 units for each variable and 1 unit for each complement sign.



Set of all prime implicants:

	$x_1 x_3'$	$x_2 x_3' x_4'$	$x_1'x_2 x_4'$	$x_1 x_2 x_4$	$x_1'x_3 x_4'$	$x_2' x_3 x_4'$	$x_1 x_2' x_4'$
Symbols:	A	B	C	D	E	F	G
Cost:	5	8	8	6	8	8	8
Covered points:	8,9,12,13	4,12	4, 6	13, 15	2, 6	2, 10	8, 10

Example (cont'd):

Set of all prime implicants:

	$x_1 x_3'$	$x_2 x_3' x_4'$	$x_1'x_2 x_4'$	$x_1 x_2 x_4$	$x_1'x_3 x_4'$	$x_2' x_3 x_4'$	$x_1 x_2' x_4'$
Symbols:	A	B	C	D	E	F	G
Costs:	5	8	8	6	8	8	8
Covered points:	8,9,12,13	4,12	4, 6	13, 15	2, 6	2, 10	8, 10

True points of the function

Prime implicants

	2	4	6	8	9	10	12	13	15	Cost
A				X	X		X	X		5
B		X					X			8
C		X	X							8
D								X	X	6
E	X		X							8
F	X					X				8
G				X		X				8

Simplification of the Prime Implicant Chart:

1. Distinguished points are determined. If a column contains only one X, then that is a distinguished point.
   The prime implicant that covers the distinguished point (essential prime implicant) is necessarily selected.
   The row of this essential prime implicant and columns that are covered by this implicant are removed from the chart (crossed out).
2. If there is an X in the i^{th} row for each X in the j^{th} row, then row i covers row j. In other words, all points covered by row j are also covered by row i.

i	X		X	4
~~j~~			~~X~~	~~5~~

   If row i covers row j AND the cost at row i is **smaller than or equal** to the cost at row j, then row j (covered row) is removed from the chart.
3. If a column covers another column, the covering column (with more X's) is removed from the chart.

i	X	~~X~~
j	X	~~X~~
k		~~X~~

These rules are applied successively until all true points are covered with the lowest cost.

Example: Simplification of prime implicant chart of the following function

$f(x_1, x_2, x_3, x_4)=\Sigma m(2, 4, 6, 8, 9, 10, 12, 13, 15)$

True points of the function

		2	4	6	8	9	10	12	13	15	Cost
√ $x_1 x_3'$	A				X	X		X	X		5
$x_2 x_3' x_4'$	B		X					X			8
$x_1' x_2 x_4'$	C		X	X							8
√ $x_1 x_2 x_4$	D								X	X	6
$x_1' x_3 x_4'$	E	X		X							8
$x_2' x_3 x_4'$	F	X					X				8
$x_1 x_2' x_4'$	G				X		X				8

1. step: In this chart 9 and 15 are the distinguished points.

As A and D are essential prime implicants, the rows and columns that they cover should be crossed out (removed from the chart).

These products are marked to show their inclusion into the final set.

	2	4	6	10	Cost
~~B~~		~~x~~			~~8~~
C		x	x		8
E	x		x		8
F	x			x	8
~~G~~				~~x~~	~~8~~

2. step: In this chart, C covers B. As the cost of C is equal to B, B (as the covered row) is crossed out (removed from the chart).

Similarly, F covers G, and they have the same cost. So, the row of G is removed from the chart. These products (B and G) will not be in the final set.

		2	4	6	10	Cost
√	C		x	x		8
	E	x		x		8
√	F	x			x	8

3. step: In this chart, 4 and 10 are distinguished points. Therefore, C and F are selected (and marked). With this selection all true points of the function are covered.

Result:

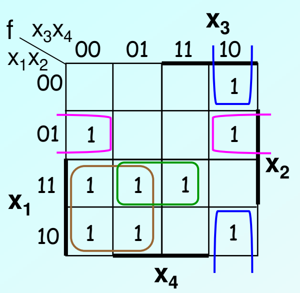
Marked prime implicants yield the expression of the function with the lowest cost.

Selected prime implicants: A + D + C + F

Total cost= 5 + 6 + 8 + 8 = 27

$f(x_1, x_2, x_3, x_4) = x_1 x_3' + x_1 x_2 x_4 + x_1'x_2 x_4' + x_2' x_3 x_4'$

Using a Karnaugh map, we can visualize the selected prime implicants.



Selected prime implicants should cover all true points, and there should be no redundancy.

Selected prime implicants should form a minimal cover. Therefore, removing any implicant should result in uncovered true point(s).

$\boxed{x_1 x_3'}$ $\boxed{x_1'x_2 x_4'}$ $\boxed{x_1 x_2 x_4}$ $\boxed{x_2' x_3 x_4'}$

Simplification of Incompletely Specified Functions

Reminder: In incompletely specified functions, the function result is undetermined (we do not care about it) for some input combinations.

These combinations may never occure in the circuit, or the designer prohibits them.

Example: BCD incrementer circuit on slide 2.26

Selection of Unspecified (Don't Care) Values (Φ):

For the don't care terms (Φ), it is desirable to choose values (0 or 1) which will help simplify the function (to find the expression with the lowest cost).

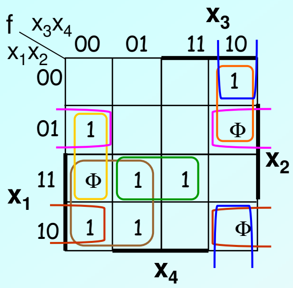
- In the process of searching **for the set of all prime implicants** of a function in SOP form, we will treat the don't care terms as if they were required minterms (**Φ = 1**).
  This way, they can be combined with other minterms to eliminate as many literals as possible (larger groups in the Karnaugh map).
- When **forming the prime implicant chart**, the "don't cares" are **not** listed at the top (**Φ = 0**) because there is no need to cover the points with unspecified values.
  This way, when the prime implicant chart is solved, all of the required minterms will be covered by one of the selected prime implicants.
  However, the don't cares are not included in the final solution unless they have been in the process of forming one of the selected implicants.

Example: Implement the following incomplete function with the lowest possible cost.

$f(x_1, x_2, x_3, x_4) = \cup_1(2, 4, 8, 9, 13, 15) + \cup_\Phi(6,10,12)$

Cost criteria: 2 units for each variable and 1 unit for each complement.

Finding the prime implicants:



When we are finding the set of all prime implicants of a function in SOP form, we **assign 1 to Φ's**.

Set of all prime implicants:

	$x_1 x_3'$	$x_2 x_3' x_4'$	$x_1' x_2 x_4'$	$x_1 x_2 x_4$	$x_1' x_3 x_4'$	$x_2' x_3 x_4'$	$x_1 x_2' x_4'$
Symbols:	A	B	C	D	E	F	G
Costs:	5	8	8	6	8	8	8
Points covered:	8,9,13	4	4	13,15	2	2	8

Forming the prime implicant chart:

Set of all prime implicants:

	$x_1 x_3'$	$x_2 x_3' x_4'$	$x_1' x_2 x_4'$	$x_1 x_2 x_4$	$x_1' x_3 x_4'$	$x_2' x_3 x_4'$	$x_1 x_2' x_4'$
Symbols:	A	B	C	D	E	F	G
Costs:	5	8	8	6	8	8	8
Points covered:	8,9,13	4	4	13,15	2	2	8

True points of the function

Prime implicants	2	4	8	9	13	15	Cost
A			X	X	X		5
B		X					8
C		X					8
D					X	X	6
E	X						8
F	X						8
G			X				8

When we are forming the prime implicant chart of a function in SOP form, we **assign 0 to Φ's.**

As there is no need to cover the points with unspecified values, these points are not placed into the prime implicant chart.

Simplifying the prime implicant chart:

True points of the function

	Prime implicants	2	4	8	9	13	15	Cost
√	A			X	X	X		5
	B		X					8
	C		X					8
√	D					X	X	6
	E	X						8
	F	X						8
	G			X				8

Step 1: In this chart, points 9 and 15 are distinguished points.

As A and D are the essential prime implicants, they are selected. The rows and columns covered by A and D are removed.

A and D are marked to show that they will be in the final set of prime implicants.

	2	4	Cost
B		x	8
C		x	8
E	x		8
F	x		8

Step 2: B and C cover the same points, and they have the same cost. Therefore, it is not possible to choose between B and C; either one can be selected.

The same situation exists for prime implicants E and F.

In the end, the same function can be implemented using any of the following expressions which have the same (lowest) cost.

$f = A + D + B + E = x_1 x_3' + x_1 x_2 x_4 + x_2 x_3' x_4' + x_1'x_3x_4'$

$f = A + D + B + F = x_1 x_3' + x_1 x_2 x_4 + x_2 x_3' x_4' + x_2' x_3x_4'$

$f = A + D + C + E = x_1 x_3' + x_1 x_2 x_4 + x_1'x_2 x_4' + x_1'x_3x_4'$

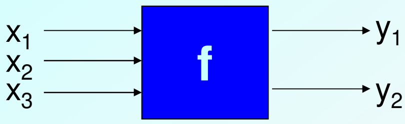
$f = A + D + C + F = x_1 x_3' + x_1 x_2 x_4 + x_1'x_2 x_4' + x_2' x_3x_4'$

All designs have the same cost (27).

Simplification of General Functions

Remark: General functions have more than one output.

x_1 x_2 x_3	y_1 y_2
0 0 0	1 1
0 0 1	1 Φ
0 1 0	0 0
0 1 1	Φ 0
1 0 0	1 Φ
1 0 1	0 1
1 1 0	0 1
1 1 1	Φ 0



$y_1 = f_1(x_1, x_2, x_3)$

$y_2 = f_2(x_1, x_2, x_3)$

During the simplification of general functions, the set of prime implicants for each output is found independently, and prime implicants are selected from these sets.

An important point is to select the common prime implicants of both outputs.

The simplification of general functions is outside the scope of this course.

Finding the Set of All Prime Implicants Using the Quine-McCluskey (Tabulation) Method

The Karnaugh map method is an effective way to simplify logic functions that have a small number of variables.

When the number of variables is large (more than four) or if several functions must be simplified, the use of a digital computer is desirable.

The Quine-McCluskey (tabulation) method provides a systematic procedure that can be readily programmed for a digital computer.

Quine-McCluskey (Tabulation) **Method:**

Remember: to find the set of all prime implicants, true points (minterms) of the function are combined (grouped). Adjacent minterms where single variable changes are grouped together (See the figure on slide 4.5).

In the tabulation method, each minterm (corresponding to 1-generating input combinations) is compared to all other minterms.

If a single variable (input) changes between two minterms, they are combined.

The variable with the changing value is removed, and a new term is obtained.

This process is repeated until no further groups can be formed.

Terms that cannot be grouped are the prime implicants.

Method (Algorithm):

Willard Van Orman Quine (1908-2000) Philosophy, logic
Edward J. McCluskey (1929-2016) Electrical engineer

1st Step: Finding the set of all prime implicants:

- Consider 1-generating input combinations (true points) in the truth table.
- Cluster the 1-generating input combinations depending on the number of 1s included in the combination. For example, 1011 has three 1s.
  This will shorten the running time of the algorithm.
- Compare combinations that are in the neighboring clusters. Group the combinations where a single variable changes value.
- The variable with the changing value will be eliminated.
- Mark the combinations that are grouped.
- Repeat the grouping on the newly formed combinations until no further groups can be formed.
- Combinations that are not grouped (items that are not marked) form the set of all prime implicants.

2nd Step: Finding the minimal covering sum

The prime implicant chart is used to select the subset of prime implicants with minimum cost that covers the function (minimal covering sum) (See 4.16).

Example 1: Find the set of all prime implicants of the following function using Quine-McCluskey method.

$f(x_1, x_2, x_3, x_4) = \Sigma_m(0, 1, 2, 8, 10, 11, 14, 15)$

1-generating (true) input combinations

Num.	x_1 x_2 x_3 x_4	
0	0 0 0 0	√
1	0 0 0 1	√
2	0 0 1 0	√
8	1 0 0 0	√
10	1 0 1 0	√
11	1 0 1 1	√
14	1 1 1 0	√
15	1 1 1 1	√

Groups with 2 points

Num.	x_1 x_2 x_3 x_4	
0,1	0 0 0 -	
0,2	0 0 - 0	√
0,8	- 0 0 0	√
2,10	- 0 1 0	√
8,10	1 0 - 0	√
10,11	1 0 1 -	√
10,14	1 - 1 0	√
11,15	1 - 1 1	√
14,15	1 1 1 -	√

Groups with 4 points

Num.	x_1 x_2 x_3 x_4
0,2,8,10	- 0 - 0
0,8,2,10	- 0 - 0
10,11,14,15	1 - 1 -
10,14,11,15	1 - 1 -

No need to rewrite the same items.

Set of all prime implicants (Not marked): $x_1' x_2' x_3'$, $x_2' x_4'$, $x_1 x_3$

To find the minimal covering sum (lowest cost), the prime implicant chart is used.

Example 2: Find the set of all prime implicants of the following function using Quine-McCluskey method.

$f(x_1, x_2, x_3, x_4) = \Sigma_m(3, 4, 7, 8, 9, 12, 13)$

1-generating (true) input combinations

Num.	x_1 x_2 x_3 x_4	
4	0 1 0 0	√
8	1 0 0 0	√
3	0 0 1 1	√
9	1 0 0 1	√
12	1 1 0 0	√
7	0 1 1 1	√
13	1 1 0 1	√

Groups with 2 points

Num.	x_1 x_2 x_3 x_4	
4,12	- 1 0 0	
8,9	1 0 0 -	√
8,12	1 - 0 0	√
3,7	0 - 1 1	
9,13	1 - 0 1	√
12,13	1 1 0 -	√

Groups with 4 points

Num.	x_1 x_2 x_3 x_4
8,9,12,13	1 - 0 -
8,12,9,13	1 - 0 -

No need to rewrite the same items.

Note: Here, there are no input combinations containing no 1s. We start with combinations containing a single 1. Also, note that we have 3 following 8, or 7 following 12 because we need to group based on number of 1s, not on the increasing order of combinations. This is how you should construct the table.

Set of all prime implicants (Not marked): $x_2 x_3' x_4'$, $x_1' x_3 x_4$, $x_1 x_3'$

Simplification of functions in Product-of-Sums (POS) form

The principle of duality allows us to use the same techniques to obtain the minimum-cost expression for functions in the product-of-sums (POS) form.

An **implicant** of a function F (in POS form) is a sum S that covers this function F ≤ S (See "Order relation" on slide 2.12).

Reminder: Each maxterm (sum) in the 2nd canonical form corresponds to a single 0-generating point. The maxterms are implicants of the function (F ≤ M).

Prime implicant (POS form):

A **prime implicant** is a sum term of a function (POS form) that cannot be simplified.

A **prime implicant** of F is an implicant that is minimal - that is, the removal of any literal from PI results in a non-implicant for F (*Willard Van Orman Quine*).

Example: Find the prime implicants of the following function in POS form (same example as on slide 4.10).

$F(A,B,C,D) = \Pi_M(1,3,4,6,7)$

We group zeros.

We use complements of the input variables.

F: AB \ CD	00	01	11	10
00	1	0	0	1
01	0	1	0	0
11	1	1	1	1
10	1	1	1	1

Prime implicants in POS form:

A+B+D' , A+B'+C' , A+B'+D , A+C'+D'

Example: Find the prime implicants of the following function in POS form. (same example as on slide 4.14)

For some functions, it is easier to work on 0-generating input combinations than on 1-generating combinations.

F: AB \ CD	00	01	11	10
00	1	0	1	1
01	1	0	1	1
11	1	1	1	1
10	0	1	1	0

Set of all prime implicants:

A+C+D' , **A'+B+D**

Since **A+C+D'** and **A'+B+D** are essential prime implicants, we must take them.

By including these prime implicants, we cover all 0-generating points.

The expression for F in POS form:

$$F = (A + C + \bar{D})(\bar{A} + B + D)$$

We can use axioms and theorems of Boolean algebra on this expression (POS) to reach expressions in SOP form given on slide 4.14.

$$F = \bar{A}\bar{D} + AD + CD + AB$$

$$F = \bar{A}\bar{D} + AD + \bar{A}C + B\bar{D}$$

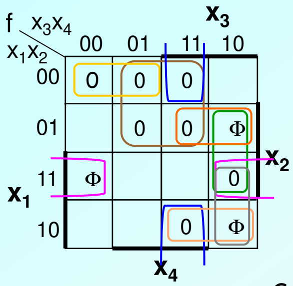
Note that all these expressions have the same truth table. They generate the same output value given the same input values.

Example: Implement the following incompletely specified function in PoS form with the lowest possible cost (same example as on slide 4.24).

$f(x_1, x_2, x_3, x_4)=\cup_0(0, 1, 3, 5, 7, 11, 14) + \cup_\Phi(6,10,12)$

Cost criteria: 2 units for each variable and 1 unit for each complement.

Finding the prime implicants:



When we are finding the set of all prime implicants of a function in POS form, we **assign 0 to Φ's.**

Set of all prime implicants:

	x_1+x_4'	$x_1+x_2+x_3$	$x_1+x_2'+x_3'$	$x_2'+x_3'+x_4$	$x_1'+x_2'+x_4$	$x_2+x_3'+x_4'$	$x_1'+x_3'+x_4$	$x_1'+x_2+x_3'$
	A	B	C	D	E	F	G	H
Cost:	5	6	8	8	8	8	8	8
	1,3,5,7	0,1	7	14	14	3,11	14	11

Forming the prime implicant chart:

Set of all prime implicants:

	x_1+x_4'	$x_1+x_2+x_3$	$x_1+x_2'+x_3'$	$x_2'+x_3'+x_4$	$x_1'+x_2'+x_4$	$x_2+x_3'+x_4'$	$x_1'+x_3'+x_4$	$x_1'+x_2+x_3'$
	A	B	C	D	E	F	G	H
Cost:	5	6	8	8	8	8	8	8
	1,3,5,7	0,1	7	14	14	3,11	14	11

False (0) points of the function

Prime implicants

	0	1	3	5	7	11	14	Cost
A		X	X	X	X			5
B	X	X						6
C					X			8
D							X	8
E							X	8
F			X			X		8
G							X	8
H						X		8

When we are forming the prime implicant chart for a function in POS form, we **assign 1 to Φ's.**

As there is no need to cover the points with unspecified values, these points are not listed in the prime implicant chart.

As A and B are essential prime implicants, they are selected. The rows and columns covered by A and B are crossed out (removed).

	11	14	Cost
D		x	8
E		x	8
F	x		8
G		x	8
H	x		8

D, E, and G cover the same point (14) and have the same cost. Therefore, it is not possible to choose between them; any one of D, E, and G can be selected.

The same is true for prime implicants F and H.

In the end, the same function can be implemented using any of the following expressions, which have the same (lowest) cost.

$$f = A \cdot B \cdot D \cdot F = (x_1+x_4')(x_1+x_2+x_3)(x_2'+x_3'+x_4)(x_2+x_3'+x_4')$$
$$f = A \cdot B \cdot D \cdot H = (x_1+x_4')(x_1+x_2+x_3)(x_2'+x_3'+x_4)(x_1'+x_2+x_3')$$
$$f = A \cdot B \cdot E \cdot F = (x_1+x_4')(x_1+x_2+x_3)(x_1'+x_2'+x_4)(x_2+x_3'+x_4')$$
$$f = A \cdot B \cdot E \cdot H = (x_1+x_4')(x_1+x_2+x_3)(x_1'+x_2'+x_4)(x_1'+x_2+x_3')$$
$$f = A \cdot B \cdot G \cdot F = (x_1+x_4')(x_1+x_2+x_3)(x_1'+x_3'+x_4)(x_2+x_3'+x_4')$$
$$f = A \cdot B \cdot G \cdot H = (x_1+x_4')(x_1+x_2+x_3)(x_1'+x_3'+x_4)(x_1'+x_2+x_3')$$

All designs have the same cost (27).

Since this function is the same as the one given on slide 4.24, we can use axioms and theorems of Boolean algebra on any of these expressions to reach the expression in the SOP form given on slide 4.27.