Math 332E (Measure Theory) **Quiz 1** **October 5, 2016**

Name:________________________
Number:______________________

1. Write the definition of a measure on a set X.

Solution: A measure is a partial function $\mu\colon 2^X \to [0,\infty)\cup\{\infty\}$ which is countably additive, i.e. if $\{E_n\}_{n\in\mathbb{N}}$ is a countable family of sets such that $E_n \cap E_m = \emptyset$ whenever $n \neq m$ then we have

$$\mu\left(\bigcup_{n=0}^{\infty} E_n\right) = \sum_{n=0}^{\infty} \mu(E_n)$$

2. Assume we defined a decreasing chain of subsets $C_n \supset C_{n+1}$ in $\mathbb{R}$ recursively:

$$C_0 = [0,1] \quad \text{and} \quad C_{n+1} = \frac{1}{4}C_n \cup \left(\frac{3}{4} + \frac{1}{4}C_n\right)$$

(a) Write the first 3 terms: C_0, C_1 and C_2.

Solution: We already have $C_0 = [0,1]$. For C_1 we apply the recursive formula. We have two parts $\frac{1}{4}[0,1] = \left[0,\frac{1}{4}\right]$, and $\frac{3}{4}+\frac{1}{4}[0,1] = \left[\frac{3}{4},1\right]$. Notice that the second part is the first part shifted to the right by $\frac{1}{4}$. So,

$$C_1 = \left[0,\frac{1}{4}\right] \cup \left[\frac{3}{4},1\right]$$

Applying the recursive formula, we again get two parts. For the first part we have

$$\frac{1}{4}\left[0,\frac{1}{4}\right] \cup \frac{1}{4}\left[\frac{3}{4},1\right] = \left[0,\frac{1}{16}\right] \cup \left[\frac{3}{16},\frac{1}{4}\right]$$

For the second part, we get this first part and shift to right by $\frac{3}{4}$.

$$\left(\frac{3}{4} + \left[0,\frac{1}{16}\right]\right) \cup \left(\frac{3}{4} + \left[\frac{3}{16},\frac{1}{4}\right]\right) = \left[\frac{12}{16},\frac{13}{16}\right] \cup \left[\frac{15}{16},1\right]$$

Combining these parts we get

$$C_2 = \left[0,\frac{1}{16}\right] \cup \left[\frac{3}{16},\frac{1}{4}\right] \cup \left[\frac{12}{16},\frac{13}{16}\right] \cup \left[\frac{15}{16},1\right]$$

(b) Calculate the measure of C_{n+1} in terms of the measure of C_n.

Solution: The recursive formula indicates $C_{n+1} = \frac{1}{4}C_n \cup \left(\frac{3}{4}+\frac{1}{4}C_n\right)$. We have seen in the class that $\mu(\lambda X) = |\lambda|\mu(X)$ and $\mu(\lambda + X) = \mu(X)$ for every $\lambda \in \mathbb{R}$, and for every subset $X \subseteq \mathbb{R}$ which can be written finite unions of intervals. Then by *Inclusion/Exclusion* principle

$$\mu(C_{n+1}) = \frac{1}{4}\mu(C_n) + \frac{1}{4}\mu(C_n) - \mu\left(\frac{1}{4}C_n \cap \left(\frac{3}{4} + \frac{1}{4}C_n\right)\right)$$

But $\frac{1}{4}C_n \cap \left(\frac{3}{4} + \frac{1}{4}C_n\right)$ is the empty set. Then

$$\mu(C_{n+1}) = \frac{1}{2}\mu(C_n)$$

(c) Calculate $\mu\left(\bigcap_{n=0}^{\infty} C_n\right)$.

Solution: The recursive formula above indicates that $\mu(C_n) = \frac{1}{2^n}\mu(C_0) = \frac{1}{2^n}$. Since the standard measure on $\mathbb{R}$ is continuous and C_n is a monotonously decreasing sequence of sets we get that

$$\mu\left(\bigcap_{n=0}^{\infty} C_n\right) = \lim_{n\to\infty} \mu(C_n) = \lim_{n\to\infty} \frac{1}{2^n} = 0$$

3. (a) Assume $\mu\colon 2^X \to [0,\infty) \cup \{\infty\}$ is a partial function such that $\mu(\emptyset) = 0$. Show that the following statements are equivalent:
   i. **(Inclusion/Exclusion Principle)** For all $A, B \subseteq X$ we have $\mu(A\cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ whenever all of these numbers are finite.
   ii. **(Finite Additivity)** For all $A, B \subseteq X$, if $A \cap B$ is empty then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Solution: ($\Longrightarrow$) Assume we have IEP for μ, and assume we have $A, B \subseteq X$ such that $A \cap B = \emptyset$. Then by IEP

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) = \mu(A) + \mu(B) - \mu(\emptyset) = \mu(A) + \mu(B)$$

Thus we have FA.
($\Longleftarrow$) Assume we have FA for μ. Assume we have $A, B \subseteq X$ but we don't know if $A \cap B$ is empty. We can write

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$$

These sets are mutually disjoint. Then by FA we get

$$\begin{aligned}\mu(A \cup B) =& \mu\left((A \setminus B) \cup (A \cap B)\right) + \mu(B \setminus A)\\ =& \mu(A \setminus B) + \mu(A \cap B) + \mu(B \setminus A)\end{aligned}$$

We also have disjoint unions of the shape

$$A = (A \setminus B) \cup (A \cap B) \quad \text{and} \quad B = (B \setminus A) \cup (A \cap B)$$

Then

$$\mu(A) = \mu(A \setminus B) + \mu(A \cap B) \quad \text{and} \quad \mu(B) = \mu(B \setminus A) + \mu(A \cap B)$$

This means

$$\begin{aligned} \mu(A \cup B) =& \mu(A \setminus B) + \mu(A \cap B) + \mu(B \setminus A) \\ =& \mu(A \setminus B) + \mu(A \cap B) + \mu(B \setminus A) + \mu(A \cap B) - \mu(A \cap B) \\ =& \mu(A) + \mu(B) - \mu(A \cap B) \end{aligned}$$

Thus we have IEP.

(b) Assume $\mu \colon 2^X \to [0, \infty)$ is a partial function which satisfies *Finite Addivity*. Show that $\mu(\emptyset) = 0$.

Solution: Since $\emptyset \cap \emptyset = \emptyset$, by FA we have

$$\mu(\emptyset) = \mu(\emptyset \cup \emptyset) = \mu(\emptyset) + \mu(\emptyset) = 2\mu(\emptyset)$$

By definition of μ, $\mu(\emptyset)$ is not ∞. Then the only real number λ which is equal to 2λ is 0.