Parametric Curves

Analytical (*nonparametrical*) representations of curves:

Line:	$y = m \cdot x + c$	explicit nonparametrical form	$y = f(x)$
Circle:	$(x - a)^2 + (y - b)^2 = r^2$	implicit nonparametrical form	$f(x, y) = 0$
Parabola:	$y = b \cdot x^2 + c$	explicit nonparametrical form	$y = f(x)$
Ellipse:	$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1$	implicit nonparametrical form	$f(x, y) = 0$
Hyperbola:	$x \cdot y = k$	implicit nonparametrical form	$f(x, y) = 0$

Although nonparametric forms of curve equations are used in some cases, they are not in general suitable for CAD because of the following reasons:

1. The equation is dependent on the choice of the coordinate system.
2. Unless additional constraints are provided, the curves are inherently unbounded.
3. The implicit form *f(x,y)=0* is inconvenient for computing points on the curve; values of *x* chosen may not lie on the curve, hence yielding no solution for *y*.
4. It is cumbersome to perform geometric transformations (rotations, translations, scaling) on the curve.

The *parametric* form of a curve equation overcomes these problems. In this form the equations are decoupled, in that there are seperate equations for each of the coordinates expressed in terms of an additional variable *u*:

$x = f(u)$ $\quad y = g(u)$ $\quad$ and for space curves $\quad z = h(u)$

where *u*, called parametric variable, varies in a convenient range, typically from *0* to *1*, *f(u)*, *g(u)*, and *h(u)* are all functions of *u*.

Since the pair *(x,y)* represents a position vector of a point, we can express the equation in vector form:

$$p(u) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(u) \\ g(u) \end{pmatrix}$$

a curve in space

$$p(u) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f(u) \\ g(u) \\ h(u) \end{pmatrix}$$

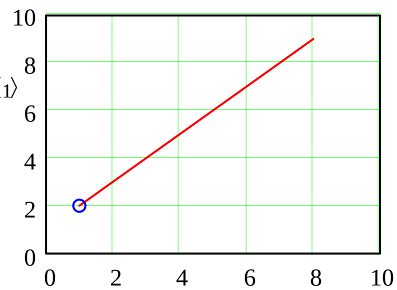
Plotting a Line

$p(u) = P_0 \cdot (1 - u) + P_1 \cdot u$

$P_0 := \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\quad P_1 := \begin{pmatrix} 8 \\ 9 \\ 7 \end{pmatrix}$ $\quad i := 0..10$ $\quad t_i := \frac{i}{10}$ $\quad j := 0..2$

$P := \begin{pmatrix} P_{0_0} & P_{1_0} \\ P_{0_1} & P_{1_1} \end{pmatrix}$ $\quad P = \begin{pmatrix} 1 & 8 \\ 2 & 9 \end{pmatrix}$ $\quad P^T = \begin{pmatrix} 1 & 2 \\ 8 & 9 \end{pmatrix}$

$p_i := \begin{bmatrix} P_{0_0} + t_i \cdot \left(P_{1_0} - P_{0_0}\right) \\ P_{0_1} + t_i \cdot \left(P_{1_1} - P_{0_1}\right) \end{bmatrix}$ $\quad p_{i,j} := P_{0_j} + t_i \cdot \left(P_{1_j} - P_{0_j}\right)$ $\quad i := \text{FRAME}$



$\left(P^T\right)^{\langle 1 \rangle}$

$p_{i,1}$

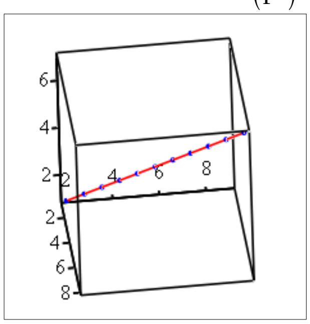
$\left(P^T\right)^{\langle 0 \rangle}, p_{i,0}$

p =

	0	1	2
0	1	2	1
1	1.7	2.7	1.6
2	2.4	3.4	2.2
3	3.1	4.1	2.8
4	3.8	4.8	3.4
5	4.5	5.5	4
6	5.2	6.2	4.6
7	5.9	6.9	5.2
8	6.6	7.6	5.8
9	7.3	8.3	6.4
10	8	9	7

t =

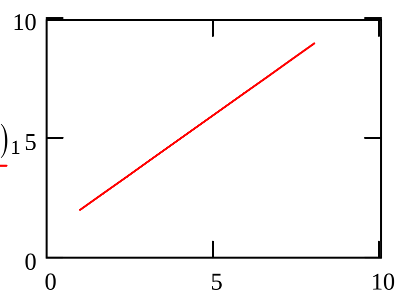
	0
0	0
1	0.1
2	0.2
3	0.3
4	0.4
5	0.5
6	0.6
7	0.7
8	0.8
9	0.9
10	1



$\left(p^{\langle 0 \rangle}, p^{\langle 1 \rangle}, p^{\langle 2 \rangle}\right)$

$i := 0..10$ $\quad u_i := \frac{i}{10}$

$p(u) := P_0 \cdot (1 - u) + P_1 \cdot u$



$p(u_i)_1$

$p(u_i)_0$

Plotting a Circle

$x_c := 2$ $\quad r := 5$ $\quad x(t) := r \cdot \cos(t) + x_c$ $\quad n := 10$ $\quad j := 0..n$ $\quad i := 0..n$

$y_c := 3$ $\quad s := 10$ $\quad y(t) := r \cdot \sin(t) + y_c$ $\quad t_j := \dfrac{j \cdot 2 \cdot \pi}{n}$

$z(t) := 0$

$P_{0,j} := x(t_j)$

$P_{1,j} := y(t_j)$

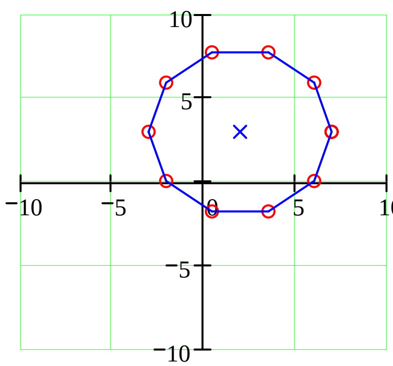
$P_{2,j} := z(t_j)$

$P_{3,j} := 1$

$\Delta\theta := \dfrac{2 \cdot \pi}{n}$

$s\Delta\theta := \sin(\Delta\theta)$

$c\Delta\theta := \cos(\Delta\theta)$



$x_0 := r \cdot \cos(0)$

$y_0 := r \cdot \sin(0)$

$P_{0,0} := x_0 + x_c$

$P_{1,0} := y_0 + y_c$

$P_{2,0} := 0$

$P_{3,0} := 1$

$$\begin{pmatrix} x_{i+1} & y_{i+1} & z_{i+1} \end{pmatrix} := \begin{pmatrix} x_i & y_i & 1 \end{pmatrix} \cdot \begin{pmatrix} c\Delta\theta & s\Delta\theta & 0 \\ -s\Delta\theta & c\Delta\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

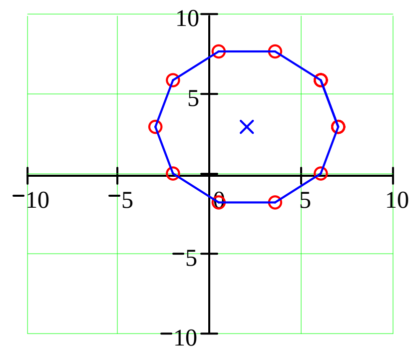
$P_{0,i+1} := x_{i+1} + x_c$

$P_{1,i+1} := y_{i+1} + y_c$

$$\begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} := \begin{pmatrix} x_i \cdot c\Delta\theta - y_i \cdot s\Delta\theta \\ y_i \cdot c\Delta\theta + x_i \cdot s\Delta\theta \end{pmatrix}$$

$P_{2,i+1} := 0$

$P_{3,i+1} := 1$



Rotating a Circle

$x_c := 2$ $\quad r := 5 \quad x(t) := r\cdot\cos(t) + x_c \quad n := 10 \quad j := 0..n \quad t_j := \frac{j\cdot 2\cdot\pi}{n}$

$y_c := 3$ $\quad s := 10 \quad y(t) := r\cdot\sin(t) + y_c$

$z(t) := 0$

$P_{0,j} := x(t_j) \quad P_{1,j} := y(t_j) \quad P_{2,j} := z(t_j) \quad P_{3,j} := 1$

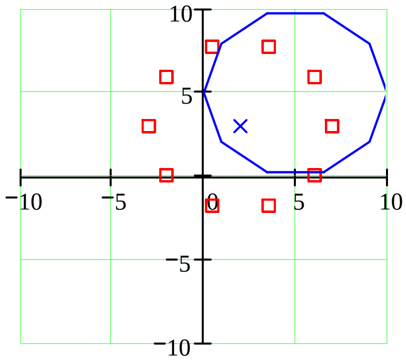
$P_{0,n+j+1} := x(t_j) \quad P_{1,n+j+1} := y(t_j) \quad P_{2,n+j+1} := z(t_j) + 5 \quad P_{3,n+j+1} := 1$

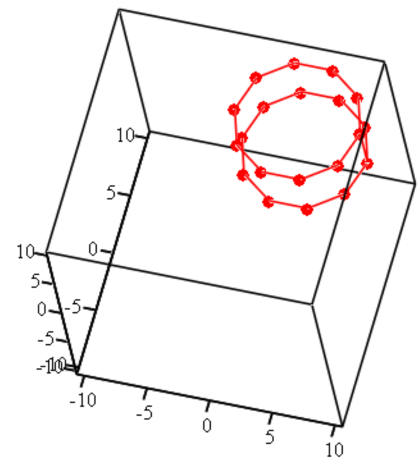
$$Rx(\theta) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad Ry(\theta) := \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T := \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Rz(\theta) := \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$p := Rz\left(\frac{2\cdot\pi\cdot 2}{10}\right)\cdot P \qquad \theta := \frac{2\cdot\pi\cdot FRAME}{10}$

$p := Rx(\theta)\cdot Rz(\theta)\cdot T\cdot P \qquad p1 := T\cdot P$





$$\left[\left(p^T\right)^{\langle 0\rangle},\left(p^T\right)^{\langle 1\rangle},\left(p^T\right)^{\langle 2\rangle}\right]$$

Parametric form of the Circle

$n := 20 \qquad i := 0..n \qquad u_i := \dfrac{i\cdot 2\cdot\pi}{n}$

$x(u) := \cos(u) \qquad y(u) := \sin(u) \qquad C(u) := \begin{pmatrix} x(u) \\ y(u) \end{pmatrix} \qquad C(u) := \begin{pmatrix} \cos(u) \\ \sin(u) \end{pmatrix}$

$x'(u) := -\sin(u) \qquad y'(u) := \cos(u) \qquad C'(u) := \begin{pmatrix} x'(u) \\ y'(u) \end{pmatrix} \qquad C'(u) := \begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix}$

$p := 0$

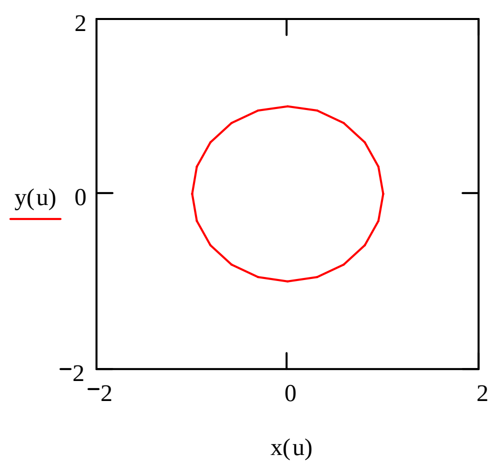
$p_{2\cdot i,0} := x\left(u_i\right)$

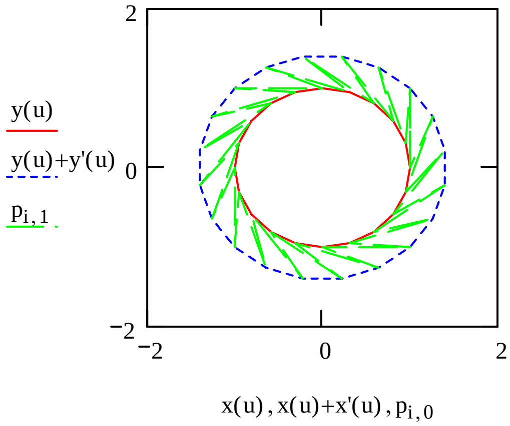
$p_{2\cdot i,1} := y\left(u_i\right)$

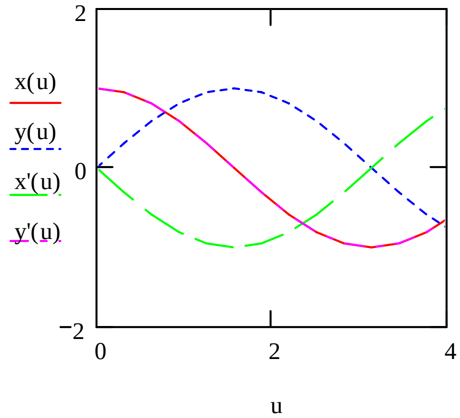
$p_{2\cdot i+1,0} := x\left(u_i\right) + x'\left(u_i\right)$

$p_{2\cdot i+1,1} := y\left(u_i\right) + y'\left(u_i\right)$

$i := 0..2\cdot n + 1$

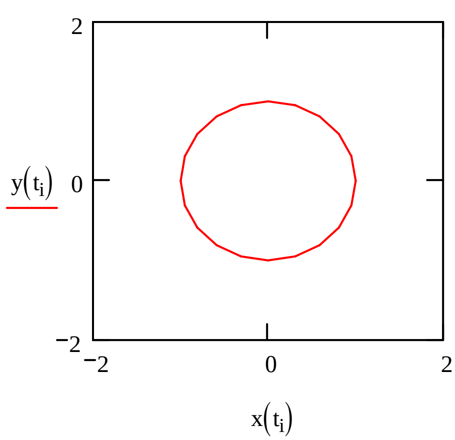






An Alternate Parametric form of the Circle

$n := 20 \qquad i := 0 .. n \qquad \alpha_i := \frac{i \cdot 2 \cdot \pi}{n} \qquad t_i := \tan\left(\frac{\alpha_i}{2}\right) \qquad x(t) := \frac{1 - t^2}{1 + t^2} \qquad y(t) := \frac{2 \cdot t}{1 + t^2}$



$$C(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$x'(t) := \frac{-4 \cdot t}{\left(1 + t^2\right)^2} \qquad y'(t) := \frac{2 \cdot \left(1 - t^2\right)}{\left(1 + t^2\right)^2}$$

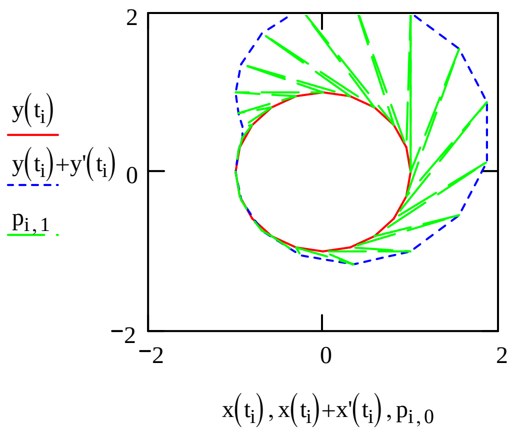
$$C'(t) := \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

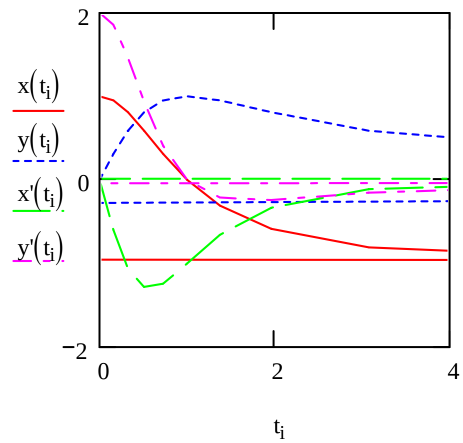
$p := 0 \qquad p_{2 \cdot i, 0} := x\left(t_i\right)$

$$p_{2\cdot i,1} := y(t_i)$$

$$p_{2\cdot i+1,0} := x(t_i) + x'(t_i)$$

$$p_{2\cdot i+1,1} := y(t_i) + y'(t_i) \qquad i := 0 .. 2\cdot n + 1$$





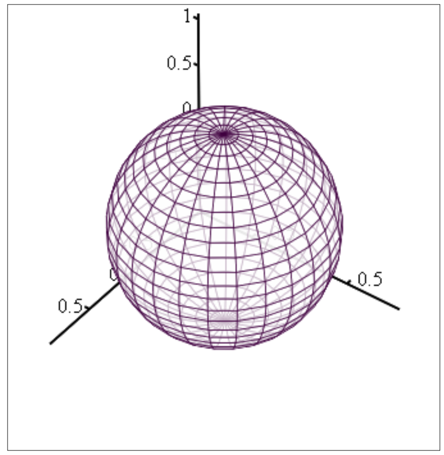
Plotting a Sphere

a parametric surface plot of a unit sphere.

$X(u,v) := \sin(u)\cdot\cos(v) \qquad 0 \le u \le \pi$

$Y(u,v) := \sin(u)\cdot\sin(v) \qquad 0 \le v \le 2\cdot\pi$

$Z(u,v) := \cos(u)$



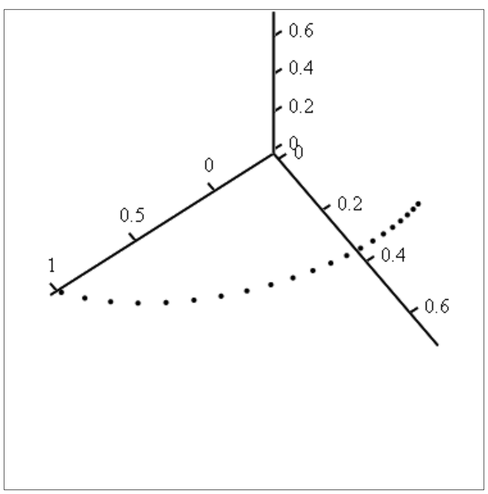
(X, Y, Z)

$n := 20 \quad j := 0..n$

$s_j := \frac{j}{n} \quad t_j := \frac{j}{n}$

Parametric Surfaces

$$x_j := \frac{1 - (s_j)^2 - (t_j)^2}{1 + (s_j)^2 + (t_j)^2} \qquad y_j := \frac{2 \cdot s_j}{1 + (s_j)^2 + (t_j)^2} \qquad z_j := \frac{2 \cdot t_j}{1 + (s_j)^2 + (t_j)^2}$$



(x, y, z)

$x :=$ xxx $\quad x := x$

$u :=$ uuu $\quad u := u \qquad x_c :=$ xxx $\quad x_c := x_c$

$v :=$ vvvv $\quad v := v \qquad y_c :=$ yyy $\quad y_c := y_c$

Intersection point parameter between *Line* and *Circle*

$x_1 := 2 \quad y_1 := 3 \quad x_2 := 8 \quad y_2 := 11 \qquad r := 4 \qquad x_c := 4 \quad y_c := 5$

$$\left[x_1 - x_c + u \cdot (x_2 - x_1)\right]^2 + \left[y_1 - y_c + u \cdot (y_2 - y_1)\right]^2 - r^2 = 0$$

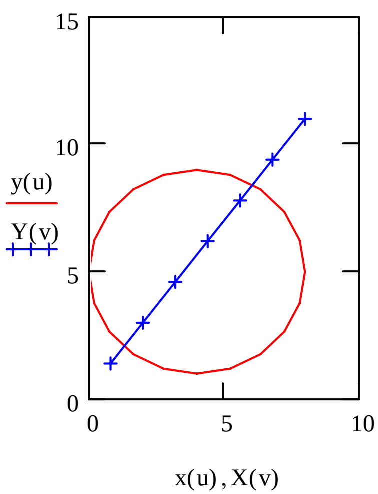
$$\left[x_1 - x_c + u\cdot(x_2 - x_1)\right]^2 + \left[y_1 - y_c + u\cdot(y_2 - y_1)\right]^2 - r^2 \;\Big|\; \begin{array}{l}\text{simplify}\\ \text{collect}, u\end{array} \rightarrow -8 - 56\cdot u + 100\cdot u^2$$

$$v := \left[x_1 - x_c + v\cdot(x_2 - x_1)\right]^2 + \left[y_1 - y_c + v\cdot(y_2 - y_1)\right]^2 - r^2 = 0 \text{ solve}, v \;\rightarrow \begin{pmatrix} \frac{7}{25} + \frac{3}{25}\cdot 11^{\frac{1}{2}} \\ \frac{7}{25} - \frac{3}{25}\cdot 11^{\frac{1}{2}} \end{pmatrix}$$

$f(x,y) := (x - x_c)^2 + (y - y_c)^2 - r^2$ $\qquad v = \begin{pmatrix} 0.678 \\ -0.118 \end{pmatrix}$ $\qquad vv := v$

$x(u) := x_c + r\cdot\cos(u)$ $\qquad X(v) := x_1 + v\cdot(x_2 - x_1)$

$y(u) := y_c + r\cdot\sin(u)$ $\qquad u := 0, \frac{\pi}{10} .. 2\pi$ $\qquad v := -0.2, 0 .. 1$ $\qquad Y(v) := y_1 + v\cdot(y_2 - y_1)$



Intersection point parameter *v* solution on Line:

$v := vv$ $\qquad v = \begin{pmatrix} 0.678 \\ -0.118 \end{pmatrix}$ $\qquad u := uuu$ $\qquad u := u$

$$X(v) = \begin{pmatrix} 6.068 \\ 1.292 \end{pmatrix}$$

$$Y(v) = \begin{pmatrix} 8.424 \\ 2.056 \end{pmatrix}$$

Intersection point parameter *u* solution on Circle:

Given $\quad X(v)_0 = x_c + r\cdot\cos(u)$ $\qquad u := \text{Find}(u) \rightarrow \text{acos}\left(\frac{-2}{25} + \frac{9}{50}\cdot 11^{\frac{1}{2}}\right)$

$x(u) =$ $y(u) =$ $u = \mathrm{acos}\left(\frac{-2}{25} + \frac{9}{50}\cdot 11^{\frac{1}{2}}\right)\mathrm{deg}$ $u := uuuu$ $u := u$

Given $Y(v)_0 = y_c + r\cdot\sin(u)$ $u := \mathrm{Find}(u) \rightarrow \mathrm{asin}\left(\frac{3}{50} + \frac{6}{25}\cdot 11^{\frac{1}{2}}\right)$

$x(u) =$ $y(u) =$ $u = \mathrm{asin}\left(\frac{3}{50} + \frac{6}{25}\cdot 11^{\frac{1}{2}}\right)\mathrm{deg}$ $u := uuuu$ $u := u$

Given $X(v)_1 = x_c + r\cdot\cos(u)$ $u := \mathrm{Find}(u) \rightarrow \mathrm{acos}\left(\frac{-2}{25} - \frac{9}{50}\cdot 11^{\frac{1}{2}}\right)$

$x(u) =$ $y(u) =$ $u = \mathrm{acos}\left(\frac{-2}{25} - \frac{9}{50}\cdot 11^{\frac{1}{2}}\right)\mathrm{deg}$ $u := uuuu$ $u := u$

Given $Y(v)_1 = y_c + r\cdot\sin(u)$ $u := \mathrm{Find}(u) \rightarrow \mathrm{asin}\left(\frac{3}{50} - \frac{6}{25}\cdot 11^{\frac{1}{2}}\right)$

$x(u) =$ $y(u) =$ $u = \mathrm{asin}\left(\frac{3}{50} - \frac{6}{25}\cdot 11^{\frac{1}{2}}\right)\mathrm{deg}$ $u := uuuu$ $u := u$