Parametric Cubic Curves

$x = a_{31}\cdot u^3 + a_{21}\cdot u^2 + a_{11}\cdot u + a_{01}$

$y = a_{32}\cdot u^3 + a_{22}\cdot u^2 + a_{12}\cdot u + a_{02}$

$z = a_{33}\cdot u^3 + a_{23}\cdot u^2 + a_{13}\cdot u + a_{03}$

$$p(u) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}\cdot\begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{01} & a_{02} & a_{03} \end{pmatrix} \quad \textit{(1)}$$

$$p(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}\cdot A$$

a_{ij} : 12 degrees of freedom (dof).

$$\frac{d}{du}\left(u^3 + u^2 + u + 1\right) \rightarrow 3\cdot u^2 + 2\cdot u + 1$$

$u = 0 .. 1$

Hermite Curve

$$\frac{d}{du}p(u) = p'(u) = \begin{pmatrix} 3\cdot u^2 & 2\cdot u & 1 & 0 \end{pmatrix}\cdot A \qquad A := \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{01} & a_{02} & a_{03} \end{pmatrix}$$

Start point (u=0): $p(0) = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}\cdot A$

End point (u=1): $p(1) = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}\cdot A$

Tangent at (u=0): $p'(0) = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}\cdot A$

Tangent at (u=1): $p'(1) = \begin{pmatrix} 3 & 2 & 1 & 0 \end{pmatrix}\cdot A$

$$\begin{pmatrix} p(0) \\ p(1) \\ p'(0) \\ p'(1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}\cdot A$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}^{-1}\cdot\begin{pmatrix} p(0) \\ p(1) \\ p'(0) \\ p'(1) \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} p(0) \\ p(1) \\ p'(0) \\ p'(1) \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$p(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}\cdot\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} p(0) \\ p(1) \\ p'(0) \\ p'(1) \end{pmatrix} \quad \textit{(2a)} \qquad p(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}\cdot A$$

$$p(u) := \left(2\cdot u^3 - 3\cdot u^2 + 1\right)\cdot p(0) + \left(-2\cdot u^3 + 3\cdot u^2\right)\cdot p(1) + \left(u^3 - 2\cdot u^2 + u\right)\cdot p'(0) + \left(u^3 - u^2\right)\cdot p'(1) \quad \textit{(2b)}$$

$F_1 = 2\cdot u^3 - 3\cdot u^2 + 1$

$F_2 = -2\cdot u^3 + 3\cdot u^2$

$F_3 = u^3 - 2\cdot u^2 + u$

$F_4 = u^3 - u^2$

Hermite Curve

$$p(u) = \begin{pmatrix} F_1 & F_2 & F_3 & F_4 \end{pmatrix}\cdot\begin{pmatrix} p(0) \\ p(1) \\ p'(0) \\ p'(1) \end{pmatrix} \quad \textit{(3)}$$

curve parameters and indexes :

$n := 10 \qquad i := 0 .. n \qquad u_i := \frac{i}{n}$

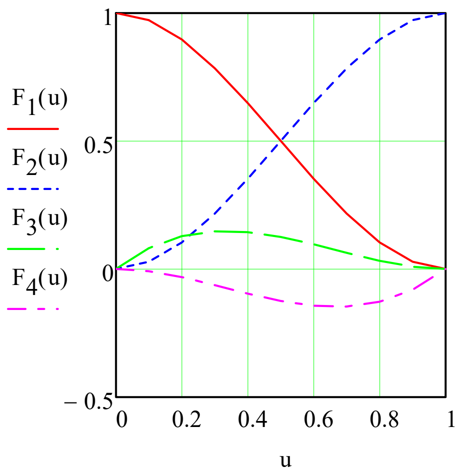
$F_1(u) := 2\cdot u^3 - 3\cdot u^2 + 1$

$F_2(u) := -2\cdot u^3 + 3\cdot u^2$

$F_3(u) := u^3 - 2\cdot u^2 + u$

$F_4(u) := u^3 - u^2$

F_i are called the blending functions. F_1 has the maximum influence on the curve shape near the start, F_2 near the end, and F_3, F_4, exert influence only in between the end points.



Instead of using tangents as the boundary conditions, we could also use two additional points on the curve. Thus the boundary conditions are specified by four points on the curve. Four points also have a total of 12 degrees of freedom. These alternative boundary conditions are shown in the following Figure.

$$p := \gamma_1 \cdot p_1 + \gamma_2 \cdot p_2 + \gamma_3 \cdot p_3 + \gamma_4 \cdot p_4 \qquad \textbf{\textit{(4)}}$$

γ_i are new blending functions that can be determined in terms of the algebraic coefficients *(1)* and *(2)*.

Hermite Curve

$$U := \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}^T \qquad U^T \rightarrow \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \qquad u := uu \qquad u := u$$

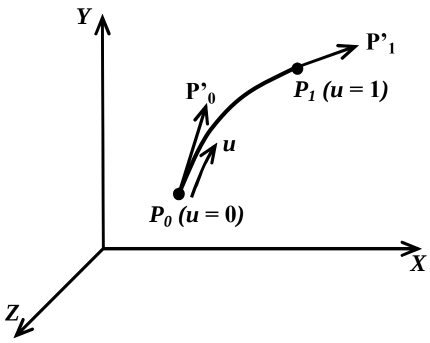
$$C := \begin{pmatrix} C_{3x} & C_{2x} & C_{1x} & C_{0x} \\ C_{3y} & C_{2y} & C_{1y} & C_{0y} \\ C_{3z} & C_{2z} & C_{1z} & C_{0z} \end{pmatrix}^T \qquad C \rightarrow \begin{pmatrix} C_{3x} & C_{3y} & C_{3z} \\ C_{2x} & C_{2y} & C_{2z} \\ C_{1x} & C_{1y} & C_{1z} \\ C_{0x} & C_{0y} & C_{0z} \end{pmatrix} \qquad p(u) := \sum_{i=0}^{3} \left(C_i \cdot u^i \right)$$

$$p(u) := U^T \cdot C \qquad p(u) := C^T \cdot U \qquad \frac{d}{du} p(u) \rightarrow \begin{pmatrix} 3 \cdot C_{3x} \cdot u^2 + 2 \cdot C_{2x} \cdot u + C_{1x} \\ 3 \cdot C_{3y} \cdot u^2 + 2 \cdot C_{2y} \cdot u + C_{1y} \\ 3 \cdot C_{3z} \cdot u^2 + 2 \cdot C_{2z} \cdot u + C_{1z} \end{pmatrix}$$

$$\begin{pmatrix} C_{3x} & C_{2x} & C_{1x} & C_{0x} \\ C_{3y} & C_{2y} & C_{1y} & C_{0y} \\ C_{3z} & C_{2z} & C_{1z} & C_{0z} \end{pmatrix} \cdot \begin{pmatrix} u^3 \\ u^2 \\ u \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} C_{3x} \cdot u^3 + C_{2x} \cdot u^2 + C_{1x} \cdot u + C_{0x} \\ C_{3y} \cdot u^3 + C_{2y} \cdot u^2 + C_{1y} \cdot u + C_{0y} \\ C_{3z} \cdot u^3 + C_{2z} \cdot u^2 + C_{1z} \cdot u + C_{0z} \end{pmatrix} \qquad C := \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

$$p(u) := \sum_{i=0}^{3} \left(C_i \cdot u^i \right) \qquad p(u) \rightarrow C_3 \cdot u^3 + C_2 \cdot u^2 + C_1 \cdot u + C_0 \qquad 0 \le u \le 1$$

$$p'(u) := \sum_{i=0}^{3} \left(C_i \cdot i \cdot u^{i-1} \right) \qquad \frac{d}{du} p(u) \rightarrow 3 \cdot C_3 \cdot u^2 + 2 \cdot C_2 \cdot u + C_1$$



$p'(u) := \frac{d}{du} p(u)$ $\qquad$ $p'(u) \rightarrow 3 \cdot C_3 \cdot u^2 + 2 \cdot C_2 \cdot u + C_1$

$P_0 := p(0)$ $\qquad$ $p(u)$ substitute, $u = 0 \rightarrow C_0$ $\qquad$ $C_0 := P_0$

$P'_0 := p'(0)$ $\qquad$ $P'_0 \rightarrow C_1$ $\qquad$ $C_1 := P'_0$

$P_1 := p(1)$ $\qquad$ $P_1 \rightarrow C_0 + C_1 + C_2 + C_3$ $\qquad$ $C_2 := P_1 - P_0 - P'_0 - C_3$

$P'_1 := p'(1)$ $\qquad$ $P'_1 \rightarrow C_1 + 2 \cdot C_2 + 3 \cdot C_3$ $\qquad$ $3 \cdot C_3 = P'_1 - C_1 - 2 \cdot C_2$

$$3 \cdot C_3 = P'_1 - P'_0 - 2 \cdot \left(P_1 - P_0 - P'_0 - C_3\right)$$

$$C_3 := 2 \cdot \left(P_0 - P_1\right) + P'_0 + P'_1$$

$$C_2 := P_1 - P_0 - P'_0 - \left[2 \cdot \left(P_0 - P_1\right) + P'_0 + P'_1\right]$$

$$C_2 := 3 \cdot \left(P_1 - P_0\right) - 2 \cdot P'_0 - P'_1$$

Erase the points and tangents for symbolic calculation.

$P_0 := P0$

$P'_0 := P'0$

$P_1 := P1$

$P'_1 := P'1$

$C_0 := P_0$

$C_1 := P'_0$

$$p(u) := C_0 + C_1 \cdot u + C_2 \cdot u^2 + C_3 \cdot u^3$$

$$p(u) := P_0 + P'_0 \cdot u + \left[3 \cdot \left(P_1 - P_0\right) - 2 \cdot P'_0 - P'_1\right] \cdot u^2 + \left[2 \cdot \left(P_0 - P_1\right) + P'_0 + P'_1\right] \cdot u^3$$

$$p(u) := \left(2 \cdot u^3 - 3 \cdot u^2 + 1\right) \cdot p(0) + \left(-2 \cdot u^3 + 3 \cdot u^2\right) \cdot p(1) + \left(u^3 - 2 \cdot u^2 + u\right) \cdot p'(0) + \left(u^3 - u^2\right) \cdot p'(1)$$

$$p(u) := \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} p(0) \\ p(1) \\ p'(0) \\ p'(1) \end{pmatrix} \qquad M_H := \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$C := M_H \cdot V$ $\qquad$ $V := M_H^{-1} \cdot C$ $\qquad$ $U^T \rightarrow \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}$ $\qquad$ $V := \begin{pmatrix} P_0 \\ P_1 \\ P'_0 \\ P'_1 \end{pmatrix}$ $\qquad$ $M_H^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}$

$p(u) := U^T \cdot C$ $\qquad$ $p(u) := U^T \cdot M_H \cdot V$

$$p(u) \rightarrow P0 \cdot \left(2 \cdot u^3 - 3 \cdot u^2 + 1\right) + P1 \cdot \left(3 \cdot u^2 - 2 \cdot u^3\right) + P'0 \cdot \left(u^3 - 2 \cdot u^2 + u\right) - P'1 \cdot \left(u^2 - u^3\right)$$

$$p(u) := \left(2 \cdot u^3 - 3 \cdot u^2 + 1\right) \cdot P_0 + \left(-2 \cdot u^3 + 3 \cdot u^2\right) \cdot P_1 + \left(u^3 - 2 \cdot u^2 + u\right) \cdot P'_0 + \left(u^3 - u^2\right) \cdot P'_1$$

$$\frac{d}{du} p(u) \rightarrow P1 \cdot \left(6 \cdot u - 6 \cdot u^2\right) - P0 \cdot \left(6 \cdot u - 6 \cdot u^2\right) - P'1 \cdot \left(2 \cdot u - 3 \cdot u^2\right) + P'0 \cdot \left(3 \cdot u^2 - 4 \cdot u + 1\right)$$

$$M'_H := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 6 & -6 & 3 & 3 \\ -6 & 6 & -4 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad V := \begin{pmatrix} P_0 \\ P_1 \\ P'_0 \\ P'_1 \end{pmatrix} \qquad M''_H := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & -12 & 6 & 6 \\ -6 & 6 & -4 & -2 \end{pmatrix}$$

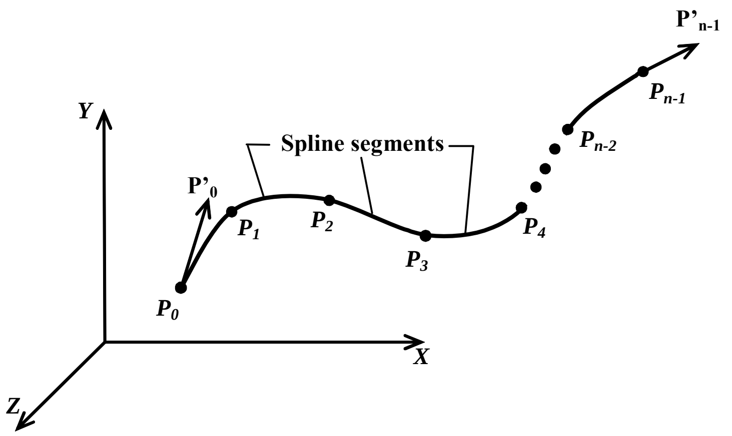
$$p'(u) := U^T \cdot M'_H \cdot V$$

$$p'(u) \rightarrow P1 \cdot \left(6 \cdot u - 6 \cdot u^2\right) - P0 \cdot \left(6 \cdot u - 6 \cdot u^2\right) - P'1 \cdot \left(2 \cdot u - 3 \cdot u^2\right) + P'0 \cdot \left(3 \cdot u^2 - 4 \cdot u + 1\right)$$

$$p''(u) := \frac{d^2}{du^2}p(u) \qquad p''(u) \rightarrow P0\cdot(12\cdot u - 6) - P1\cdot(12\cdot u - 6) + P'1\cdot(6\cdot u - 2) + P'0\cdot(6\cdot u - 4)$$

For curvature continuity between the first two segments, we can write

$$p''\left(u_1 = 1\right) = p''\left(u_2 = 0\right) \qquad P_0 \quad P_1 \quad \cdots \quad P_{n-1} \qquad P'_0 \qquad P'_{n-1}$$



The above equation is for one cubic spline segment. It can be generalized for any two adjacent spline segments of a spline curve that are to fit a given number of data points. This introduces the problem of blending or joining cubic spline segments which can be stated as follows. Given a set of n points P_0, P_1,...P_{n-1} and the end tangent vectors P'_0, P'_{n-1} connect the points with a cubic spline curve.

$$V := \begin{pmatrix} P_1 \\ P_2 \\ P'_1 \\ P'_2 \end{pmatrix}$$

$$p''(1) \rightarrow 6\cdot P0 - 6\cdot P1 + 2\cdot P'0 + 4\cdot P'1$$

$$p(u) := U^T\cdot M_H\cdot V$$

$$p(u) \rightarrow P1\cdot\left(2\cdot u^3 - 3\cdot u^2 + 1\right) + P_2\cdot\left(3\cdot u^2 - 2\cdot u^3\right) + P'1\cdot\left(u^3 - 2\cdot u^2 + u\right) - P'_2\cdot\left(u^2 - u^3\right)$$

$$p''(u) := \frac{d^2}{du^2}p(u) \qquad p''(u) \rightarrow P1\cdot(12\cdot u - 6) + P'1\cdot(6\cdot u - 4) - P_2\cdot(12\cdot u - 6) + P'_2\cdot(6\cdot u - 2)$$

$$p''(0) \rightarrow 6\cdot P_2 - 4\cdot P'1 - 6\cdot P1 - 2\cdot P'_2$$

$$p''\left(u_1 = 1\right) = p''\left(u_2 = 0\right)$$

$$6\cdot P0 - 6\cdot P1 + 2\cdot P'0 + 4\cdot P'1 = -6\cdot P1 + 6\cdot P_2 - 4\cdot P'1 - 2\cdot P'_2$$

$$6\cdot P0 - 6\cdot P1 + 2\cdot P'0 + 4\cdot P'1 = -6\cdot P1 + 6\cdot P_2 - 4\cdot P'1 - 2\cdot P'_2 \text{ solve}, P'1 \rightarrow \frac{3\cdot P_2}{4} - \frac{P'0}{4} - \frac{3\cdot P0}{4} - \frac{P'_2}{4}$$

$$P'1 := \frac{-3}{4}\cdot P0 - \frac{1}{4}\cdot P'0 + \frac{3}{4}\cdot P_2 - \frac{1}{4}\cdot P'_2$$

$$V := \begin{pmatrix} P_2 \\ P_3 \\ P'_2 \\ P'_3 \end{pmatrix}$$

$$p''(1) \rightarrow 6\cdot P1 + 2\cdot P'1 - 6\cdot P_2 + 4\cdot P'_2$$

$$p(u) := U^T\cdot M_H\cdot V$$

$$p(u) \rightarrow P_2\cdot\left(2\cdot u^3 - 3\cdot u^2 + 1\right) + P_3\cdot\left(3\cdot u^2 - 2\cdot u^3\right) + P'_2\cdot\left(u^3 - 2\cdot u^2 + u\right) - P'_3\cdot\left(u^2 - u^3\right)$$

$$p''(u) := \frac{d^2}{du^2}p(u) \qquad p''(u) \rightarrow P_2\cdot(12\cdot u - 6) - P_3\cdot(12\cdot u - 6) + P'_3\cdot(6\cdot u - 2) + P'_2\cdot(6\cdot u - 4)$$

$$p''(0) \rightarrow 6\cdot P_3 - 6\cdot P_2 - 4\cdot P'_2 - 2\cdot P'_3$$

$$p''(u_2 = 1) = p''(u_3 = 0)$$

$$6 \cdot P1 - 6 \cdot P_2 + 2 \cdot P'1 + 4 \cdot P'_2 = -6 \cdot P_2 + 6 \cdot P_3 - 4 \cdot P'_2 - 2 \cdot P'_3$$

$$6 \cdot P1 - 6 \cdot P_2 + 2 \cdot P'1 + 4 \cdot P'_2 = -6 \cdot P_2 + 6 \cdot P_3 - 4 \cdot P'_2 - 2 \cdot P'_3 \text{ solve}, P'_2 \rightarrow \frac{P0}{5} - \frac{4 \cdot P1}{5} + \frac{P'0}{15} - \frac{P_2}{5} + \frac{4 \cdot P_3}{5} - \frac{4 \cdot P'_3}{15}$$

Thus, the geometric information of a cubic spline database consists of the set of the data points (P_0, P_1, P_2, P_3, . . . P_{n-1}) and the two end tangent vectors (P'_0, P'_{n-1}).

Interpolation Curve:

If we have m-1 segments on cubic spline defined by $P_0 \dots P_{m-1}$ points.

a) if the end point tangents are known

b) and the second derivatives at $P_0 \dots P_{m-1}$ end points are equal to 0,

this curve is named as **Natural Cubic Spline**.

$$p''(u_1 = 1) = p''(u_2 = 0)$$

$$p''(u_2 = 1) = p''(u_3 = 0)$$

$$P'_{i-1} + 4 \cdot P'_i + P'_{i+1} = 3 \cdot (P_{i+1} - P_{i-1})$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} P'_0 \\ P'_1 \\ P'_2 \\ P'_3 \\ P'_{m-2} \\ P'_{m-1} \end{pmatrix} = \begin{bmatrix} P'_0 \\ 3 \cdot (P_2 - P_0) \\ 3 \cdot (P_3 - P_1) \\ 3 \cdot (P_{m-2} - P_2) \\ 3 \cdot (P_{m-1} - P_{m-3}) \\ P'_{m-1} \end{bmatrix}$$

$$\begin{pmatrix} P'_0 \\ P'_1 \\ P'_2 \\ P'_3 \\ P'_{m-2} \\ P'_{m-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{bmatrix} P'_0 \\ 3 \cdot (P_2 - P_0) \\ 3 \cdot (P_3 - P_1) \\ 3 \cdot (P_{m-2} - P_2) \\ 3 \cdot (P_{m-1} - P_{m-3}) \\ P'_{m-1} \end{bmatrix}$$

$$P'_i := M_{Gs}^{-1} \cdot G_{Cs}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 15 & 0 & 0 & 0 \\ -4 & 4 & -1 & 1 \\ 1 & -1 & 4 & -4 \\ 0 & 0 & 0 & 15 \end{pmatrix} \cdot \frac{1}{15}$$

SAMPLE PROBLEM:

Calculate the P_1 and P_2 tangent vectors of the curve defined by the following interpolation points.

$$P_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad P_1 := \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad P_2 := \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad P_3 := \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad P'_0 := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad P'_3 := \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix}P'_0\\ P'_1\\ P'_2\\ P'_3\end{pmatrix}\ \begin{pmatrix}P'0_0 & P'0_1\\ P'1_0 & P'1_1\\ P'2_0 & P'2_1\\ P'3_0 & P'3_1\end{pmatrix} := \begin{pmatrix}1&0&0&0\\ 1&4&1&0\\ 0&1&4&1\\ 0&0&0&1\end{pmatrix}^{-1}\cdot\begin{bmatrix}P'0_0 & P'0_1\\ 3\cdot\left(P2_0-P0_0\right) & 3\cdot\left(P2_1-P0_1\right)\\ 3\cdot\left(P3_0-P1_0\right) & 3\cdot\left(P3_1-P1_1\right)\\ P'3_0 & P'3_1\end{bmatrix}\qquad P'_1=\begin{pmatrix}\frac{31}{15}\\ \frac{16}{5}\end{pmatrix}\qquad P'_2=\begin{pmatrix}\frac{41}{15}\\ -\frac{9}{5}\end{pmatrix}$$

TANGENT OF A CURVE :

$u := uu \qquad u := u \qquad p := \left(p_0\ \ p_1\ \ p_2\ \ p_3\right)^T$

$$p(u) := p_0\cdot(1-u)^3 + p_1\cdot 3\cdot u\cdot(1-u)^2 + p_2\cdot 3\cdot u^2\cdot(1-u) + p_3\cdot u^3 \qquad p'(u) := \frac{d}{du}p(u)$$

$$p'(u) \rightarrow 3\cdot p_3\cdot u^2 - 3\cdot p_2\cdot u^2 - 3\cdot p_0\cdot(u-1)^2 + 3\cdot p_1\cdot(u-1)^2 + 3\cdot p_1\cdot u\cdot(2\cdot u-2) - 6\cdot p_2\cdot u\cdot(u-1)$$

Control Points : $n := 10 \quad i := 0..n \quad u_i := \frac{i}{n} \quad p_0 := \begin{pmatrix}1\\1\\6\end{pmatrix} \quad p_1 := \begin{pmatrix}7\\6\\4\end{pmatrix} \quad p_2 := \begin{pmatrix}2\\4\\19\end{pmatrix} \quad p_3 := \begin{pmatrix}8\\8\\10\end{pmatrix}$

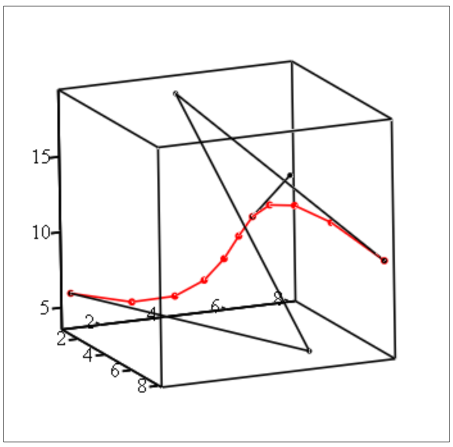
$P := \mathrm{stack}\left(p_0{}^T, p_1{}^T, p_2{}^T, p_3{}^T\right) \qquad Px := P^{\langle 0\rangle} \quad Py := P^{\langle 1\rangle} \quad Pz := P^{\langle 2\rangle}$

$$P(u) := p_0\cdot(1-u)^3 + p_1\cdot 3\cdot u\cdot(1-u)^2 + p_2\cdot 3\cdot u^2\cdot(1-u) + p_3\cdot u^3$$

$x_i := P(u_i)_0 \qquad y_i := P(u_i)_1 \qquad z_i := P(u_i)_2$

$$P'(u) := -3\cdot p_0\cdot(1-u)^2 + 3\cdot p_1\cdot(1-u)^2 - 6\cdot p_1\cdot u\cdot(1-u) + 6\cdot p_2\cdot u\cdot(1-u) - 3\cdot p_2\cdot u^2 + 3\cdot p_3\cdot u^2$$

$PP'(u) := \mathrm{stack}\left[P(u)^T, \left(P(u) + 3\cdot\frac{P'(u)}{\left|P'(u)\right|}\right)^T\right] \qquad u := 0.6 \quad X := PP'(u)^{\langle 0\rangle} \quad Y := PP'(u)^{\langle 1\rangle} \quad Z := PP'(u)^{\langle 2\rangle}$

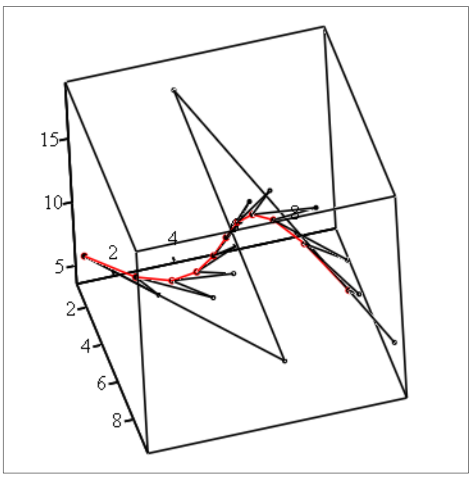


(x,y,z),(X,Y,Z),(Px,Py,Pz)

$u_i := \frac{i}{n} \qquad X_{2\cdot i} := \left(PP'(u_i)^{\langle 0\rangle}\right)_0 \qquad X_{2\cdot i+1} := \left(PP'(u_i)^{\langle 0\rangle}\right)_1$

$Y_{2\cdot i} := \left(PP'(u_i)^{\langle 1\rangle}\right)_0 \qquad Y_{2\cdot i+1} := \left(PP'(u_i)^{\langle 1\rangle}\right)_1$

$Z_{2\cdot i} := \left(PP'(u_i)^{\langle 2\rangle}\right)_0 \qquad Z_{2\cdot i+1} := \left(PP'(u_i)^{\langle 2\rangle}\right)_1$



(x,y,z),(X,Y,Z),(Px,Py,Pz)

Bezier Curve by using Hermite curve definition:

$$\begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}\cdot\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} p0 \\ p3 \\ p_1 - p0 \\ p3 - p2 \end{pmatrix}$$

tangents for start and end points

$$\left(u^2 - u^3\right)\cdot(p2 - p3) - \left(p0 - p_1\right)\cdot\left(u^3 - 2\cdot u^2 + u\right) + p0\cdot\left(2\cdot u^3 - 3\cdot u^2 + 1\right) + p3\cdot\left(3\cdot u^2 - 2\cdot u^3\right)$$

$$\left(u^3 - u^2 - u + 1\right)\cdot p0 + \left(u^2 - u^3\right)\cdot p2 + \left(2\cdot u^2 - u^3\right)\cdot p3 + p_1\cdot\left(u^3 - 2\cdot u^2 + u\right)$$

$$p(u) \qquad \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -2 & 1 & 2 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

Bezier Curve by using Hermite curve definition:

$$\begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}\cdot\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\cdot\begin{bmatrix} p0 \\ p3 \\ 3\cdot\left(p_1 - p0\right) \\ 3\cdot(p3 - p2) \end{bmatrix}$$

tangents for start and end points

$$p0\cdot\left(2\cdot u^3 - 3\cdot u^2 + 1\right) + p3\cdot\left(3\cdot u^2 - 2\cdot u^3\right) - \left(3\cdot p0 - 3\cdot p_1\right)\cdot\left(u^3 - 2\cdot u^2 + u\right) + \left(u^2 - u^3\right)\cdot(3\cdot p2 - 3\cdot p3)$$

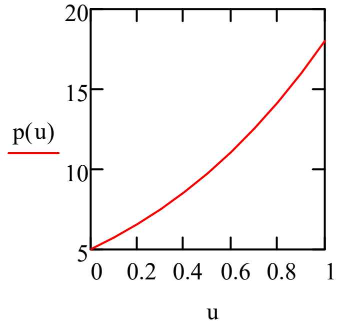
$$\left(3\cdot u^2 - u^3 - 3\cdot u + 1\right)\cdot p0 + \left(3\cdot u^2 - 3\cdot u^3\right)\cdot p2 + u^3\cdot p3 + 3\cdot p_1\cdot\left(u^3 - 2\cdot u^2 + u\right)$$

$$p(u) \qquad \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}\cdot\begin{pmatrix} -1 & 3 & -3 & 1 \\ -3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

curve parameters and indexes : $n := 10 \quad i := 0..n \quad u_i := \frac{i}{n} \quad m := 3 \quad j := 0..m$

$a := 2 \quad b := 4 \quad c := 7 \quad d := 5$

General Polynomial Curve Equation : $p(u) := a \cdot u^3 + b \cdot u^2 + c \cdot u + d$



BEZIER CURVE

Approximated Curve

Control Points (characteristic polygon) :

$$p_0 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad p_1 := \begin{pmatrix} 3 \\ 6 \\ 12 \end{pmatrix} \quad p_2 := \begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix} \quad p_3 := \begin{pmatrix} 8 \\ 8 \\ 10 \end{pmatrix} \quad P := \text{stack}\left(p_0^T, p_1^T, p_2^T, p_3^T\right) \quad P0 := P$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 6 & 12 \\ 5 & 8 & 9 \\ 8 & 8 & 10 \end{pmatrix} \qquad X := P^{\langle 0 \rangle} \quad Y := P^{\langle 1 \rangle} \quad Z := P^{\langle 2 \rangle}$$

$$p1(u) := (1-u) \cdot p_0 + u \cdot p_1$$

$$p2(u) := p_0 \cdot (1-u)^2 + p_1 \cdot 2 \cdot u \cdot (1-u) + p_2 \cdot u^2$$

$$p2(u) := (1-u)\left[(1-u) \cdot p_0 + u \cdot p_1\right] + u \cdot \left[(1-u) \cdot p_1 + u \cdot p_2\right]$$

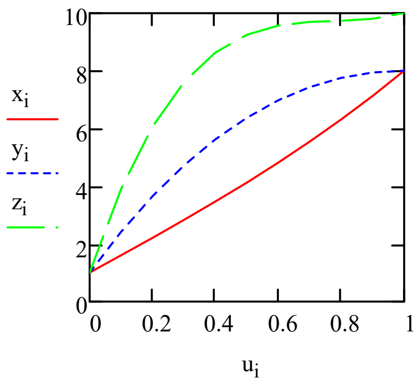
$$p30(u) := (1-u) \cdot \left[(1-u) \cdot \left[(1-u) \cdot p_0 + u \cdot p_1\right] + u \cdot \left[(1-u) \cdot p_1 + u \cdot p_2\right]\right]$$

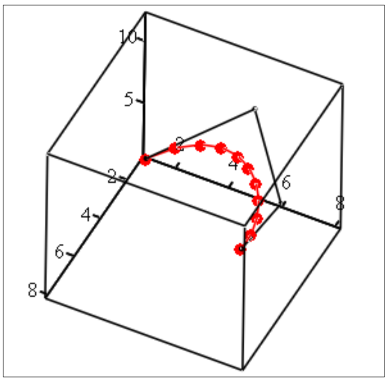
$$p3(u) := p30 + u \cdot \left[(1-u) \cdot \left[(1-u) \cdot p_1 + u \cdot p_2\right] + u \cdot \left[(1-u) \cdot p_2 + u \cdot p_3\right]\right]$$

$$p(u) := p_0 \cdot (1-u)^3 + p_1 \cdot 3 \cdot u \cdot (1-u)^2 + p_2 \cdot 3 \cdot u^2 \cdot (1-u) + p_3 \cdot u^3$$

$$x_i := p(u_i)_0 \quad y_i := p(u_i)_1 \quad z_i := p(u_i)_2$$

Home Work 2 :
Drive this Bezier Curve polinom for 5 and 6 points interpolation.
Due date: Oct. 31, 2002 Thursday





(x, y, z), (X, Y, Z)

$$p(u) = \left(3 \cdot p_1 - p_0 - 3 \cdot p_2 + p_3\right) \cdot u^3 + \left(3 \cdot p_0 + 3 \cdot p_2 - 6 \cdot p_1\right) \cdot u^2 + \left(-3 \cdot p_0 + 3 \cdot p_1\right) \cdot u + p_0$$

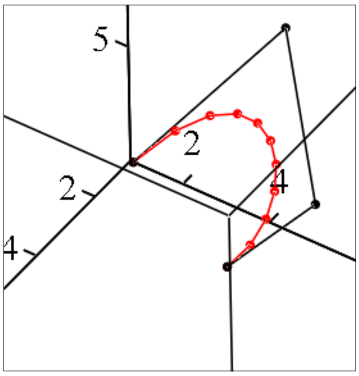
Matrix Representation of Bezier Curve :

$$U(u) := \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \qquad B := \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad p(u) := U(u) \cdot B \cdot P$$

$x_i := p(u_i)_{0,0}$

$y_i := p(u_i)_{0,1}$

$z_i := p(u_i)_{0,2}$



$(x,y,z),(X,Y,Z)$

$u := u$

$p := p$

$p_i := p_{n-i}$

$n := 4 \quad i := 0..n \quad j := 0..n \quad U_{0,i} := u^{n-i}$

$B := 0 \quad C(n,i) := \dfrac{n!}{i!\cdot(n-i)!} \quad U \rightarrow \begin{pmatrix} u^4 & u^3 & u^2 & u & 1 \end{pmatrix}$

$B_{i,j} := \text{if}\left[0 \le i+j \le n, \left[C(n,j)\cdot C(n-j, n-i-j)\cdot(-1)^{n-i-j}\right], 0\right]$

$$B = \begin{pmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad p \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$n := 10 \quad i := 0..n \quad n := 3 \quad j := 0..3 \quad p(u) := U\cdot B\cdot p$

$p(u) \rightarrow 0$

Bernstien Polynomials as blending functions

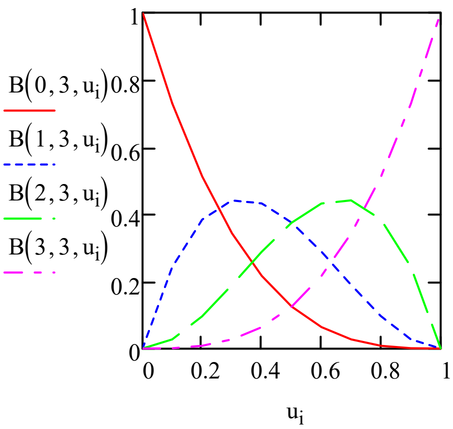
Binom Representation of Bezier Curve :

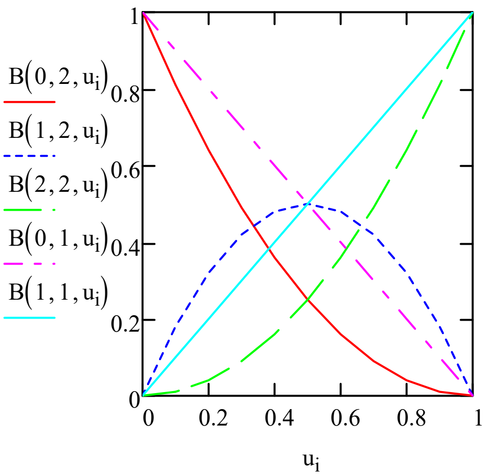
$C(n,i) := \dfrac{n!}{i!\cdot(n-i)!} \quad B(i,n,u) := C(n,i)\cdot u^i\cdot(1-u)^{n-i}$

$x(u) := \sum_{i=0}^{n} \left[\left(p^{\langle 0\rangle}\right)_i\cdot B(i,n,u)\right] \quad X_i := x(u_i) \quad XX_i := X_i$

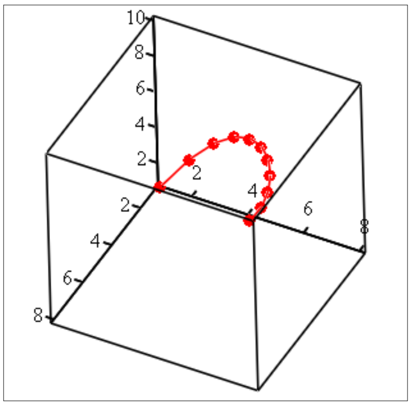
$y(u) := \sum_{i=0}^{n} \left[\left(p^{\langle 1\rangle}\right)_i\cdot B(i,n,u)\right] \quad Y_i := y(u_i) \quad YY_i := Y_i$

$z(u) := \sum_{i=0}^{n} \left[\left(p^{\langle 2\rangle}\right)_i\cdot B(i,n,u)\right] \quad Z_i := z(u_i) \quad ZZ_i := Z_i$





$i =$	$B(0,3,u_i) + B(1,3,u_i) + B(2,3,u_i) + B(3,3,u_i) =$
0	1
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	1



(X,Y,Z)

$u := u$

$B(0,3,u) \rightarrow -(u-1)^3$

$B(1,3,u) \rightarrow 3 \cdot u \cdot (u-1)^2$

$B(2,3,u) \rightarrow -3 \cdot u^2 \cdot (u-1)$

$B(3,3,u) \rightarrow u^3$

$p(u) := p_0 \cdot B(0,3,u) + p_1 \cdot B(1,3,u) + p_2 \cdot B(2,3,u) + p_3 \cdot B(3,3,u)$

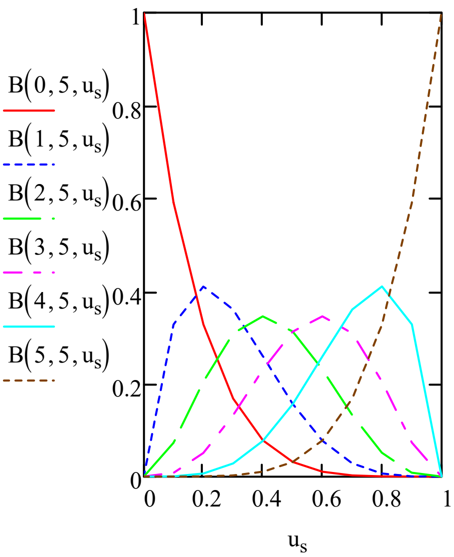
$p(u) := p_0 \cdot (1-u)^3 + p_1 \cdot 3 \cdot u \cdot (1-u)^2 + p_2 \cdot 3 \cdot u^2 \cdot (1-u) + p_3 \cdot u^3$

BEZIER CURVE :

$r := 0..2$ $\quad P := 0$ $\quad t := 10$ $\quad n := 5$ $\quad i := 0..n$ $\quad u := 0$

$C(n,i) := \dfrac{n!}{i! \cdot (n-i)!}$ $\quad B(i,n,u) := C(n,i) \cdot u^i \cdot (1-u)^{n-i}$ $\quad s := 0..t$ $\quad u_s := \dfrac{s}{t}$

$p_0 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad p_1 := \begin{pmatrix} 3 \\ 7 \\ 16 \end{pmatrix}$ $\quad p_2 := \begin{pmatrix} 5 \\ 12 \\ 6 \end{pmatrix}$ $\quad p_3 := \begin{pmatrix} 5 \\ 12 \\ 6 \end{pmatrix}$ $\quad p_4 := \begin{pmatrix} 5 \\ 12 \\ 6 \end{pmatrix}$ $\quad p_5 := \begin{pmatrix} 8 \\ 8 \\ 10 \end{pmatrix}$



$P_{i,r} := \left[\left[\left(p_i\right)^T\right]^{\langle r \rangle}\right]_{0,0}$

$P = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 7 & 16 \\ 5 & 12 & 6 \\ 5 & 12 & 6 \\ 5 & 12 & 6 \\ 8 & 8 & 10 \end{pmatrix}$

u =

	0
0	0
1	0.1
2	0.2
3	0.3
4	0.4
5	0.5
6	0.6
7	0.7
8	0.8
9	0.9
10	1

$u := u$

$B(0,n,u) \;\Big|\; \begin{matrix} \text{simplify} \\ \text{collect}, u \end{matrix} \rightarrow 5 \cdot u^4 - u^5 - 10 \cdot u^3 + 10 \cdot u^2 - 5 \cdot u + 1$

$B(1,1,u) \;\Big|\; \begin{matrix} \text{simplify} \\ \text{collect}, u \end{matrix} \rightarrow u$

$B(0,3,u) \rightarrow -(u-1)^3$

$$x(u) := \sum_{i=0}^{n} \left[\left(P^{\langle 0 \rangle}\right)_i \cdot B(i,n,u) \right] \quad X_s := x\left(u_s\right) \quad X_{t+i} := \left(P^{\langle 0 \rangle}\right)_{n-i}$$

$$y(u) := \sum_{i=0}^{n} \left[\left(P^{\langle 1 \rangle}\right)_i \cdot B(i,n,u) \right] \quad Y_s := y\left(u_s\right) \quad Y_{t+i} := \left(P^{\langle 1 \rangle}\right)_{n-i}$$

$$PP := P$$

$$z(u) := \sum_{i=0}^{n} \left[\left(P^{\langle 2 \rangle}\right)_i \cdot B(i,n,u) \right] \quad Z_s := z\left(u_s\right) \quad Z_{t+i} := \left(P^{\langle 2 \rangle}\right)_{n-i}$$

$$P := \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{pmatrix}$$

$$B(0,5,u) \rightarrow -(u-1)^5$$

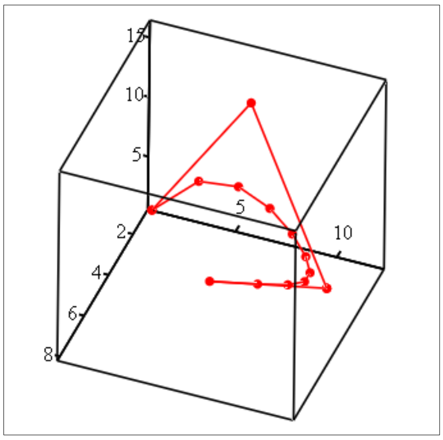
$$\frac{d}{du} B(0,5,u) \rightarrow -5\cdot(u-1)^4$$

$$\frac{d^2}{du^2} B(0,5,u) \rightarrow -20\cdot(u-1)^3$$

$$\sum_{i=0}^{n} B(i,n,($$

$$n := 3 \qquad i := 0..n \qquad P_i := P_{n-i} \qquad P \rightarrow \begin{pmatrix} \{2,1\} \\ \{2,1\} \\ \{2,1\} \\ \{2,1\} \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{pmatrix}$$

$$p(u) := \sum_{i=0}^{n} \left(B(i,n,u)\cdot P_i\right)$$



(X, Y, Z)

Rational Bezier Curve Equation:

$$h := \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$pR(u) := \frac{\displaystyle\sum_{i=0}^{n} \left(B(i,n,u)\cdot h_i \cdot P_i\right)}{\displaystyle\sum_{i=0}^{n} \left(B(i,n,u)\cdot h_i\right)}$$

$$p(u) \rightarrow \begin{bmatrix} 6\cdot u^3 + 12\cdot u\cdot(u-1)^2 - 12\cdot u^2\cdot(u-1) - 6\cdot(u-1)^3 \\ 12\cdot u\cdot(u-1)^2 - 12\cdot u^2\cdot(u-1) \end{bmatrix}$$

take the value saved in variable

$$P := PP$$

polinom curve points :

$$x := 0, 0.1..5 \qquad n := 3 \qquad i := 0..n \qquad m := 5 \qquad j := 0..m \qquad u := 0 \qquad u_j := \frac{j}{m}$$

$$Y_i := A_0 + A_1\cdot X_i + A_2\cdot\left(X_i\right)^2 + A_3\cdot\left(X_i\right)^3 \qquad Y = M\cdot A$$

$$X := \begin{pmatrix} 0 \\ 1.6 \\ 3.6 \\ 5.0 \end{pmatrix} \quad Y := \begin{pmatrix} 3.0 \\ 2.6 \\ 5.6 \\ 9.0 \end{pmatrix} \quad M := \begin{bmatrix} 1 & X_0 & \left(X_0\right)^2 & \left(X_0\right)^3 \\ 1 & X_1 & \left(X_1\right)^2 & \left(X_1\right)^3 \\ 1 & X_2 & \left(X_2\right)^2 & \left(X_2\right)^3 \\ 1 & X_3 & \left(X_3\right)^2 & \left(X_3\right)^3 \end{bmatrix} \quad A := M^{-1}\cdot Y \quad A = \begin{pmatrix} 3 \\ -1.273 \\ 0.708 \\ -0.043 \end{pmatrix}$$

$$y(x) := A_0 + A_1\cdot x + A_2\cdot x^2 + A_3\cdot x^3 \qquad p(u) := A_0 + A_1\cdot u + A_2\cdot u^2 + A_3\cdot u^3 \qquad y(X) := \sum_{i} \left(A_i\cdot X^i\right)$$

$$P_x(u) := \sum_{i=0}^{n} \left(X_i\cdot B(i,n,u)\right) \qquad P_y(u) := \sum_{i=0}^{n} \left(Y_i\cdot B(i,n,u)\right)$$

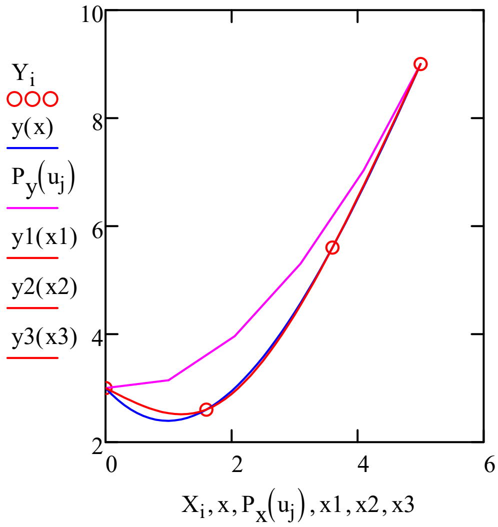
interpolation with three cubic b-spline curve :

Composite Curves

$y1(x) := 0.1396 \cdot x^3 - 0.6074 \cdot x + 3$ $x1 := X_0, X_0 + 0.1 .. X_1$

$y2(x) := -0.0763 \cdot x^3 + 1.0362 \cdot x^2 - 2.2651 \cdot x + 3.8842$ $x2 := X_1, X_1 + 0.1 .. X_2$

$y3(x) := -0.0506 \cdot x^3 + 0.7591 \cdot x^2 - 1.2677 \cdot x + 2.6871$ $x3 := X_2, X_2 + 0.1 .. X_3$



C^0 (point) continuity: $p_1{}^A = p_0{}^B$

C^1 (tangent) continuity: $p_1{}^A = p_0{}^B$

$p_1{}^{uA} = a \cdot p_0{}^{uB}$

save the values in variables for use later

$xxx := x$

$x := xx$

$x := x$

$YYY := Y$

C^2 (curvature) continuity:

$p_1{}^A = p_0{}^B$ $Y(x) := A0 + A1 \cdot x + A2 \cdot (x)^2 + A3 \cdot (x)^3$

$p_1{}^{uA} = a \cdot p_0{}^{uB}$ $\frac{d}{dx} Y(x) \rightarrow 3 \cdot A3 \cdot x^2 + 2 \cdot A2 \cdot x + A1$

$p_1{}^{uuA} = b \cdot p_0{}^{uuB}$ $\frac{d^2}{dx^2} Y(x) \rightarrow 2 \cdot A2 + 6 \cdot A3 \cdot x$

take the values saved in variables

$x := xxx$ $Y := YYY$

natural cubic spline interpolation :

$$M := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad PP := \begin{bmatrix} 3 \cdot (X_1 - X_0) & 3 \cdot (Y_1 - Y_0) \\ 3 \cdot (X_2 - X_0) & 3 \cdot (Y_2 - Y_0) \\ 3 \cdot (X_3 - X_1) & 3 \cdot (Y_3 - Y_1) \\ 3 \cdot (X_3 - X_2) & 3 \cdot (Y_3 - Y_2) \end{bmatrix} \quad PT := M^{-1} \cdot PP$$

$$PT = \begin{pmatrix} 1.453 & -1.28 \\ 1.893 & 1.36 \\ 1.773 & 3.64 \\ 1.213 & 3.28 \end{pmatrix}$$

$$M_H := \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$M'_H := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 6 & -6 & 3 & 3 \\ -6 & 6 & -4 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$U(u) := \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \quad MH := \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$M''_H := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & -12 & 6 & 6 \\ -6 & 6 & -4 & -2 \end{pmatrix}$$

$$G := \begin{pmatrix} X_0 & Y_0 \\ X_1 & Y_1 \\ PT_{0,0} & PT_{0,1} \\ PT_{1,0} & PT_{1,1} \end{pmatrix} \qquad p(u) := U(u)\cdot MH\cdot G$$

$$xx(u) := U(u)\cdot MH\cdot G^{\langle 0\rangle} \qquad X1_j := xx(u_j)$$

$$yy(u) := U(u)\cdot MH\cdot G^{\langle 1\rangle} \qquad Y1_j := yy(u_j)$$

$$G := \begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \\ PT_{1,0} & PT_{1,1} \\ PT_{2,0} & PT_{2,1} \end{pmatrix}$$

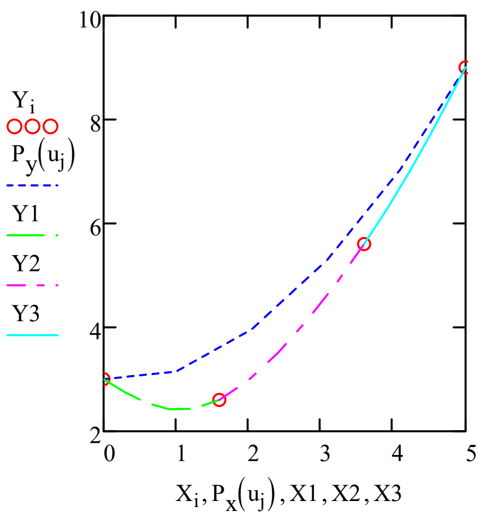
$$xx(u) := U(u)\cdot MH\cdot G^{\langle 0\rangle} \qquad X2_j := xx(u_j)$$

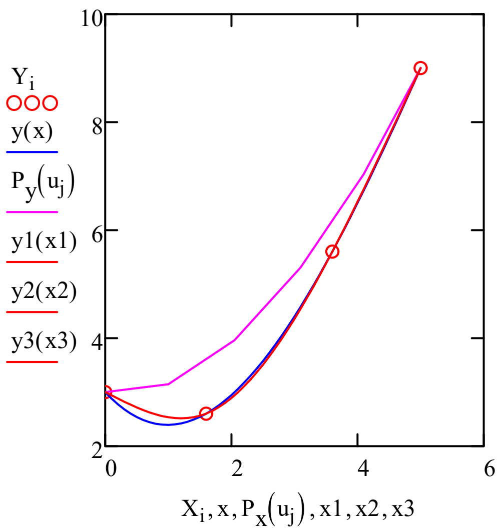
$$yy(u) := U(u)\cdot MH\cdot G^{\langle 1\rangle} \qquad Y2_j := yy(u_j)$$

$$G := \begin{pmatrix} X_2 & Y_2 \\ X_3 & Y_3 \\ PT_{2,0} & PT_{2,1} \\ PT_{3,0} & PT_{3,1} \end{pmatrix}$$

$$xx(u) := U(u)\cdot MH\cdot G^{\langle 0\rangle} \qquad X3_j := xx(u_j)$$

$$yy(u) := U(u)\cdot MH\cdot G^{\langle 1\rangle} \qquad Y3_j := yy(u_j)$$





cubic spline interpolation :

$$M := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 7 & 16 \\ 5 & 12 & 6 \\ 5 & 12 & 6 \\ 5 & 12 & 6 \\ 8 & 8 & 10 \end{pmatrix}$$

natural cubic spline interpolation :

$$M := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 7 & 16 \\ 5 & 12 & 6 \\ 5 & 12 & 6 \\ 5 & 12 & 6 \\ 8 & 8 & 10 \end{pmatrix}$$

$$PP := \begin{bmatrix} 3\cdot(P_{1,0}-P_{0,0}) & 3\cdot(P_{1,1}-P_{0,1}) & 3\cdot(P_{1,2}-P_{0,2}) \\ 3\cdot(P_{2,0}-P_{0,0}) & 3\cdot(P_{2,1}-P_{0,1}) & 3\cdot(P_{2,2}-P_{0,2}) \\ 3\cdot(P_{3,0}-P_{1,0}) & 3\cdot(P_{3,1}-P_{1,1}) & 3\cdot(P_{3,2}-P_{1,2}) \\ 3\cdot(P_{3,0}-P_{2,0}) & 3\cdot(P_{3,1}-P_{2,1}) & 3\cdot(P_{3,2}-P_{2,2}) \end{bmatrix}$$

$$PT := M^{-1} \cdot PP \qquad PT = \begin{pmatrix} 1.867 & 5.933 & 22.333 \\ 2.267 & 6.133 & 0.333 \\ 1.067 & 2.533 & -8.667 \\ -0.533 & -1.267 & 4.333 \end{pmatrix} \qquad MH := \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$U(u) := \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}$$

$$p(u) := U(u) \cdot MH \cdot G$$

$$G := \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} \\ P_{1,0} & P_{1,1} & P_{1,2} \\ PT_{0,0} & PT_{0,1} & PT_{0,2} \\ PT_{1,0} & PT_{1,1} & PT_{1,2} \end{pmatrix}$$

$$x(u) := U(u) \cdot MH \cdot G^{\langle 0 \rangle} \qquad X_j := x(u_j)$$

$$y(u) := U(u) \cdot MH \cdot G^{\langle 1 \rangle} \qquad Y_j := y(u_j)$$

$$z(u) := U(u) \cdot MH \cdot G^{\langle 2 \rangle} \qquad Z_j := z(u_j)$$

$$G := \begin{pmatrix} P_{1,0} & P_{1,1} & P_{1,2} \\ P_{2,0} & P_{2,1} & P_{2,2} \\ PT_{1,0} & PT_{1,1} & PT_{1,2} \\ PT_{2,0} & PT_{2,1} & PT_{2,2} \end{pmatrix}$$

$$x(u) := U(u) \cdot MH \cdot G^{\langle 0 \rangle} \qquad X_{m+j} := x(u_j) \qquad XX := xxxx$$

$$y(u) := U(u) \cdot MH \cdot G^{\langle 1 \rangle} \qquad Y_{m+j} := y(u_j) \qquad YY := yyyy$$

$$z(u) := U(u) \cdot MH \cdot G^{\langle 2 \rangle} \qquad Z_{m+j} := z(u_j) \qquad ZZ := zzzz$$

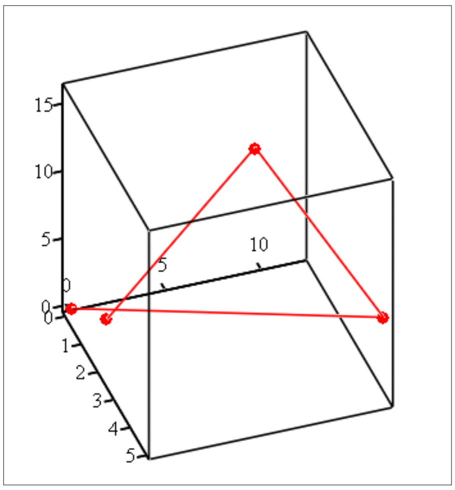
$$G := \begin{pmatrix} P_{2,0} & P_{2,1} & P_{2,2} \\ P_{3,0} & P_{3,1} & P_{3,2} \\ PT_{2,0} & PT_{2,1} & PT_{2,2} \\ PT_{3,0} & PT_{3,1} & PT_{3,2} \end{pmatrix}$$

$$x(u) := U(u) \cdot MH \cdot G^{\langle 0 \rangle} \qquad X_{2 \cdot m+j} := x(u_j) \qquad X_{3 \cdot m+i} := \left(P^{\langle 0 \rangle}\right)_{n-i}$$

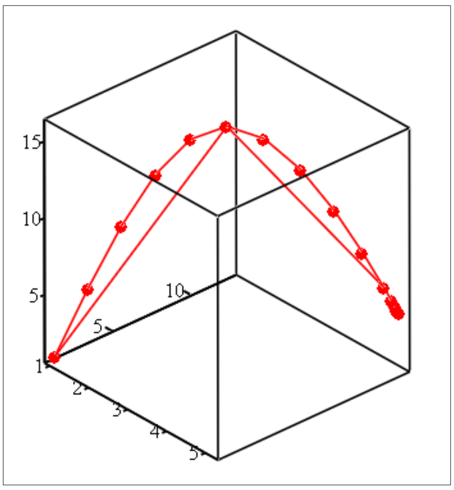
$$y(u) := U(u) \cdot MH \cdot G^{\langle 1 \rangle} \qquad Y_{2 \cdot m+j} := y(u_j) \qquad Y_{3 \cdot m+i} := \left(P^{\langle 1 \rangle}\right)_{n-i}$$

$$z(u) := U(u) \cdot MH \cdot G^{\langle 2 \rangle} \qquad Z_{2 \cdot m+j} := z(u_j) \qquad Z_{3 \cdot m+i} := \left(P^{\langle 2 \rangle}\right)_{n-i}$$

$$XX_{m+i} := \left(P^{\langle 0 \rangle}\right)_{n-i} \qquad YY_{m+i} := \left(P^{\langle 1 \rangle}\right)_{n-i} \qquad ZZ_{m+i} := \left(P^{\langle 2 \rangle}\right)_{n-i}$$



(XX, YY, ZZ)



(X, Y, Z)

B-Spline Curve

$$P(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m}\left[P_{ij}\cdot N_{i,k}(u)\cdot N_{j,l}(v)\right] \qquad \begin{array}{l} 0 \le u \le u_{max} \\ 0 \le v \le v_{max}\end{array}$$

$$N_{i,k}(u) = (u - u_i)\cdot\frac{N_{i,k-1}}{u_{i+k-1} - u_i} + (u_{i+k} - u)\cdot\frac{N_{i+1,k-1}}{u_{i+k} - u_{i+1}}$$

$$P(u) = \sum_{i=0}^{n}\left[P_i\cdot N_{i,k}(u)\right] \qquad 0 \le u \le u_{max}$$

$N_{i,1} = 1$ if $u_i \le u \le u_{i+1}$

$N_{i,1} = 0$ otherwise

Properties

Partition of unity: $\displaystyle\sum_{i=0}^{n}\left[N_{i,k}(u)\right] = 1$

Positivity: $N_{i,k}(u) \ge 0$

Local support: $N_{i,k}(u) = 0$ if $u \ne (u_i, u_{i+k+1})$

Continuity: $N_{i,k}(u)$ is (k - 2) times continuously differentiable

$u_j = 0$ if $j < k$

$u_j = j - k + 1$ if $k \le j \le n$

$u_j = n - k + 2$ if $j > n$

where

$0 \le j \le n + k$

and the range of u is

$0 \le u \le n - k + 2$

The computation of $N_{1,3}$

$$\begin{array}{cccc}
N_{0,0} & & & \\
& N_{0,1} & & \\
N_{1,0} & & N_{0,2} & \\
& N_{1,1} & & N_{0,3} \\
N_{2,0} & & N_{1,2} & \\
& N_{2,1} & & N_{1,3} \\
N_{3,0} & & N_{2,2} & \\
& N_{3,1} & \vdots & N_{2,3} \\
N_{4,0} & \vdots & & \vdots
\end{array}$$

The computation of $N_{0,3}$, $N_{1,3}$, $N_{2,3}$, $N_{3,3}$

$$\begin{array}{cccc}
N_{0,0} & & & \\
& N_{0,1} & & \\
N_{1,0} & & N_{0,2} & \\
& N_{1,1} & & N_{0,3} \\
N_{2,0} & & N_{1,2} & \\
& N_{2,1} & & N_{1,3} \\
N_{3,0} & & N_{2,2} & \\
& N_{3,1} & & N_{2,3} \\
N_{4,0} & & N_{3,2} & \\
& N_{4,1} & & N_{3,3}
\end{array}$$

Open B-Spline Curve

$p := 0 \quad X := 0 \quad Y := 0 \quad Z := 0 \quad u := 0 \quad N := 0 \quad r := 0..2 \quad P := 0 \quad t := 19$

$n := 5 \quad ks := 3 \quad k := ks \quad m := n - k + 2 \quad i := 0..n \quad j := 0..n + ks \quad kk := 2..k$

$u_j := \mathrm{if}(j < k, 0, \mathrm{if}(j \le n, j - k + 1, n - k + 2)) \qquad s := 0..t \qquad uu_s := \frac{s\cdot m}{t}$

$$p_0 := \begin{pmatrix}1\\1\\1\end{pmatrix} \quad p_1 := \begin{pmatrix}3\\7\\10\end{pmatrix} \quad p_2 := \begin{pmatrix}5\\12\\6\end{pmatrix} \quad p_3 := \begin{pmatrix}5\\12\\6\end{pmatrix}$$

$$p_4 := \begin{pmatrix}7\\12\\6\end{pmatrix} \quad p_5 := \begin{pmatrix}8\\8\\4\end{pmatrix} \quad P_{i,r} := \left[\left[\left(p_i\right)^T\right]^{\langle r\rangle}\right]_{0,0} \quad P = \begin{pmatrix}1&1&1\\3&7&10\\5&12&6\\5&12&6\\7&12&6\\8&8&4\end{pmatrix}$$

$u_j =$

0
0
0
1
2
3
4
4
4

$uu_s =$

0
0.211
0.421
0.632
0.842
1.053
1.263
1.474
1.684
1.895
2.105
2.316
2.526
2.737
2.947
...

$s =$

0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
...

$$Ns(i,k,v) := \mathrm{if}\left[k = 1, \mathrm{if}\left(u_i \le v \le u_{i+1}, 1, 0\right), \left(v - u_i\right)\cdot\frac{Ns(i,k-1,v)\cdot\left(u_{i+k-1} - u_i\right)}{\left(u_{i+k-1} - u_i\right)^2} \ldots + \left(u_{i+k} - v\right)\cdot\frac{Ns(i+1,k-1,v)\cdot\left(u_{i+k} - u_{i+1}\right)}{\left(u_{i+k} - u_{i+1}\right)^2}\right]$$

CALCULATION OF CURVE POINTS

$$X := 0 \qquad x(u) := \sum_{i=0}^{n}\left[\left(P^{\langle 0\rangle}\right)_i\cdot Ns(i,k,u)\right] \qquad X_s := x\left(uu_s\right)$$

$$Y := 0 \qquad y(u) := \sum_{i=0}^{n}\left[\left(P^{\langle 1\rangle}\right)_i\cdot Ns(i,k,u)\right] \qquad Y_s := y\left(uu_s\right)$$

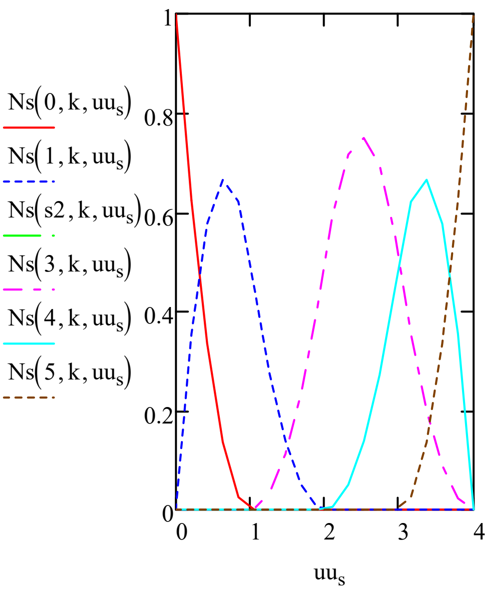
$$Z := 0 \qquad z(u) := \sum_{i=0}^{n}\left[\left(P^{\langle 2\rangle}\right)_i\cdot Ns(i,k,u)\right] \qquad Z_s := z\left(uu_s\right)$$

$\sum_{i=0}^{5} Ns\left(i,k,uu_s\right) =$

1
1
1
1
1
1
1
...

$\sum_{i} Ns\left(i,k,uu_s\right) =$

1
1
1
1
1
1
1
...



$Ns\left(0,k,uu_s\right) =$

1
0.623
0.335
0.136
0.025
0
0
0
0
0
0
0
0
0
0
...

$j =$

0
1
2
3
4
5
6
7
8

$u_j =$

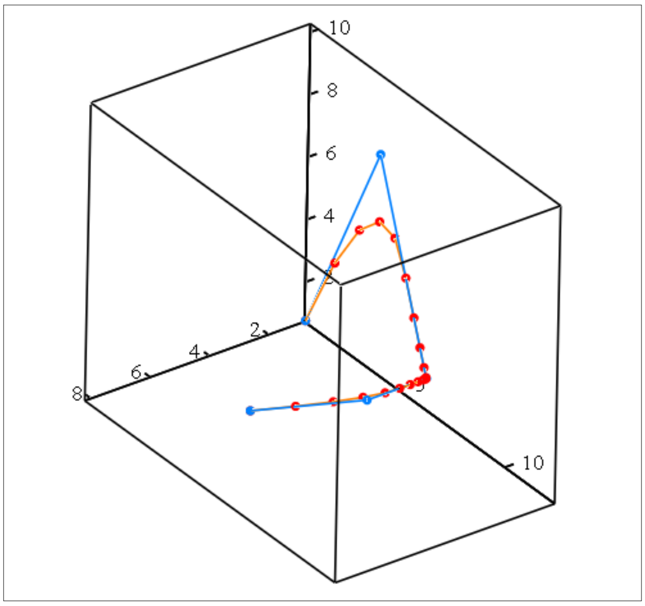
0
0
0
1
2
3
4
4
4

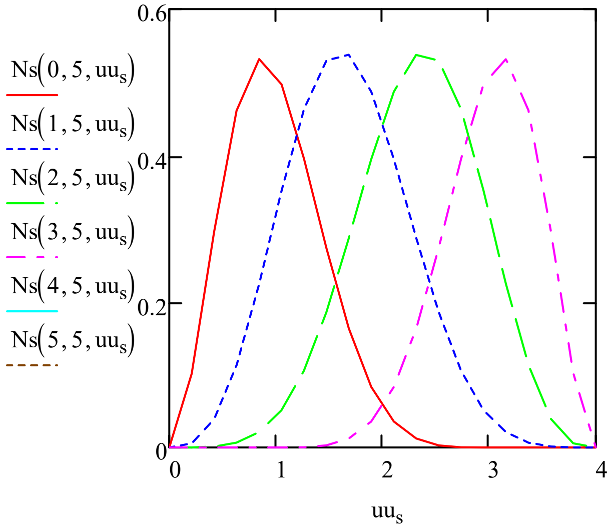
$uu_s =$

0
0.211
0.421
0.632
0.842
1.053
1.263
1.474
1.684
1.895
2.105
2.316
2.526
2.737
2.947
...

$s =$

0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
...

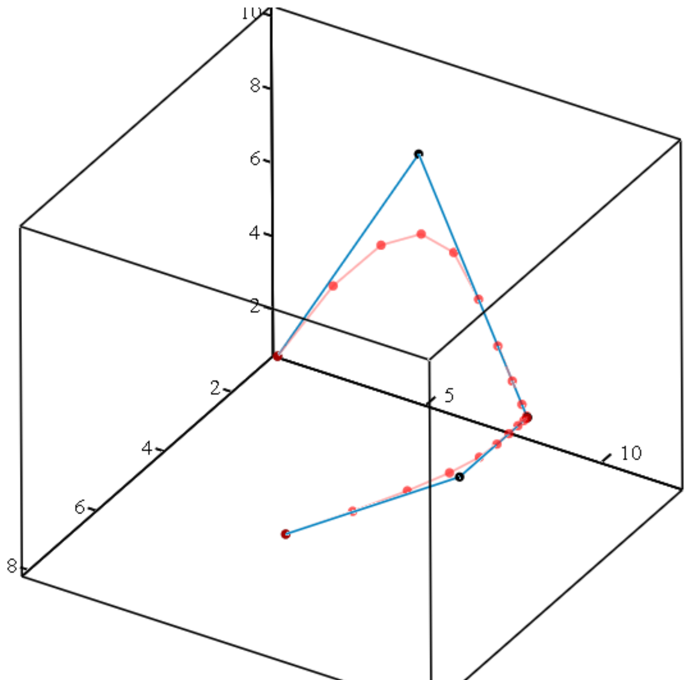




$(X,Y,Z),\left(P^{\langle 0\rangle},P^{\langle 1\rangle},P^{\langle 2\rangle}\right)$

CALCULATION OF CURVE POINTS

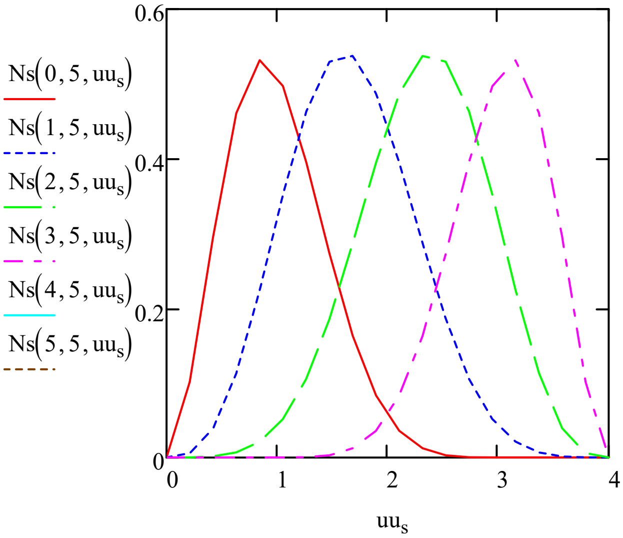
$N_{i,s} := Ns\left(i,k,uu_s\right)$ $\qquad x_s := \sum_{i=0}^{n}\left[\left(p_i\right)_0 \cdot N_{i,s}\right]$ $\qquad y_s := \sum_{i=0}^{n}\left[\left(p_i\right)_1 \cdot N_{i,s}\right]$ $\qquad z_s := \sum_{i=0}^{n}\left[\left(p_i\right)_2 \cdot N_{i,s}\right]$



B-Spline Surface

$(x,y,z),\left(P^{\langle 0\rangle},P^{\langle 1\rangle},P^{\langle 2\rangle}\right)$

$Ns(i,k,v) := \text{if}\left[k = 1, \text{if}\left(u_i \le v \le u_{i+1}, 1, 0\right), \text{if}\left[\left(u_{i+k-1} - u_i\right) = 0 \wedge \left(u_{i+k} - u_{i+1}\right) = 0, 0, \text{if}\left[\left(u_{i+k-1} - u_i\right) = 0, \left(u_{i+k} - v\right) \cdot \right.\right.\right.$



$$U := \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix}^T$$

$$V := \begin{pmatrix} P_0 \\ P_1 \\ P'_0 \\ P'_1 \end{pmatrix} \quad P_B := \begin{pmatrix} P_3 \\ P_2 \\ P_1 \\ P_0 \end{pmatrix} \quad P_S := \begin{pmatrix} V_{i-1} \\ V_i \\ V_{i+1} \\ V_{i+2} \end{pmatrix}$$

CONVERSION BETWEEN REPRESENTATIONS :

Hermite

$$M_H := \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Bezier

$$M_B := \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

B-Spline

$$M_S := \frac{1}{6} \cdot \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}$$

$$p(u) := U \cdot M_H \cdot V \qquad p(u) := U \cdot M_B \cdot P_B \qquad P_i(t) := U \cdot M_S \cdot P_S$$

$$HtoB := \frac{1}{3} \cdot \begin{pmatrix} -3 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 3 & 0 & 0 \end{pmatrix} \quad BtoH := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix} \quad BStoH := \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -3 & 0 & 3 & 0 \\ 0 & -3 & 0 & 3 \end{pmatrix}$$

$$HtoBS := \frac{1}{3} \cdot \begin{pmatrix} -3 & 6 & -7 & -2 \\ 6 & -3 & 2 & 1 \\ -3 & 6 & -1 & -2 \\ 6 & -3 & 2 & 7 \end{pmatrix} \quad BtoBS := \begin{pmatrix} 6 & -7 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -7 & 6 \end{pmatrix} \quad BStoB := \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix}$$

Cubic Bezier representation

Conversion to B-Spline representation

$$P := \begin{pmatrix} -6 & 0 & 0 & 1 \\ -3 & 4 & 0 & 1 \\ 3 & -4 & 0 & 1 \\ 6 & 0 & 0 & 1 \end{pmatrix} \qquad V_S := BtoBS \cdot P \qquad V_S = \begin{pmatrix} -9 & -36 & 0 & 1 \\ -9 & 12 & 0 & 1 \\ 9 & -12 & 0 & 1 \\ 9 & 36 & 0 & 1 \end{pmatrix}$$

$$u := uu$$

$$u := u$$

$$p := \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \end{pmatrix}^T$$

$$p(u) := p_0 \cdot (1-u)^3 + p_1 \cdot 3 \cdot u \cdot (1-u)^2 + p_2 \cdot 3 \cdot u^2 \cdot (1-u) + p_3 \cdot u^3$$

$$p'(u) := \frac{d}{du} p(u) \qquad p'(0) \rightarrow 3 \cdot p_1 - 3 \cdot p_0$$

$$p'(1) \rightarrow 3 \cdot p_3 - 3 \cdot p_2$$

$$M_H^{-1}\cdot M_B \qquad \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix}$$

Hermite <--- Bezier

$$V = M_H^{-1}\cdot M_B\cdot P_B \qquad V := \begin{pmatrix} P_0 \\ P_1 \\ P'_0 \\ P'_1 \end{pmatrix} \qquad V := \begin{bmatrix} P_0 \\ P_1 \\ 3\cdot(P_1 - P_0) \\ 3\cdot(P_3 - P_2) \end{bmatrix} \qquad M_H^{-1}\cdot M_B = \overset{\text{BtoH}}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix}} \qquad P := \begin{pmatrix} P_3 \\ P_2 \\ P_1 \\ P_0 \end{pmatrix}$$

$$BtoH := M_H^{-1}\cdot M_B$$

$$M_B^{-1}\cdot M_H \qquad \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 3 & 0 & 0 \end{pmatrix}\cdot\frac{1}{3}$$

Bezier <--- Hermite

$$P = M_B^{-1}\cdot M_H\cdot V \qquad P := \begin{pmatrix} P_3 \\ P_2 \\ P_1 \\ P_0 \end{pmatrix} \qquad P = \begin{pmatrix} P_0 \\ P_0 + \frac{1}{3}\cdot P'_0 \\ P_1 - \frac{1}{3}\cdot P'_1 \\ P_1 \end{pmatrix} \qquad M_B^{-1}\cdot M_H = \overset{\text{HtoB}}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 \end{pmatrix}} \qquad V := \begin{pmatrix} P_0 \\ P_1 \\ P'_0 \\ P'_1 \end{pmatrix}$$

$$HtoB := M_B^{-1}\cdot M_H$$

$$M_B^{-1}\cdot M_S \qquad \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1} \cdot\frac{1}{6}\cdot \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix}\cdot\frac{1}{6}$$

$$M_S^{-1}\cdot M_B \qquad \frac{6}{1}\cdot \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -7 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -7 & 6 \end{pmatrix}$$

$$p(u) := U\cdot B\cdot P$$

$$U\cdot M_B\cdot P_B = U\cdot M_S\cdot P_S$$

$$M_B\cdot P_B = M_S\cdot P_S$$

$$P_B := M_B^{-1}\cdot M_S\cdot P_S \qquad P_B = \frac{1}{6}\cdot \overset{\text{BStoB}}{\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix}}\cdot P_S \qquad P_S := M_S^{-1}\cdot M_B\cdot P_B \qquad P_S = \overset{\text{BtoBS}}{\begin{pmatrix} 6 & -7 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -7 & 6 \end{pmatrix}}\cdot P_B$$

$$M_H^{-1}\cdot M_S \qquad \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1} \cdot \frac{1}{6} \cdot \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -3 & 0 & 3 & 0 \\ 0 & -3 & 0 & 3 \end{pmatrix} \cdot \frac{1}{6}$$

$$M_S^{-1}\cdot M_H \qquad \frac{6}{1} \cdot \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 6 & -7 & -2 \\ 6 & -3 & 2 & 1 \\ -3 & 6 & -1 & -2 \\ 6 & -3 & 2 & 7 \end{pmatrix} \cdot \frac{1}{3}$$

BStoH

$$V = M_H^{-1}\cdot M_S \cdot P_S \qquad V = \frac{1}{6} \cdot \begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -3 & 0 & 3 & 0 \\ 0 & -3 & 0 & 3 \end{pmatrix} \cdot P_S$$

HtoBS

$$P_S := M_S^{-1}\cdot M_H \cdot V \qquad P_S = \frac{1}{3} \cdot \begin{pmatrix} -3 & 6 & -7 & -2 \\ 6 & -3 & 2 & 1 \\ -3 & 6 & -1 & -2 \\ 6 & -3 & 2 & 7 \end{pmatrix} \cdot V$$

Power Basis Form of a Curve

$$k := 0 .. 10 \qquad u_k := \frac{k}{10}$$

$$X := 0 \qquad Y := 0 \qquad Z := 0$$

$$C(u) := \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} \qquad \begin{pmatrix} a_{0,1} & a_{1,1} & a_{2,1} & a_{3,1} \\ a_{0,2} & a_{1,2} & a_{2,2} & a_{3,2} \\ a_{0,3} & a_{1,3} & a_{2,3} & a_{3,3} \end{pmatrix} := \begin{pmatrix} 1 & 3 & 5 & 8 \\ 1 & 6 & 8 & 8 \\ 1 & 12 & 9 & 10 \end{pmatrix} \qquad a = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 3 & 6 & 12 \\ 0 & 5 & 8 & 9 \\ 0 & 8 & 8 & 10 \end{pmatrix}$$

$$n := \mathrm{cols}(a) - 1$$

$$n = 3 \qquad i := 0 .. n$$

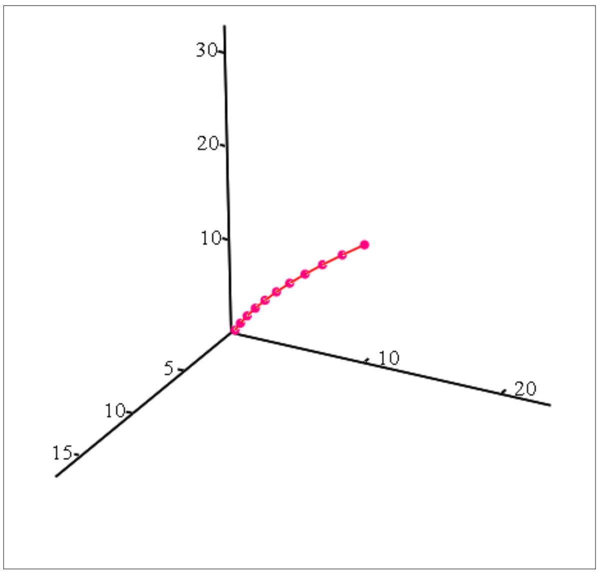
$$m := \mathrm{rows}(a) - 1$$

$$j := 0 .. m$$

$$x(u) := a_{0,1} + a_{1,1}\cdot u + a_{2,1}\cdot u^2 + a_{3,1}\cdot u^3 \qquad X_k := x(u_k) \qquad x := a^{\langle 1 \rangle}$$

$$y(u) := a_{0,2} + a_{1,2}\cdot u + a_{2,2}\cdot u^2 + a_{3,2}\cdot u^3 \qquad Y_k := y(u_k) \qquad y := a^{\langle 2 \rangle}$$

$$z(u) := a_{0,3} + a_{1,3}\cdot u + a_{2,3}\cdot u^2 + a_{3,3}\cdot u^3 \qquad Z_k := z(u_k) \qquad z := a^{\langle 3 \rangle}$$



(X, Y, Z)

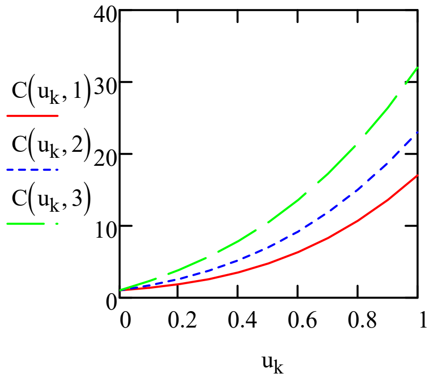
$$C(u) := \sum_{i=0}^{n} \left(a_i \cdot u^i\right)$$

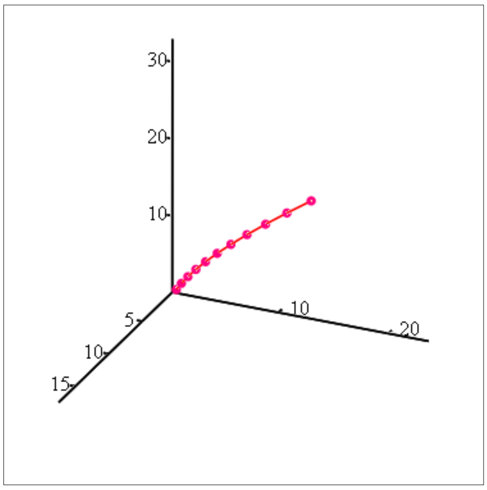
$$x(u) := \sum_{i=0}^{n} \left(x_i \cdot u^i\right) \qquad X_k := x(u_k)$$

$$y(u) := \sum_{i=0}^{n} \left(y_i \cdot u^i\right) \qquad Y_k := y(u_k)$$

$$z(u) := \sum_{i=0}^{n} \left(z_i \cdot u^i\right) \qquad Z_k := z(u_k)$$

$$C(u,m) := a^{\langle m \rangle} \cdot \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix}$$





(X, Y, Z)

Horner's Method

for degree = 1: $C(u) := a_{1,m} \cdot u + a_{0,m}$

for degree = 2: $C(u) := \left(a_{2,m} \cdot u + a_{1,m}\right) \cdot u + a_{0,m}$

for degree = 3: $C(u) := \left[\left(a_{3,m} \cdot u + a_{2,m}\right) \cdot u + a_{1,m}\right] \cdot u + a_{0,m}$

for degree = n: $C(u) := \left[\left(a_{n,m} \cdot u + a_{n-1,m}\right) \cdot u + a_{n-2,m}\right] \cdot u + \blacksquare + a_{0,m}$

Horner1(a, n, u0, C)

```
C := | a ← a
     | m ← 3
     | n ← 3
     | u0 ← 0.7
     | C ← a[n,m]
     | for i ∈ n − 1 .. 0
     |   C ← C·u0 + a[n,m]
     | C ← C
```

$C = 25.33$

Inverse - Point Solution

$u := uu \quad u := u$

$a := 2 \quad b := 4 \quad c := 7 \quad d := 5$

$p(u) := a \cdot u^3 + b \cdot u^2 + c \cdot u + d \qquad u := 0.3$

$$\begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{01} & a_{02} & a_{03} \end{pmatrix} := \begin{pmatrix} 2 & 4 & 1 \\ 4 & 8 & 5 \\ 8 & 3 & 2 \\ 12 & 2 & 4 \end{pmatrix}$$

$x := a_{31} \cdot u^3 + a_{21} \cdot u^2 + a_{11} \cdot u + a_{01}$

$y := a_{32} \cdot u^3 + a_{22} \cdot u^2 + a_{12} \cdot u + a_{02}$

$z := a_{33} \cdot u^3 + a_{23} \cdot u^2 + a_{13} \cdot u + a_{03}$

$$p(u) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{01} & a_{02} & a_{03} \end{pmatrix}$$

$x = 14.814 \quad y = 3.728 \quad z = 5.077 \quad u := uu$

$u := u$

$x := 14.814 \quad y := 3.728 \quad z := 5.077$

Given

$x = a_{31} \cdot u^3 + a_{21} \cdot u^2 + a_{11} \cdot u + a_{01}$

$y = a_{32} \cdot u^3 + a_{22} \cdot u^2 + a_{12} \cdot u + a_{02}$

$z = a_{33} \cdot u^3 + a_{23} \cdot u^2 + a_{13} \cdot u + a_{03}$

$u := \text{Find}(u) \rightarrow 0.3 \qquad u = 0.3$

$u := 0.3 \quad n := 10 \quad i := 0 .. n \quad u := \text{uuu} \quad u := u$

$p_0 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad p_1 := \begin{pmatrix} 3 \\ 6 \\ 12 \end{pmatrix} \quad p_2 := \begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix} \quad p_3 := \begin{pmatrix} 8 \\ 8 \\ 10 \end{pmatrix} \quad P := \text{stack}\left(p_0^T, p_1^T, p_2^T, p_3^T\right) \quad P = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 6 & 12 \\ 5 & 8 & 9 \\ 8 & 8 & 10 \end{pmatrix}$

$p(u) := p_0 \cdot (1-u)^3 + p_1 \cdot 3 \cdot u \cdot (1-u)^2 + p_2 \cdot 3 \cdot u^2 \cdot (1-u) + p_3 \cdot u^3 \qquad u := \text{uuu} \quad u := u$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} := p(u) \text{ simplify } \rightarrow \begin{pmatrix} u^3 + 6 \cdot u + 1 \\ u^3 - 9 \cdot u^2 + 15 \cdot u + 1 \\ 18 \cdot u^3 - 42 \cdot u^2 + 33 \cdot u + 1 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \blacksquare \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} 2.827 \\ 4.717 \\ 7.606 \end{pmatrix}$

Given

$x = u^3 + 6 \cdot u + 1$

$y = u^3 - 9 \cdot u^2 + 15 \cdot u + 1$

$z = 18 \cdot u^3 - 42 \cdot u^2 + 33 \cdot u + 1$

$u := \text{Find}(u) \rightarrow 0.3 \qquad u = 0.3$

$\begin{pmatrix} 2.827 \\ 4.717 \\ 7.606 \end{pmatrix} := \begin{pmatrix} 1 & 0 & 6 & 1 \\ 1 & -9 & 15 & 1 \\ 18 & -42 & 33 & 1 \end{pmatrix} \cdot \begin{pmatrix} u^3 \\ u^2 \\ u \\ 1 \end{pmatrix} \qquad \begin{pmatrix} u3 \\ u2 \\ u \\ 1 \end{pmatrix} := \begin{pmatrix} 1 & 0 & 6 & 1 \\ 1 & -9 & 15 & 1 \\ 18 & -42 & 33 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2.827 \\ 4.717 \\ 7.606 \end{pmatrix}$

$u := \text{uuu} \quad u := u \qquad x = 2.827$

$x = u^3 + 6 \cdot u + 1$

$p(u) := u^3 + 6 \cdot u + 1 - x$

$v := p(u) \text{ coeffs}, u \rightarrow \begin{pmatrix} -1.827 \\ 6 \\ 0 \\ 1 \end{pmatrix} \qquad rx := \text{polyroots}(v) \qquad rx = \begin{pmatrix} -0.15 - 2.463i \\ -0.15 + 2.463i \\ 0.3 \end{pmatrix}$

$p(u) := u^3 - 9 \cdot u^2 + 15 \cdot u + 1 - y$

$$v := p(u)\ \text{coeffs}, u \rightarrow \begin{pmatrix} -3.717 \\ 15 \\ -9 \\ 1 \end{pmatrix} \qquad ry := \text{polyroots}(v) \qquad ry = \begin{pmatrix} 0.3 \\ 1.794 \\ 6.906 \end{pmatrix}$$

$$p(u) := 18 \cdot u^3 - 42 \cdot u^2 + 33 \cdot u + 1 - z$$

$$v := p(u)\ \text{coeffs}, u \rightarrow \begin{pmatrix} -6.606 \\ 33 \\ -42 \\ 18 \end{pmatrix} \qquad rz := \text{polyroots}(v) \qquad rz = \begin{pmatrix} 0.3 \\ 1.017 - 0.436i \\ 1.017 + 0.436i \end{pmatrix}$$

```
u := | u ← 0
     | j ← 0
     | while u = 0 ∧ j < 2
     |   | i ← 0
     |   | while u = 0 ∧ i < 2
     |   |   | u ← if(rx_j = ry_i, ry_i, 0)
     |   |   | k ← 0
     |   |   | v ← 0
     |   |   | while (u ≠ 0) ∧ (v = 0) ∧ (k < 2)
     |   |   |   | v ← if(rx_j = rz_k, rz_k, 0)
     |   |   |   | k ← k + 1
     |   |   | i ← i + 1
     |   |   | u ← v
     |   | j ← j + 1
     | u
```

Inverse-Point Solution :

$u := uu \qquad u := u \qquad p := (p_0\ \ p_1\ \ p_2\ \ p_3)^T$

$$p(u) := p_0 \cdot (1-u)^3 + p_1 \cdot 3 \cdot u \cdot (1-u)^2 + p_2 \cdot 3 \cdot u^2 \cdot (1-u) + p_3 \cdot u^3 \qquad p'(u) := \frac{d}{du} p(u)$$

Control Points :

$$n := 10 \quad i := 0..n \quad u_i := \frac{i}{n} \quad p_0 := \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix} \quad p_1 := \begin{pmatrix} 7 \\ 6 \\ 4 \end{pmatrix} \quad p_2 := \begin{pmatrix} 2 \\ 4 \\ 19 \end{pmatrix} \quad p_3 := \begin{pmatrix} 8 \\ 8 \\ 10 \end{pmatrix} \quad q := \begin{pmatrix} 2 \\ 4 \\ 19 \end{pmatrix}$$

$$P := \text{stack}\left(p_0^T, p_1^T, p_2^T, p_3^T\right) \qquad Px := P^{\langle 0 \rangle} \qquad Py := P^{\langle 1 \rangle} \qquad Pz := P^{\langle 2 \rangle}$$

$u := uu \qquad u := u$

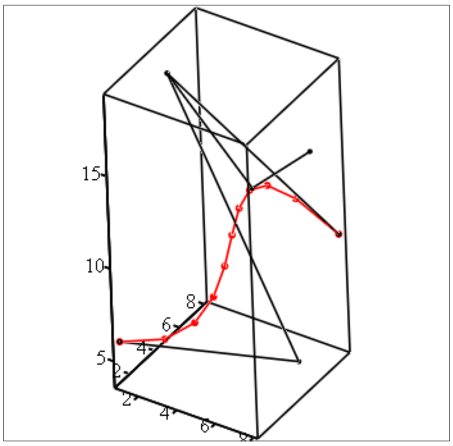
$$P(u) := p_0 \cdot (1-u)^3 + p_1 \cdot 3 \cdot u \cdot (1-u)^2 + p_2 \cdot 3 \cdot u^2 \cdot (1-u) + p_3 \cdot u^3$$

$$P(u) \rightarrow \begin{bmatrix} 8 \cdot u^3 + 21 \cdot u \cdot (u-1)^2 - 6 \cdot u^2 \cdot (u-1) - (u-1)^3 \\ 8 \cdot u^3 + 18 \cdot u \cdot (u-1)^2 - 12 \cdot u^2 \cdot (u-1) - (u-1)^3 \\ 10 \cdot u^3 + 12 \cdot u \cdot (u-1)^2 - 57 \cdot u^2 \cdot (u-1) - 6 \cdot (u-1)^3 \end{bmatrix}$$

$$P'(u) := -3 \cdot p_0 \cdot (1-u)^2 + 3 \cdot p_1 \cdot (1-u)^2 - 6 \cdot p_1 \cdot u \cdot (1-u) + 6 \cdot p_2 \cdot u \cdot (1-u) - 3 \cdot p_2 \cdot u^2 + 3 \cdot p_3 \cdot u^2$$

$u_i := \frac{i}{n}$ $x_i := P(u_i)_0$ $y_i := P(u_i)_1$ $z_i := P(u_i)_2$ $u := 0.71932$

$PP'(u) := \text{stack}\left[q^T, P(u)^T, \left(P(u) + 3\cdot\frac{P'(u)}{|P'(u)|}\right)^T\right]$ $X := PP'(u)^{\langle 0\rangle}$ $Y := PP'(u)^{\langle 1\rangle}$ $Z := PP'(u)^{\langle 2\rangle}$



$(x,y,z),(X,Y,Z),(Px,Py,Pz)$

$P(u) = \begin{pmatrix} 5.061 \\ 5.762 \\ 12.813 \end{pmatrix}$ $q = \begin{pmatrix} 2 \\ 4 \\ 19 \end{pmatrix}$

$(P(u) - q)\cdot P'(u) = 0.000214$

$u := uuu$ $u := u$ $u = \blacksquare$ erase u parameter for symbolic calculation.

$(P(u) - q)\cdot P'(u) = 0$

$u := (P(u) - q)\cdot P'(u) = 0 \text{ solve}, u \rightarrow \text{Union}\left[\text{Union}\left[\text{ImageSet}\left[V4\cdot i + \_x, \_x, \text{intersect}\left[\mathbb{R}, \text{RootOf}\left[\_z1^5 + \_z1^4\cdot\left(V4\cdot i - \frac{257}{116}\right.\right.\right.\right.\right.\right.$