DISCRETE-TIME SIGNALS AND SYSTEMS

2.0 INTRODUCTION

The term *signal* is generally applied to something that conveys information. Signals generally convey information about the state or behavior of a physical system, and often, signals are synthesized for the purpose of communicating information between humans or between humans and machines.
either continuous or discrete. *Continuous-time signals* are defined along a continuum of times and thus are represented by a continuous independent variable. Continuous-time signals are often referred to as *analog signals*. *Discrete-time signals* are defined at discrete times, and thus, the independent variable has discrete values; i.e., discrete-time signals are represented as sequences of numbers. Signals such as speech or images may have either a continuous- or a discrete-variable representation, and if certain conditions hold, these representations are entirely equivalent. Besides the independent variables being either continuous or discrete, the signal amplitude may be either continuous or discrete. *Digital signals* are those for which both time and amplitude are discrete.

Discrete-time signals may arise by sampling a continuous-time signal, or they may be generated directly by some discrete-time process. Whatever the origin of the discrete-time signals, discrete-time signal-processing systems have many attractive features. They can be realized with great flexibility with a variety of technologies, such as charge transport devices, surface acoustic wave devices, general-purpose digital computers, or high-speed microprocessors. Complete signal-processing systems can be implemented using VLSI techniques. Discrete-time systems can be used to simulate analog systems or, more importantly, to realize signal transformations that cannot be implemented with continuous-time hardware. Thus, discrete-time representations of signals are often desirable when sophisticated and flexible signal processing is required.

2.1 DISCRETE-TIME SIGNALS: SEQUENCES

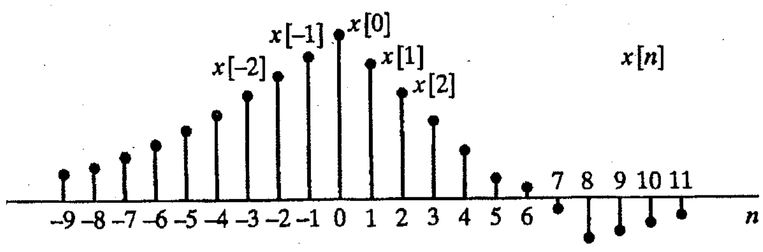
Discrete-time signals are represented mathematically as sequences of numbers. A sequence of numbers x, in which the nth number in the sequence is denoted $x[n]$,[1] is formally written as

$$x = \{x[n]\}, \qquad -\infty < n < \infty, \tag{2.1}$$

where n is an integer. In a practical setting, such sequences can often arise from periodic sampling of an analog signal. In this case, the numeric value of the nth number in the sequence is equal to the value of the analog signal, $x_a(t)$, at time nT; i.e.,

$$x[n] = x_a(nT), \qquad -\infty < n < \infty. \tag{2.2}$$

The quantity T is called the *sampling period,* and its reciprocal is the *sampling frequency.*



2.1.1 Basic Sequences and Sequence Operations

In the analysis of discrete-time signal-processing systems, sequences are manipulated in several basic ways. The product and sum of two sequences $x[n]$ and $y[n]$ are defined as the sample-by-sample product and sum, respectively. Multiplication of a sequence $x[n]$ by a number α is defined as multiplication of each sample value by α. A sequence $y[n]$ is said to be a delayed or shifted version of a sequence $x[n]$ if

$$y[n] = x[n - n_0], \tag{2.3}$$

where n_0 is an integer.

The *unit sample sequence* (Figure 2.3a) is defined as the sequence

$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases} \tag{2.4}$$

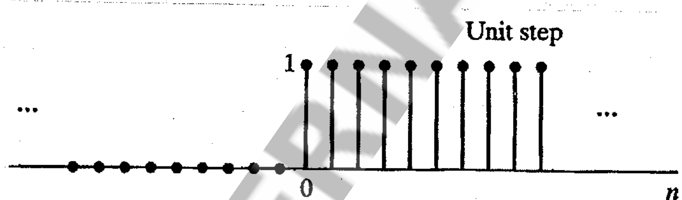
As we will see, the unit sample sequence plays the same role for discrete-time signals and systems that the unit impulse function (Dirac delta function) does for continuous-time signals and systems.

More generally, any sequence can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \tag{2.6}$$

The *unit step sequence* (Figure 2.3b) is given by

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{2.7}$$



The unit step is related to the impulse by

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]; \tag{2.8}$$

or

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{2.9b}$$

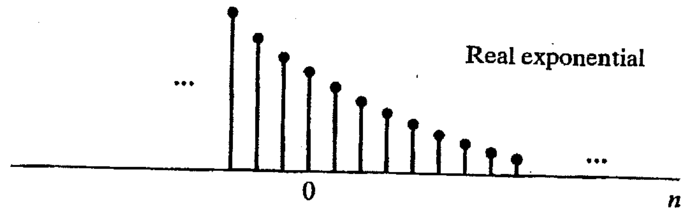
Conversely, the impulse sequence can be expressed as the first backward difference of the unit step sequence, i.e.,

$$\delta[n] = u[n] - u[n-1]. \tag{2.10}$$

Exponential sequences are extremely important in representing and analyzing linear time-invariant discrete-time systems. The general form of an exponential sequence is

$$x[n] = A\alpha^n. \tag{2.11}$$

If A and α are real numbers, then the sequence is real. If $0 < \alpha < 1$ and A is positive, then the sequence values are positive and decrease with increasing n, as in Figure 2.3(c).

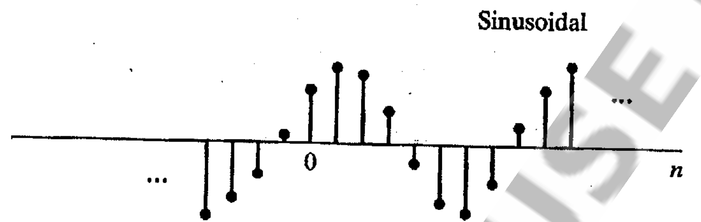


For $-1 < \alpha < 0$, the sequence values alternate in sign, but again decrease in magnitude with increasing n. If $|\alpha| > 1$, then the sequence grows in magnitude as n increases.

Sinusoidal sequences are also very important. A sinusoidal sequence has the general form

$$x[n] = A\cos(\omega_0 n + \phi), \qquad \text{for all } n, \tag{2.13}$$

with A and ϕ real constants, and is illustrated in Figure 2.3(d).



Specifically, if $\alpha = |\alpha|e^{j\omega_0}$ and $A = |A|e^{j\phi}$, the sequence $A\alpha^n$ can be expressed in any of the following ways:

$$\begin{aligned} x[n] = A\alpha^n &= |A|e^{j\phi}|\alpha|^n e^{j\omega_0 n} \\ &= |A|\,|\alpha|^n e^{j(\omega_0 n+\phi)} \\ &= |A|\,|\alpha|^n \cos(\omega_0 n + \phi) + j|A|\,|\alpha|^n \sin(\omega_0 n + \phi). \end{aligned} \tag{2.14}$$

The sequence oscillates with an exponentially growing envelope if $|\alpha| > 1$ or with an exponentially decaying envelope if $|\alpha| < 1$. (As a simple example, consider the case $\omega_0 = \pi$.)

When $|\alpha| = 1$, the sequence is referred to as a *complex exponential sequence* and has the form

$$x[n] = |A|e^{j(\omega_0 n+\phi)} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi); \tag{2.15}$$

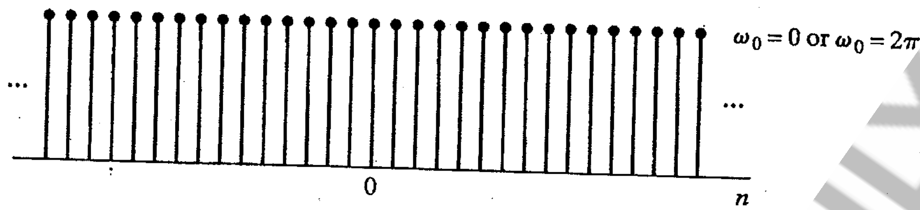
that is, the real and imaginary parts of $e^{j\omega_0 n}$ vary sinusoidally with n.

An important difference between continuous-time and discrete-time complex sinusoids is seen when we consider a frequency $(\omega_0 + 2\pi)$. In this case,

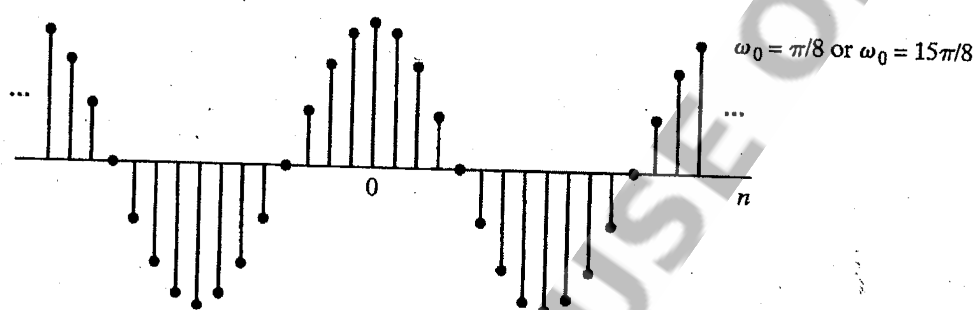
$$\begin{aligned} x[n] &= Ae^{j(\omega_0+2\pi)n} \\ &= Ae^{j\omega_0 n}e^{j2\pi n} = Ae^{j\omega_0 n}. \end{aligned} \tag{2.16}$$

More generally, we can easily see that complex exponential sequences with frequencies $(\omega_0 + 2\pi r)$, where r is an integer, are indistinguishable from one another.

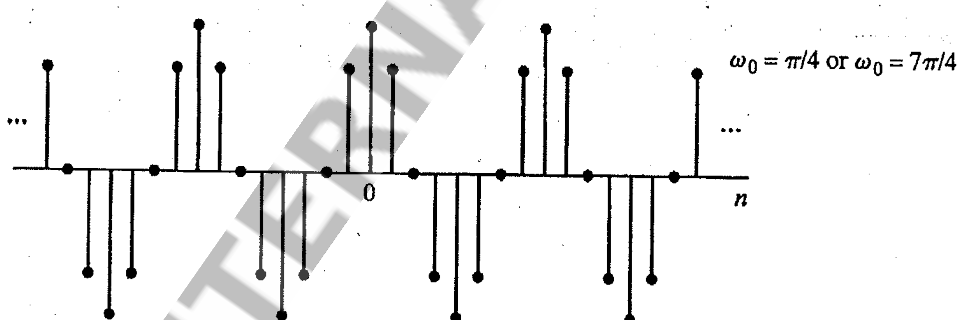
For now, we simply conclude that, when discussing complex exponential signals of the form $x[n] = Ae^{j\omega_0 n}$ or real sinusoidal signals of the form $x[n] = A\cos(\omega_0 n + \phi)$, we need only consider frequencies in an interval of length 2π, such as $-\pi < \omega_0 \le \pi$ or $0 \le \omega_0 < 2\pi$.


$\omega_0 = 0$ or $\omega_0 = 2\pi$
...
0
n

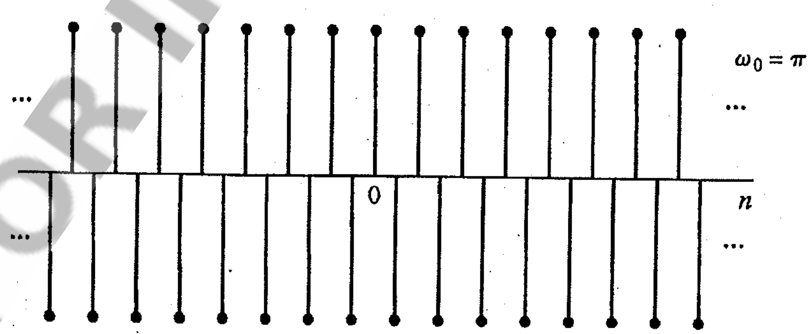
(a)


$\omega_0 = \pi/8$ or $\omega_0 = 15\pi/8$
...
0
n

(b)


$\omega_0 = \pi/4$ or $\omega_0 = 7\pi/4$
...
0
n

(c)


$\omega_0 = \pi$
...
0
n

(d)

In the discrete-time case, a periodic sequence is a sequence for which

$$x[n] = x[n+N], \qquad \text{for all } n, \tag{2.18}$$

where the period N is necessarily an integer. If this condition for periodicity is tested for the discrete-time sinusoid, then

$$A\cos(\omega_0 n + \phi) = A\cos(\omega_0 n + \omega_0 N + \phi), \tag{2.19}$$

which requires that

$$\omega_0 N = 2\pi k, \tag{2.20}$$

where k is an integer.

The integer restriction on n causes some sinusoidal signals not to be periodic at all. For example, there is no integer N such that the signal $x_3[n] = \cos(n)$ satisfies the condition $x_3[n+N] = x_3[n]$ for all n. These and other properties of discrete-time sinusoids that run counter to their continuous-time counterparts are caused by the limitation of the time index n to integers for discrete-time signals and systems.

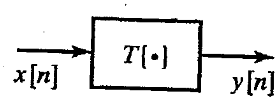
When we combine the condition of Eq. (2.20) with our previous observation that ω_0 and $(\omega_0 + 2\pi r)$ are indistinguishable frequencies, it becomes clear that there are N distinguishable frequencies for which the corresponding sequences are periodic with period N. One set of frequencies is $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. These properties of complex exponential and sinusoidal sequences are basic to both the theory and the design of computational algorithms for discrete-time Fourier analysis.

For the discrete-time sinusoidal signal $x[n] = A\cos(\omega_0 n + \phi)$, as ω_0 increases from $\omega_0 = 0$ toward $\omega_0 = \pi$, $x[n]$ oscillates more and more rapidly. However, as ω_0 increases from $\omega_0 = \pi$ to $\omega_0 = 2\pi$, the oscillations become slower. As a consequence, for sinusoidal and complex exponential signals, values of ω_0 in the vicinity of $\omega_0 = 2\pi k$ for any integer value of k are typically referred to as low frequencies (relatively slow oscillations), while values of ω_0 in the vicinity of $\omega_0 = (\pi + 2\pi k)$ for any integer value of k are typically referred to as high frequencies (relatively rapid oscillations).

2.2 DISCRETE-TIME SYSTEMS

A discrete-time system is defined mathematically as a transformation or operator that maps an input sequence with values $x[n]$ into an output sequence with values $y[n]$. This can be denoted as

$$y[n] = T\{x[n]\} \tag{2.22}$$



The ideal delay system is defined by the equation

$$y[n] = x[n - n_d], \qquad -\infty < n < \infty, \tag{2.23}$$

where n_d is a fixed positive integer called the delay of the system.

The general moving-average system is defined by the equation

$$\begin{aligned} y[n] &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k] \\ &= \frac{1}{M_1 + M_2 + 1} \{x[n+M_1] + x[n+M_1-1] + \cdots + x[n] \\ &\quad + x[n-1] + \cdots + x[n-M_2]\} \end{aligned} \tag{2.24}$$

2.2.1 Memoryless Systems

A system is referred to as memoryless if the output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n.

$$y[n] = (x[n])^2, \qquad \text{for each value of } n.$$

2.2.2 Linear Systems

The class of *linear systems* is defined by the principle of superposition. If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs, then the system is linear if and only if

$$T\{x_1[n] + x_2[n]\} = T\{x_2[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n] \tag{2.26a}$$

and

$$T\{ax[n]\} = aT\{x[n]\} = ay[n], \tag{2.26b}$$

where a is an arbitrary constant.

This equation can be generalized to the superposition of many inputs. Specifically, if

$$x[n] = \sum_k a_k x_k[n], \tag{2.28a}$$

then the output of a linear system will be

$$y[n] = \sum_k a_k y_k[n], \tag{2.28b}$$

where $y_k[n]$ is the system response to the input $x_k[n]$.

The system defined by the input–output equation

$$y[n] = \sum_{k=-\infty}^{n} x[k] \tag{2.29}$$

is called the *accumulator* system, since the output at time n is just the sum of the present and all previous input samples. The accumulator system is a linear system.

Consider the system defined by

$$w[n] = \log_{10}(|x[n]|). \tag{2.36}$$

This system is not linear.

2.2.3 Time-Invariant Systems

Then the system is said to be time invariant if, for all n_0, the input sequence with values $x_1[n] = x[n - n_0]$ produces the output sequence with values $y_1[n] = y[n - n_0]$.

Consider the accumulator from Example 2.6. We define $x_1[n] = x[n - n_0]$.

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k] \tag{2.38}$$

$$= \sum_{k=-\infty}^{n} x[k - n_0]. \tag{2.39}$$

Substituting the change of variables $k_1 = k - n_0$ into the summation gives

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] = y[n - n_0]. \tag{2.40}$$

Thus, the accumulator is a time-invariant system.

The system defined by the relation

$$y[n] = x[Mn], \qquad -\infty < n < \infty, \tag{2.41}$$

with M a positive integer, is called a *compressor.*

Comparing these two outputs, we see that $y[n-n_0]$ is not equal to $y_1[n]$ for all M and n_0, and therefore, the system is not time invariant.

2.2.4 Causality

A system is causal if, for every choice of n_0, the output sequence value at the index $n = n_0$ depends only on the input sequence values for $n \le n_0$. This implies that if $x_1[n] = x_2[n]$ for $n \le n_0$, then $y_1[n] = y_2[n]$ for $n \le n_0$. That is, the system is *nonanticipative.*

Consider the *forward difference system* defined by the relationship

$$y[n] = x[n+1] - x[n]. \tag{2.44}$$

This system is not causal, since the current value of the output depends on a future value of the input.

The *backward difference system,* defined as

$$y[n] = x[n] - x[n-1], \tag{2.45}$$

has an output that depends only on the present and past values of the input. Because there is no way for the output at a specific time $y[n_0]$ to incorporate values of the input for $n > n_0$, the system is causal.

2.2.5 Stability

A system is stable in the bounded-input, bounded-output (BIBO) sense if and only if every bounded input sequence produces a bounded output sequence. The input $x[n]$ is bounded if there exists a fixed positive finite value B_x such that

$$|x[n]| \le B_x < \infty, \qquad \text{for all } n. \tag{2.46}$$

The accumulator, as defined in Example 2.6 by Eq. (2.29), is also not stable. For example, consider the case when $x[n] = u[n]$, which is clearly bounded by $B_x = 1$. For this input, the output of the accumulator is

$$y[n] = \sum_{k=-\infty}^{n} u[k] \tag{2.48}$$

$$= \begin{cases} 0, & n < 0, \\ (n+1), & n \ge 0. \end{cases} \tag{2.49}$$

There is no finite choice for B_y such that $(n+1) \le B_y < \infty$ for all n; thus, the system is unstable.

2.3 LINEAR TIME-INVARIANT SYSTEMS

$$y[n] = T\left\{ \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right\}. \tag{2.50}$$

From the principle of superposition in Eq. (2.27), we can write

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n]. \tag{2.51}$$

According to Eq. (2.51), the system response to any input can be expressed in terms of the responses of the system to the sequences $\delta[n-k]$.

The property of time invariance implies that if $h[n]$ is the response to $\delta[n]$, then the response to $\delta[n-k]$ is $h[n-k]$. With this additional constraint, Eq. (2.51) becomes

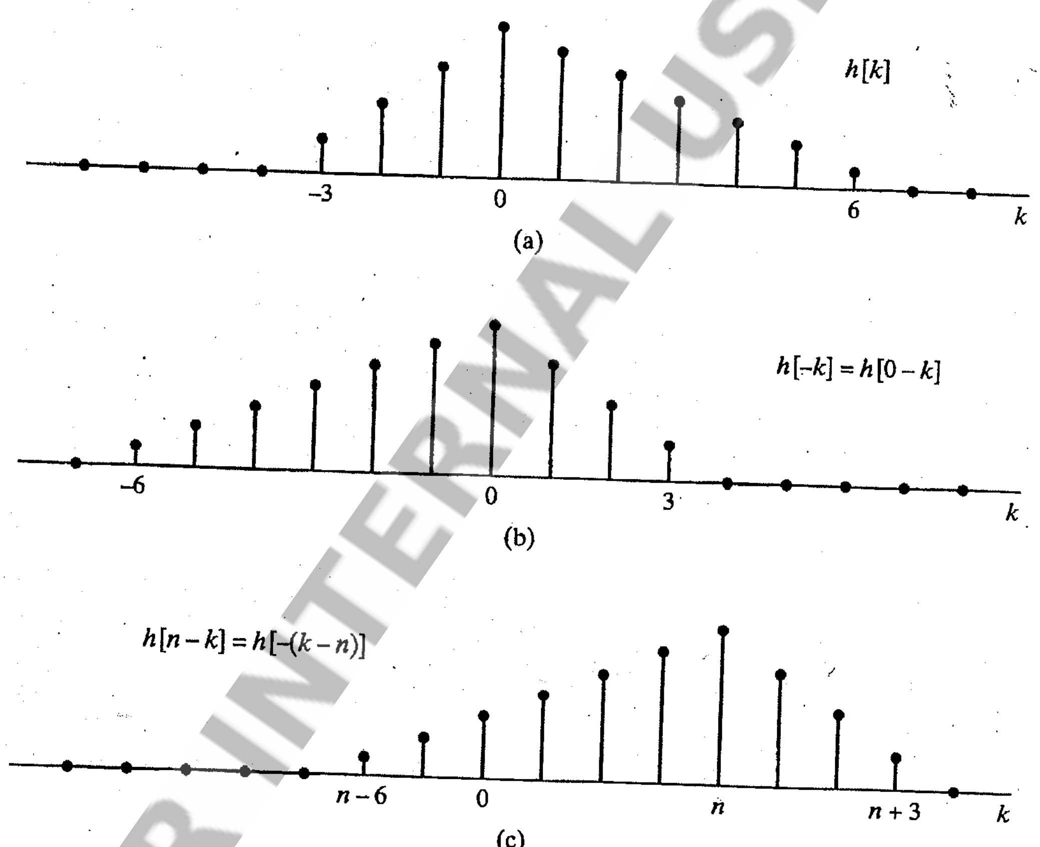
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \tag{2.52}$$

As a consequence of Eq. (2.52), a linear time-invariant system (which we will sometimes abbreviate as LTI) is completely characterized by its impulse response $h[n]$ in the sense that, given $h[n]$, it is possible to use Eq. (2.52) to compute the output $y[n]$ due to *any* input $x[n]$.

Equation (2.52) is commonly called the *convolution sum*. If $y[n]$ is a sequence whose values are related to the values of two sequences $h[n]$ and $x[n]$ as in Eq. (2.52), we say that $y[n]$ is the convolution of $x[n]$ with $h[n]$ and represent this by the notation

$$y[n] = x[n] * h[n]. \tag{2.53}$$

Example 2.12 Computation of the Convolution Sum



From Example 2.3, it should be clear that, in general, the sequence $h[n-k]$, $-\infty < k < \infty$, is obtained by

1. reflecting $h[k]$ about the origin to obtain $h[-k]$;
2. shifting the origin of the reflected sequence to $k = n$.

2.4 PROPERTIES OF LINEAR TIME-INVARIANT SYSTEMS

Some general properties of the class of linear time-invariant systems can be found by considering properties of the convolution operation. For example, the convolution operation is commutative:

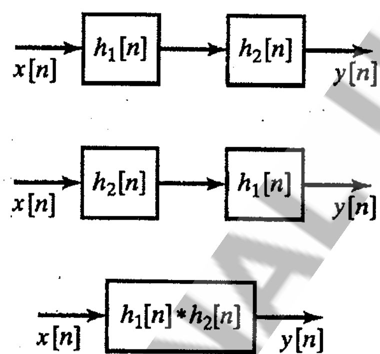
$$x[n] * h[n] = h[n] * x[n]. \qquad (2.61)$$

The convolution operation also distributes over addition; i.e.,

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

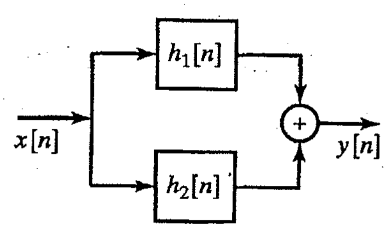
This follows in a straightforward way from Eq. (2.52) and is a direct result of the linearity and commutativity of convolution.

In a *cascade connection* of systems, the output of the first system is the input to the second, the output of the second is the input to the third, etc. The output of the last system is the overall output.

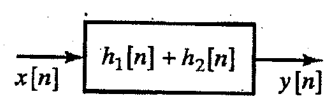


In a *parallel connection,* the systems have the same input, and their outputs are summed to produce an overall output. It follows from the distributive property of convolution that the connection of two linear time-invariant systems in parallel is equivalent to a single system whose impulse response is the sum of the individual impulse responses; i.e.,

$$h[n] = h_1[n] + h_2[n]. \qquad (2.64)$$



(a)



(b)

Linear time-invariant systems are stable if and only if the impulse response is absolutely summable, i.e., if

$$S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty. \tag{2.65}$$

This can be shown as follows. From Eq. (2.62),

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|. \tag{2.66}$$

If $x[n]$ is bounded, so that

$$|x[n]| \leq B_x,$$

then substituting B_x for $|x[n-k]|$ can only strengthen the inequality. Hence,

$$|y[n]| \leq B_x \sum_{k=-\infty}^{\infty} |h[k]|. \tag{2.67}$$

The class of causal systems was defined in Section 2.2.4 as those systems for which the output $y[n_0]$ depends only on the input samples $x[n]$, for $n \leq n_0$. It follows from Eq. (2.52) or Eq. (2.62) that this definition implies the condition

$$h[n] = 0, \qquad n < 0, \tag{2.70}$$

for causality of linear time-invariant systems. (See Problem 2.62.) For this reason, it is sometimes convenient to refer to a sequence that is zero for $n < 0$ as a *causal sequence*, meaning that it could be the impulse response of a causal system.

Ideal Delay (Example 2.3)

$$h[n] = \delta[n - n_d], \qquad n_d \text{ a positive fixed integer.} \tag{2.71}$$

Moving Average (Example 2.4)

$$\begin{aligned} h[n] &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k] \\ &= \begin{cases} \dfrac{1}{M_1 + M_2 + 1}, & -M_1 \leq n \leq M_2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.72}$$

Accumulator (Example 2.6)

$$\begin{aligned} h[n] &= \sum_{k=-\infty}^{n} \delta[k] \\ &= \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases} \\ &= u[n]. \end{aligned} \tag{2.73}$$

Forward Difference (Example 2.10)

$$h[n] = \delta[n+1] - \delta[n]. \tag{2.74}$$

Backward Difference (Example 2.10)

$$h[n] = \delta[n] - \delta[n-1]. \tag{2.75}$$

The impulse response of the accumulator is infinite in duration. This is an example of the class of systems referred to as *infinite-duration impulse response* (IIR) systems. An example of an IIR system that is stable is a system whose impulse response is $h[n] = a^n u[n]$ with $|a| < 1$. In this case,

$$S = \sum_{n=0}^{\infty} |a|^n. \tag{2.76}$$

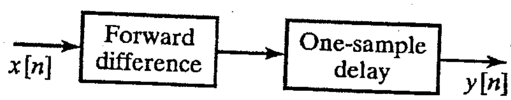
If $|a| < 1$, the formula for the sum of the terms of an infinite geometric series gives

$$S = \frac{1}{1 - |a|} < \infty. \tag{2.77}$$

If, on the other hand, $|a| \geq 1$, the sum is infinite and the system is unstable.

Since the output of the delay system is $y[n] = x[n-n_d]$, and since the delay system has impulse response $h[n] = \delta[n - n_d]$, it follows that

$$x[n] * \delta[n - n_d] = \delta[n - n_d] * x[n] = x[n - n_d]. \tag{2.78}$$



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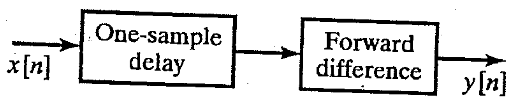

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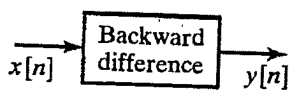
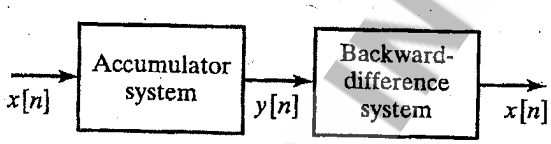


(c)

$$\begin{aligned} h[n] &= (\delta[n+1] - \delta[n]) * \delta[n-1] \\ &= \delta[n-1] * (\delta[n+1] - \delta[n]) \\ &= \delta[n] - \delta[n-1]. \end{aligned}$$



$$\begin{aligned} h[n] &= u[n] * (\delta[n] - \delta[n-1]) \\ &= u[n] - u[n-1] \\ &= \delta[n]. \end{aligned}$$

In general, if a linear time-invariant system has impulse response $h[n]$, then its inverse system, if it exists, has impulse response $h_i[n]$ defined by the relation

$$h[n] * h_i[n] = h_i[n] * h[n] = \delta[n]. \tag{2.81}$$

Inverse systems are useful in many situations in which it is necessary to compensate for the effects of a linear system.

2.5 LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

An important subclass of linear time-invariant systems consists of those systems for which the input $x[n]$ and the output $y[n]$ satisfy an Nth-order linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]. \tag{2.82}$$

An example of the class of linear constant-coefficient difference equations is the accumulator system defined by

$$y[n] = \sum_{k=-\infty}^{n} x[k]. \tag{2.83}$$

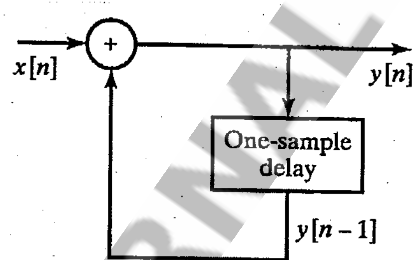
By separating the term $x[n]$ from the sum, we can rewrite Eq. (2.83) as

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]. \tag{2.85}$$

$$y[n] = x[n] + y[n-1], \tag{2.86}$$

from which the desired form of the difference equation can be obtained by grouping all the input and output terms on separate sides of the equation:

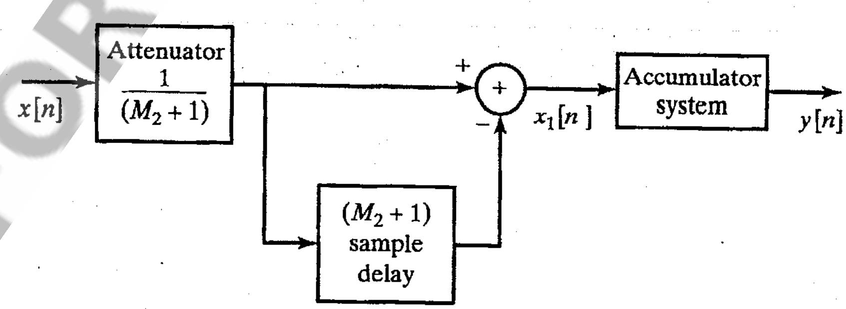
$$y[n] - y[n-1] = x[n]. \tag{2.87}$$



Consider the moving-average system of Example 2.4, with $M_1 = 0$ so that the system is causal. In this case,

$$y[n] = \frac{1}{(M_2+1)} \sum_{k=0}^{M_2} x[n-k], \tag{2.89}$$

which is a special case of Eq. (2.82), with $N = 0$, $a_0 = 1$, $M = M_2$, and $b_k = 1/(M_2+1)$ for $0 \le k \le M_2$.



Alternatively, if the auxiliary conditions are a set of auxiliary values of $y[n]$, the other values of $y[n]$ can be generated by rewriting Eq. (2.82) as a recurrence formula, i.e., in the form

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]. \tag{2.97}$$

If the input $x[n]$, together with a set of auxiliary values, say, $y[-1], y[-2], \ldots, y[-N]$, is specified, then $y[0]$ can be determined from Eq. (2.97). With $y[0], y[-1], \ldots, y[-N+1]$ available, $y[1]$ can then be calculated, and so on. When this procedure is used, $y[n]$ is said to be computed *recursively;* i.e., the output computation involves not only the input sequence, but also previous values of the output sequence.

To summarize, for a system for which the input and output satisfy a linear constant-coefficient difference equation:

- The output for a given input is not uniquely specified. Auxiliary information or conditions are required.
- If the auxiliary information is in the form of N sequential values of the output, later values can be obtained by rearranging the difference equation as a recursive relation running forward in n, and prior values can be obtained by rearranging the difference equation as a recursive relation running backward in n.
- Linearity, time invariance, and causality of the system will depend on the auxiliary conditions. If an additional condition is that the system is initially at rest, then the system will be linear, time invariant, and causal.

$$h[n] = \begin{cases} \left(\dfrac{b_n}{a_0}\right), & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases} \tag{2.107}$$

The impulse response is obviously finite in duration.

2.6 FREQUENCY-DOMAIN REPRESENTATION OF DISCRETE-TIME SIGNALS AND SYSTEMS

To demonstrate the eigenfunction property of complex exponentials for discrete-time systems, consider an input sequence $x[n] = e^{j\omega n}$ for $-\infty < n < \infty$, i.e., a complex exponential of radian frequency ω. From Eq. (2.62), the corresponding output of a linear time-invariant system with impulse response $h[n]$ is

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} \\ &= e^{j\omega n} \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right). \end{aligned} \tag{2.108}$$

If we define

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}, \tag{2.109}$$

Eq. (2.108) becomes

$$y[n] = H(e^{j\omega}) e^{j\omega n}. \tag{2.110}$$

The eigenvalue $H(e^{j\omega})$ is called the *frequency response* of the system. In general, $H(e^{j\omega})$ is complex and can be expressed in terms of its real and imaginary parts as

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}) \tag{2.111}$$

or in terms of magnitude and phase as

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\sphericalangle H(e^{j\omega})}. \tag{2.112}$$

As a simple example of how we can find the frequency response of a linear time-invariant system, consider the ideal delay system defined by

$$y[n] = x[n - n_d], \tag{2.113}$$

$$y[n] = e^{j\omega(n-n_d)} = e^{-j\omega n_d}e^{j\omega n}.$$

$$H(e^{j\omega}) = e^{-j\omega n_d}. \tag{2.114}$$

The magnitude and phase are

$$|H(e^{j\omega})| = 1, \tag{2.116a}$$

$$\sphericalangle H(e^{j\omega}) = -\omega n_d. \tag{2.116b}$$

Since it is simple to express a sinusoid as a linear combination of complex exponentials, let us consider a sinusoidal input

$$x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}. \tag{2.119}$$

Thus, the total response is

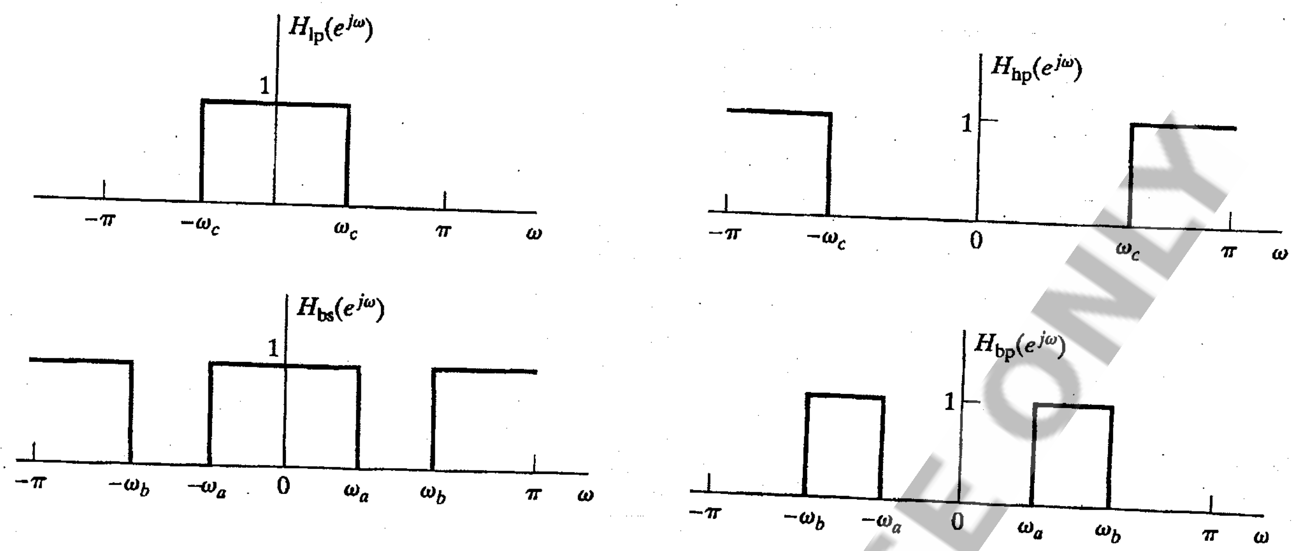
$$y[n] = \frac{A}{2}[H(e^{j\omega_0})e^{j\phi}e^{j\omega_0 n} + H(e^{-j\omega_0})e^{-j\phi}e^{-j\omega_0 n}]. \tag{2.121}$$

If $h[n]$ is real, it can be shown (see Problem 2.71) that $H(e^{-j\omega_0}) = H^*(e^{j\omega_0})$. Consequently,

$$y[n] = A|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \theta), \tag{2.122}$$

where $\theta = \sphericalangle H(e^{j\omega_0})$ is the phase of the system function at frequency ω_0.

Since $H(e^{j\omega})$ is periodic with period 2π, and since the frequencies ω and $\omega + 2\pi$ are indistinguishable, it follows that we need only specify $H(e^{j\omega})$ over an interval of length 2π, e.g., $0 \le \omega \le 2\pi$ or $-\pi < \omega \le \pi$. The inherent periodicity defines the frequency response everywhere outside the chosen interval. For simplicity and for consistency with the continuous-time case, it is generally convenient to specify $H(e^{j\omega})$ over the interval $-\pi < \omega \le \pi$. With respect to this interval, the "low frequencies" are frequencies close to zero, while the "high frequencies" are frequencies close to $\pm\pi$. Recalling that frequencies differing by an integer multiple of 2π are indistinguishable, we might generalize the preceding statement as follows: The "low frequencies" are those that are close to an even multiple of π, while the "high frequencies" are those that are close to an odd multiple of π.



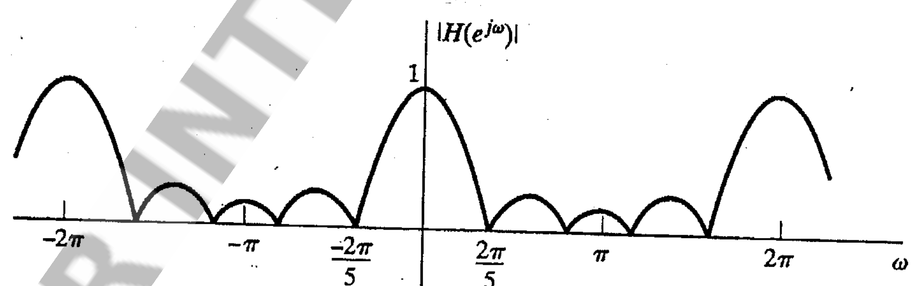
The impulse response of the moving-average system of Example 2.4 is

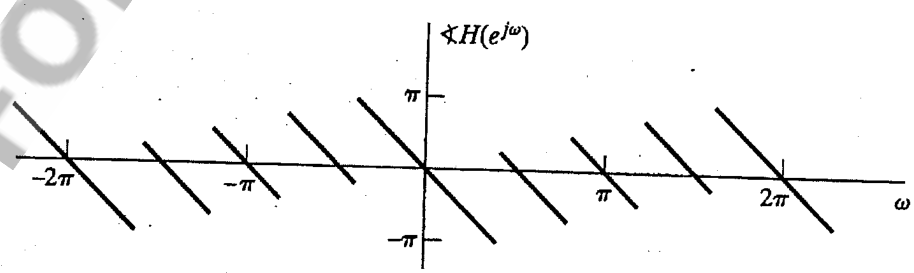
$$h[n] = \begin{cases} \dfrac{1}{M_1 + M_2 + 1}, & -M_1 \le n \le M_2, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the frequency response is

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-j\omega n}. \tag{2.127}$$

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega M_1} - e^{-j\omega(M_2+1)}}{1 - e^{-j\omega}} = \frac{1}{M_1 + M_2 + 1} \frac{\sin[\omega(M_1 + M_2 + 1)/2]}{\sin(\omega/2)} e^{-j\omega(M_2 - M_1)/2}.$$





The magnitude and phase of $H(e^{j\omega})$ are plotted in Figure 2.19 for $M_1 = 0$ and $M_2 = 4$. Note that $H(e^{j\omega})$ is periodic, as is required of the frequency response of a discrete-time system. Note also that $|H(e^{j\omega})|$ falls off at "high frequencies" and $\sphericalangle H(e^{j\omega})$, i.e., the phase of $H(e^{j\omega})$, varies linearly with ω.

Many sequences can be represented by a Fourier integral of the form

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \tag{2.133}$$

where

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \tag{2.134}$$

Equations (2.133) and (2.134) together form a *Fourier representation* for the sequence. Equation (2.133), the *inverse Fourier transform,* is a *synthesis* formula.

Equation (2.134), the *Fourier transform,*[3] is an expression for computing $X(e^{j\omega})$ from the sequence $x[n]$, i.e., for *analyzing* the sequence $x[n]$ to determine how much of each frequency component is required to synthesize $x[n]$ using Eq. (2.133).

[3] Sometimes we will refer to Eq. (2.134) more explicitly as the discrete-time Fourier transform, or DTFT, particularly when it is important to distinguish it from the continuous-time Fourier transform.

In general, the Fourier transform is a complex-valued function of ω. As with the frequency response, we may either express $X(e^{j\omega})$ in rectangular form as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) \tag{2.135a}$$

or in polar form as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}. \tag{2.135b}$$

The phase $\sphericalangle X(e^{j\omega})$ is not uniquely specified by Eq. (2.135b), since any integer multiple of 2π may be added to $\sphericalangle X(e^{j\omega})$ at any value of ω without affecting the result of the complex exponentiation. When we specifically want to refer to the principal value, i.e., $\sphericalangle X(e^{j\omega})$ restricted to the range of values between $-\pi$ and $+\pi$, we will denote this as ARG$[X(e^{j\omega})]$. If we want to refer to a phase function that is a continuous function of ω for $0 < \omega < \pi$, we will use the notation arg$[X(e^{j\omega})]$.

By comparing Eqs. (2.109) and (2.134), we can see that the frequency response of a linear time-invariant system is simply the Fourier transform of the impulse response and that, therefore, the impulse response can be obtained from the frequency response by applying the inverse Fourier transform integral; i.e.,

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega. \tag{2.136}$$

Thus, if $x[n]$ is *absolutely summable,* then $X(e^{j\omega})$ exists. Furthermore, in this case, the series can be shown to converge uniformly to a continuous function of ω.

Let $x[n] = a^n u[n]$. The Fourier transform of this sequence is

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{1}{1 - ae^{-j\omega}} \qquad \text{if } |ae^{-j\omega}| < 1 \quad \text{or} \quad |a| < 1.$$

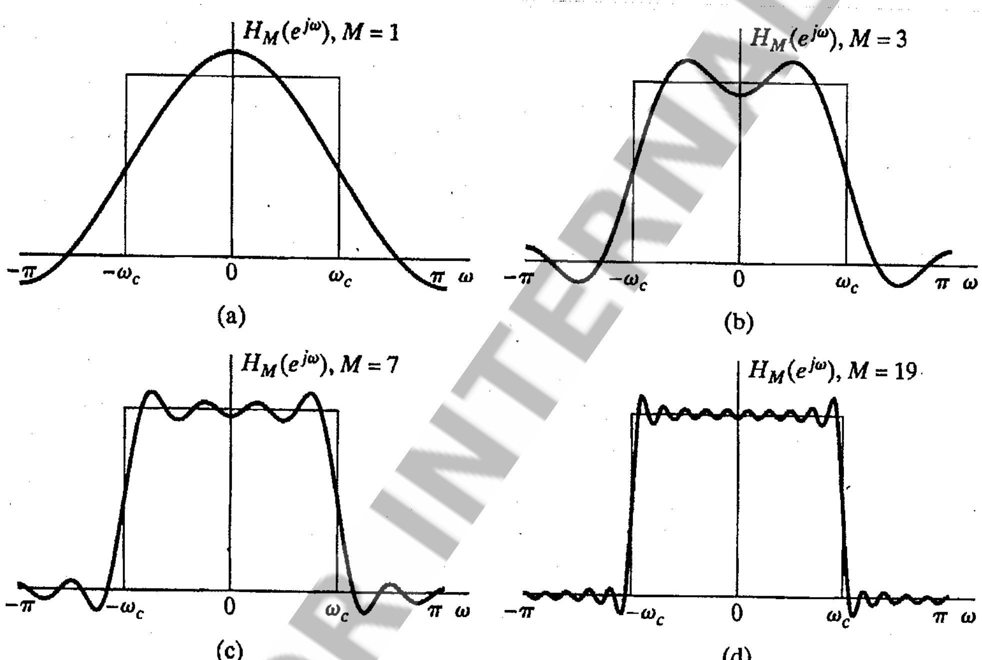
Clearly, the condition $|a| < 1$ is the condition for the absolute summability of $x[n]$; i.e.,

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty \qquad \text{if } |a| < 1. \tag{2.140}$$

Let us determine the impulse response of the ideal lowpass filter discussed in Example 2.19. The frequency response is

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases} \tag{2.144}$$

$$\begin{aligned} h_{\text{lp}}[n] &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi jn} \left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn} (e^{j\omega_c n} - e^{-j\omega_c n}) \\ &= \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty. \end{aligned} \tag{2.145}$$



$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}.$$

However, $h_{\text{lp}}[n]$, as given in Eq. (2.145), is square summable, and correspondingly, $H_M(e^{j\omega})$ converges in the mean-square sense to $H_{\text{lp}}(e^{j\omega})$; i.e.,

$$\lim_{M \to \infty} \int_{-\pi}^{\pi} |H_{\text{lp}}(e^{j\omega}) - H_M(e^{j\omega})|^2 d\omega = 0.$$

Fourier Transform of a Constant

Consider the sequence $x[n] = 1$ for all n. This sequence is neither absolutely summable nor square summable, and Eq. (2.134) does not converge in either the uniform or mean-square sense for this case. However, it is possible and useful to define the Fourier transform of the sequence $x[n]$ to be the periodic impulse train[4]

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r). \qquad (2.147)$$

Another sequence that is neither absolutely summable nor square summable is the unit step sequence $u[n]$. Although it is not completely straightforward to show, this sequence can be represented by the following Fourier transform:

$$U(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} + \sum_{r=-\infty}^{\infty} \pi\delta(\omega + 2\pi r). \qquad (2.153)$$

SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

A *conjugate-symmetric sequence* $x_e[n]$ is defined as a sequence for which $x_e[n] = x_e^*[-n]$, and a *conjugate-antisymmetric sequence* $x_o[n]$ is defined as a sequence for which $x_o[n] = -x_o^*[-n]$, where * denotes complex conjugation. Any sequence $x[n]$ can be expressed as a sum of a conjugate-symmetric and conjugate-antisymmetric sequence. Specifically,

$$x[n] = x_e[n] + x_o[n], \qquad (2.154a)$$

where

$$x_e[n] = \tfrac{1}{2}(x[n] + x^*[-n]) = x_e^*[-n] \qquad (2.154b)$$

and

$$x_o[n] = \tfrac{1}{2}(x[n] - x^*[-n]) = -x_o^*[-n]. \qquad (2.154c)$$

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega}),$$

$$X_e(e^{j\omega}) = \tfrac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})]$$

$$X_o(e^{j\omega}) = \tfrac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})].$$

$$X_e(e^{j\omega}) = X_e^*(e^{-j\omega})$$

$$X_o(e^{j\omega}) = -X_o^*(e^{-j\omega}).$$

Illustration of Symmetry Properties

$$x[n] = a^n u[n]$$

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \qquad \text{if } |a| < 1.$$

Then, from the properties of complex numbers, it follows that

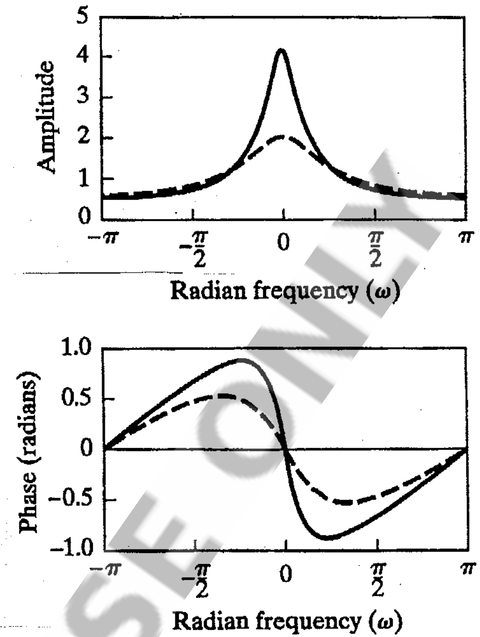
$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} = X^*(e^{-j\omega})$$

$$X_R(e^{j\omega}) = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} = X_R(e^{-j\omega})$$

$$X_I(e^{j\omega}) = \frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega} = -X_I(e^{-j\omega})$$

$$|X(e^{j\omega})| = \frac{1}{(1 + a^2 - 2a\cos\omega)^{1/2}} = |X(e^{-j\omega})|$$

$$\sphericalangle X(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right) = -\sphericalangle X(e^{-j\omega})$$



2.9 FOURIER TRANSFORM THEOREMS

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}).$$

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$	
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	**Linearity**
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$	**Time Shifting**
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$	**Frequency Shifting**
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.	**Time Reversal**
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$	**Differentiation in Frequency**
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$	**The Convolution Theorem**
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	
Parseval's theorem:		
8. $\sum_{n=-\infty}^{\infty} \|x[n]\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|X(e^{j\omega})\|^2 d\omega$		**Parseval's Theorem**
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$		

The function $|X(e^{j\omega})|^2$ is called the *energy density spectrum.*

Thus, convolution of sequences implies multiplication of the corresponding Fourier transforms. Note that the time-shifting property is a special case of the convolution property, since

$$\delta[n-n_d] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d} \tag{2.170}$$

and if $h[n]=\delta[n-n_d]$, then $y[n]=x[n]*\delta[n-n_d]=x[n-n_d]$. Therefore,

$$H(e^{j\omega})=e^{-j\omega n_d} \quad \text{and} \quad Y(e^{j\omega})=e^{-j\omega n_d}X(e^{j\omega}).$$

The duality inherent in most Fourier transform theorems is evident when we compare the convolution and modulation theorems. Specifically, discrete-time convolution of sequences (the convolution sum) is equivalent to multiplication of corresponding periodic Fourier transforms, and multiplication of sequences is equivalent to *periodic* convolution of corresponding Fourier transforms.

FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty<n<\infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert<1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty}\pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert<1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p}u[n] \quad (\lvert r\rvert<1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega})=\begin{cases}1, & \lvert\omega\rvert<\omega_c,\\ 0, & \omega_c<\lvert\omega\rvert\le\pi\end{cases}$
9. $x[n]=\begin{cases}1, & 0\le n\le M\\ 0, & \text{otherwise}\end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty}[\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

THE Z-TRANSFORM

The z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z). \tag{3.3}$$

With this interpretation, the z-transform operator is seen to transform the sequence $x[n]$ into the function $X(z)$, where z is a continuous complex variable.

This is one motivation for the notation $X(e^{j\omega})$ for the Fourier transform; when it exists, the Fourier transform is simply $X(z)$ with $z = e^{j\omega}$. This corresponds to restricting z to have unity magnitude; i.e., for $|z| = 1$, the z-transform corresponds to the Fourier transform. More generally, we can express the complex variable z in polar form as

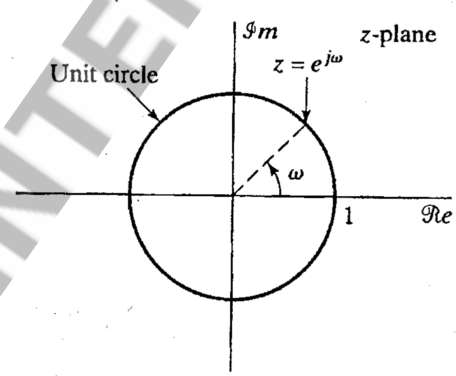
$$z = re^{j\omega}.$$

With z expressed in this form, Eq. (3.2) becomes

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}. \tag{3.6}$$

Equation (3.6) can be interpreted as the Fourier transform of the product of the original sequence $x[n]$ and the exponential sequence r^{-n}. Obviously, for $r = 1$, Eq. (3.6) reduces to the Fourier transform of $x[n]$.

The z-transform evaluated on the unit circle corresponds to the Fourier transform. Note that ω is the angle between the vector to a point z on the unit circle and the real axis of the complex z-plane.



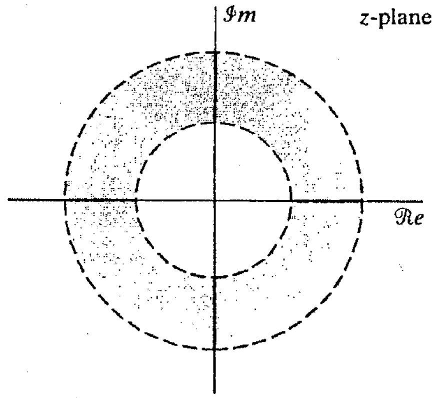
With this interpretation, the inherent periodicity in frequency of the Fourier transform is captured naturally, since a change of angle of 2π radians in the z-plane corresponds to traversing the unit circle once and returning to exactly the same point.

For any given sequence, the set of values of z for which the z-transform converges is called the *region of convergence,* which we abbreviate ROC.

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty \tag{3.7}$$

for convergence of the z-transform.

The region of convergence will consist of a ring in the z-plane.



For example, the sequence $x[n] = u[n]$ is not absolutely summable, and therefore, the Fourier transform does not converge absolutely. However, $r^{-n}u[n]$ is absolutely summable if $r > 1$. This means that the z-transform for the unit step exists with a region of convergence $|z| > 1$.

If the ROC includes the unit circle, this of course implies convergence of the z-transform for $|z| = 1$, or equivalently, the Fourier transform of the sequence converges. Conversely, if the ROC does not include the unit circle, the Fourier transform does not converge absolutely.

Among the most important and useful z-transforms are those for which $X(z)$ is a rational function inside the region of convergence, i.e.,

$$X(z) = \frac{P(z)}{Q(z)}, \tag{3.9}$$

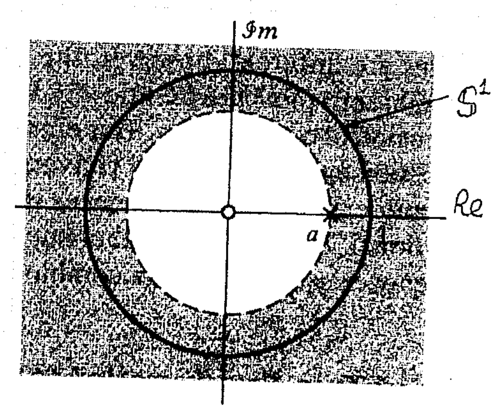
where $P(z)$ and $Q(z)$ are polynomials in z. The values of z for which $X(z) = 0$ are called the *zeros* of $X(z)$, and the values of z for which $X(z)$ is infinite are referred to as the *poles* of $X(z)$. The poles of $X(z)$ for finite values of z are the roots of the denominator polynomial. In addition, poles may occur at $z = 0$ or $z = \infty$.

Right-Sided Exponential Sequence

Consider the signal $x[n] = a^n u[n]$. Because it is nonzero only for $n \geq 0$, this is an example of a *right-sided* sequence. From Eq. (3.2),

$$X(z) = \sum_{n=-\infty}^{\infty} = a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|. \tag{3.10}$$



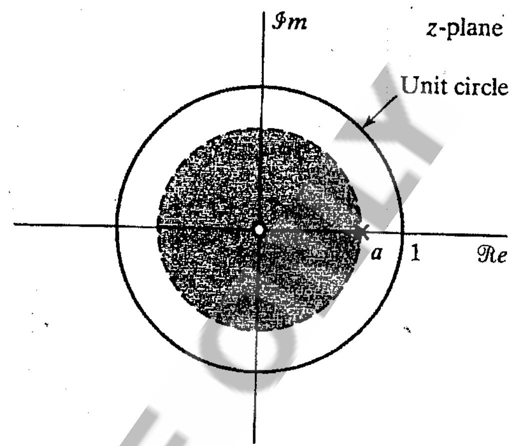
Here we have used the familiar formula for the sum of terms of a geometric series. The z-transform has a region of convergence for any finite value of $|a|$. The Fourier transform of $x[n]$, on the other hand, converges only if $|a| < 1$. For $a = 1$, $x[n]$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1. \tag{3.11}$$

Left-Sided Exponential Sequence

Now let $x[n] = -a^n u[-n-1]$. Since the sequence is nonzero only for $n \le -1$, this is a *left-sided* sequence. Then

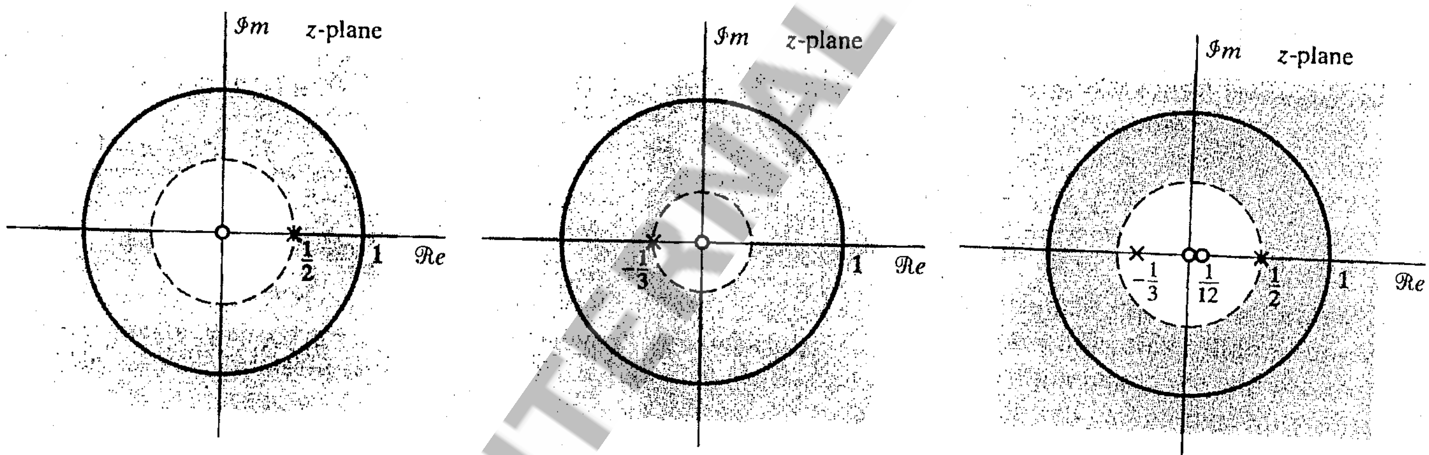
$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|.$$



Sum of Two Exponential Sequences

Consider a signal that is the sum of two real exponentials:

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n]. \qquad (3.14)$$



(a) (b) (c)

$$\left(\tfrac{1}{2}\right)^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad |z| > \tfrac{1}{2}, \qquad (3.17)$$

$$\left(-\tfrac{1}{3}\right)^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 + \frac{1}{3}z^{-1}}, \qquad |z| > \tfrac{1}{3}, \qquad (3.18)$$

and, consequently,

$$\left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}, \qquad |z| > \tfrac{1}{2}, \qquad (3.19)$$

$$= \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}.$$

Two-Sided Exponential Sequence

Consider the sequence

$$x[n] = \left(-\tfrac{1}{3}\right)^n u[n] - \left(\tfrac{1}{2}\right)^n u[-n-1]. \tag{3.20}$$

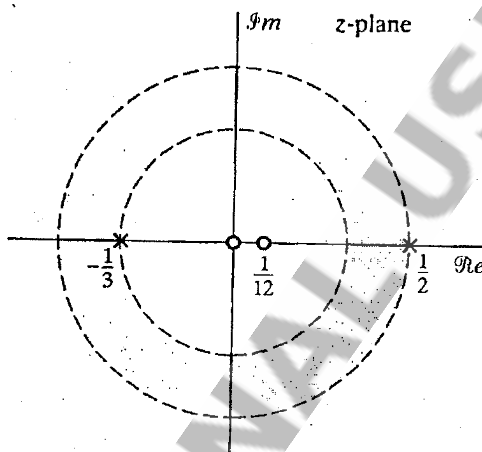
Note that this sequence grows exponentially as $n \to -\infty$.

$$\left(-\tfrac{1}{3}\right)^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \tfrac{1}{3},$$

$$-\left(\tfrac{1}{2}\right)^n u[-n-1] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \tfrac{1}{2}.$$

Thus, by the linearity of the z-transform,

$$\begin{aligned} X(z) &= \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \tfrac{1}{3} < |z|, \quad |z| < \tfrac{1}{2}, \\ &= \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}. \end{aligned} \tag{3.21}$$



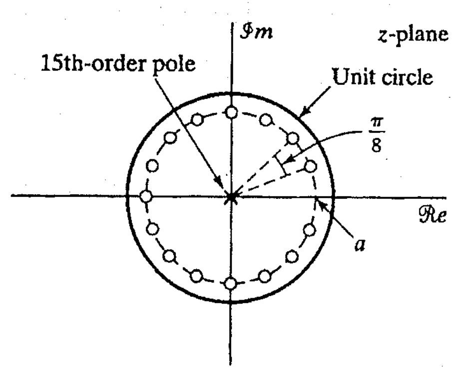
Finite-Length Sequence

Consider the signal

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n \\ &= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}, \end{aligned} \tag{3.23}$$



Specifically, the N roots of the numerator polynomial are at

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 0, 1, \ldots, N-1. \tag{3.24}$$

(Note that these values satisfy the equation $z^N = a^N$, and when $a = 1$, these complex values are the Nth roots of unity.) The zero at $k = 0$ cancels the pole at $z = a$.

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

3.2 PROPERTIES OF THE REGION OF CONVERGENCE FOR THE z-TRANSFORM

PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

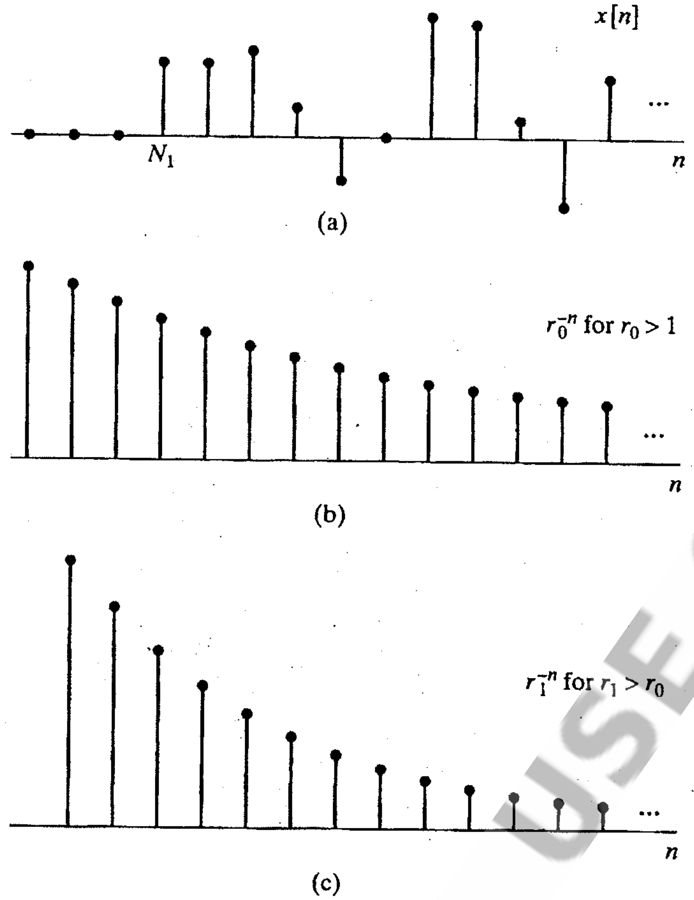
PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

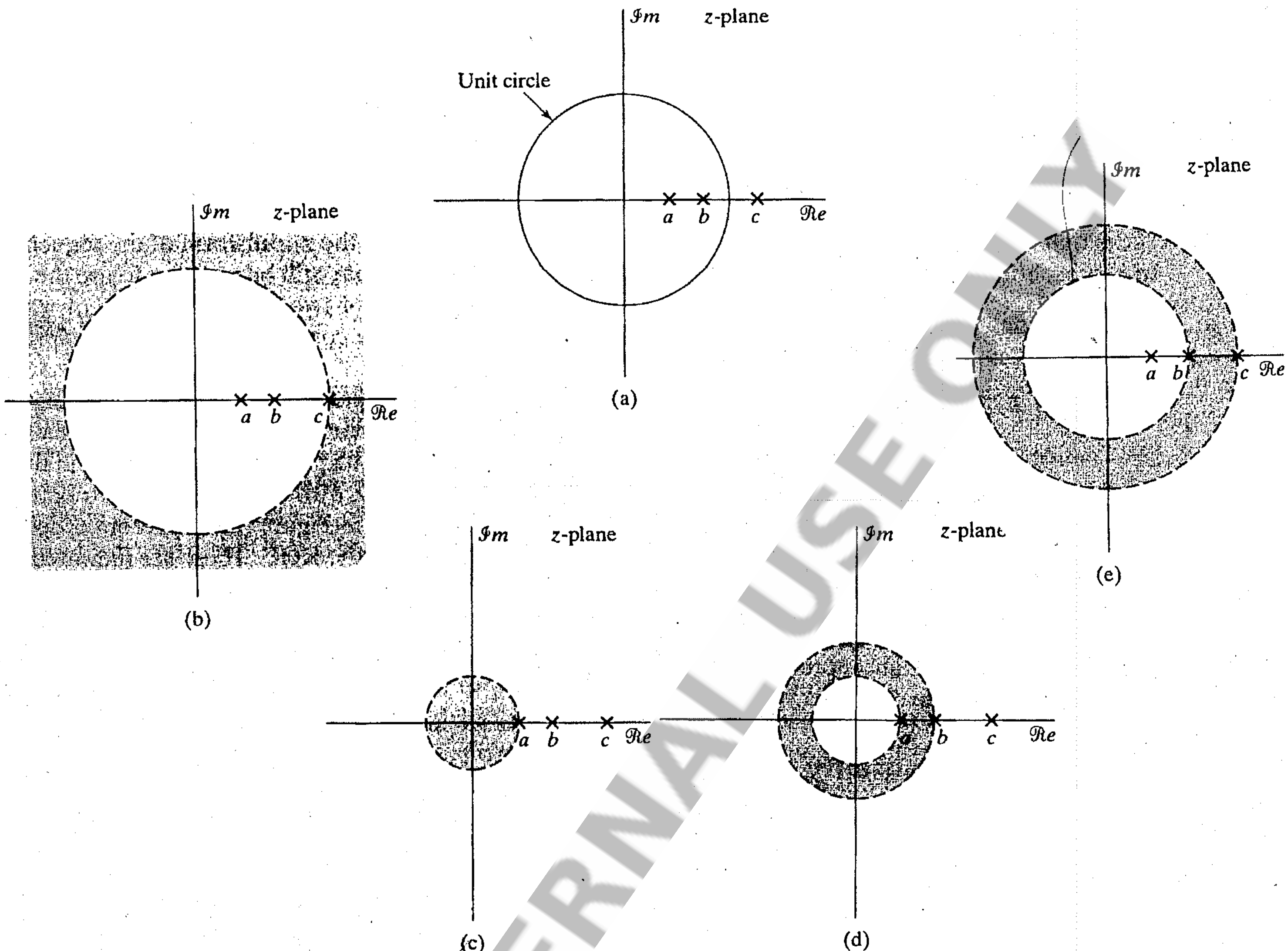
PROPERTY 8: The ROC must be a connected region.



Property 5 can be interpreted in a somewhat similar manner. Figure illustrates a right-sided sequence and the exponential sequence r^{-n} for two different values of r. A right-sided sequence is zero prior to some value of n, say, N_1. If the circle $|z| = r_0$ is in the ROC, then $x[n]r_0^{-n}$ is absolutely summable, or equivalently, the Fourier transform of $x[n]r_0^{-n}$ converges. Since $x[n]$ is right sided, the sequence $x[n]r_1^{-n}$ will also be absolutely summable if r_1^{-n} decays faster than r_0^{-n}. Specifically, as illustrated in Figure , this more rapid exponential decay will further attenuate sequence values for positive values of n and cannot cause sequence values for negative values of n to become unbounded, since $x[n]z^{-n} = 0$ for $n < N_1$. Based on this property, we can conclude that, for a right-sided sequence, the ROC extends outward from some circle in the z-plane, concentric with the origin.

For right-sided sequences, the ROC is dictated by the exponential weighting required to have all exponential terms decay to zero for increasing n; for left-sided sequences, the exponential weighting must be such that all exponential terms decay to zero for decreasing n. For two-sided sequences, the exponential weighting needs to be balanced, since if it decays too fast for increasing n, it may grow too quickly for decreasing n and vice versa. More specifically, for two-sided sequences, some of the poles contribute only for $n > 0$ and the rest only for $n < 0$. The region of convergence is bounded on the inside by the pole with the largest magnitude that contributes for $n > 0$ and on the outside by the pole with the smallest magnitude that contributes for $n < 0$.

Examples of four z- transforms with the same pole–zero locations, illustrating the different possibilities for the region of convergence. Each ROC corresponds to a different sequence: (b) to a right-sided sequence, (c) to a left-sided sequence, (d) to a two-sided sequence, and (e) to a two-sided sequence.

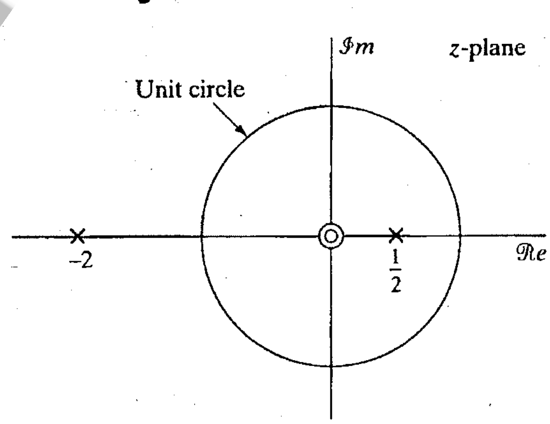


If the sequence is nonzero only in the interval $N_1 \leq n \leq N_2$, the z-transform

$$X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n} \tag{3.22}$$

has no problems of convergence, as long as each of the terms $|x[n]z^{-n}|$ is finite.

Stability, Causality, and the ROC



There are three possible ROC's consistent with properties 1–8 that can be associated with this pole-zero plot. However, if we state in addition that the system is stable (or equivalently, that $h[n]$ is absolutely summable and therefore has a Fourier transform), then the ROC must include the unit circle. Thus, stability of the system and properties 1–8 imply that the ROC is the region $\frac{1}{2} < |z| < 2$. Note that as a consequence, $h[n]$ is two sided, and therefore, the system is not causal.

If we state instead that the system is causal, and therefore that $h[n]$ is right sided, then property 5 would require that the ROC be the region $|z| > 2$. Under this condition, the system would not be stable; i.e., for this specific pole–zero plot, there is no ROC that would imply that the system is both stable and causal.

3.3 THE INVERSE z-TRANSFORM

Partial Fraction Expansion

To see how to obtain a partial fraction expansion, let us assume that $X(z)$ is expressed as a ratio of polynomials in z^{-1}; i.e.,

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \tag{3.37}$$

Such z-transforms arise frequently in the study of linear time-invariant systems. An equivalent expression is

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}. \tag{3.38}$$

Equation (3.38) explicitly shows that for such functions, there will be M zeros and N poles at nonzero locations in the z-plane. In addition, there will be either $M-N$ poles at $z=0$ if $M>N$ or $N-M$ zeros at $z=0$ if $N>M$. In other words, z-transforms of the form of Eq. (3.37) always have the same number of poles and zeros in the finite z-plane, and there are no poles or zeros at $z=\infty$. To obtain the partial fraction expansion of $X(z)$ in Eq. (3.37), it is most convenient to note that $X(z)$ could be expressed in the form

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}, \tag{3.39}$$

If $M<N$ and the poles are all first order, then $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}. \tag{3.40}$$

A_k, can be found from

$$A_k = (1-d_k z^{-1})X(z)\big|_{z=d_k}. \tag{3.41}$$

Consider a sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad |z| > \frac{1}{2}. \tag{3.42}$$

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$

From Eq. (3.41),

$$A_1 = \left(1 - \tfrac{1}{4}z^{-1}\right) X(z)\big|_{z=1/4} = -1,$$

$$A_2 = \left(1 - \tfrac{1}{2}z^{-1}\right) X(z)\big|_{z=1/2} = 2.$$

Therefore,

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$

Since $x[n]$ is right sided, the ROC for each term extends outward from the outermost pole. From Table 3.1 and the linearity of the z-transform, it then follows that

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n].$$

Clearly, the numerator that would result from adding the terms in Eq. (3.40) would be at most of degree $(N-1)$ in the variable z^{-1}. If $M \geq N$, then a polynomial must be added to the right-hand side of Eq. (3.40), the order of which is $(M-N)$. Thus, for $M \geq N$, the complete partial fraction expansion would have the form

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}. \tag{3.43}$$

If $X(z)$ has multiple-order poles and $M \geq N$, Eq. (3.43) must be further modified. In particular, if $X(z)$ has a pole of order s at $z = d_i$ and all the other poles are first-order, then Eq. (3.43) becomes

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}. \tag{3.44}$$

The coefficients A_k and B_r are obtained as before. The coefficients C_m are obtained from the equation

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} [(1 - d_i w)^s X(w^{-1})] \right\}_{w=d_i^{-1}}. \tag{3.45}$$

The terms $B_r z^{-r}$ correspond to shifted and scaled impulse sequences, i.e., terms of the form $B_r \delta[n-r]$. The fractional terms correspond to exponential sequences. To decide whether a term

$$\frac{A_k}{1 - d_k z^{-1}}$$

corresponds to $(d_k)^n u[n]$ or $-(d_k)^n u[-n-1]$, we must use the properties of the region of convergence that were discussed in Section 3.2. From that discussion, it follows that if $X(z)$ has only simple poles and the ROC is of the form $r_R < |z| < r_L$, then a given pole d_k will correspond to a right-sided exponential $(d_k)^n u[n]$ if $|d_k| < r_R$, and it will correspond to a left-sided exponential if $|d_k| > r_L$. Thus, the region of convergence can be used to sort the poles. Multiple-order poles also are divided into left-sided and right-sided contributions in the same way.

Finite-Length Sequence

Suppose $X(z)$ is given in the form

$$X(z) = z^2\left(1 - \tfrac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1}). \tag{3.50}$$

However, by multiplying the factors of Eq. (3.50), we can express $X(z)$ as

$$X(z) = z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}.$$

Therefore, by inspection, $x[n]$ is seen to be

$$x[n] = \begin{cases} 1, & n = -2, \\ -\frac{1}{2}, & n = -1, \\ -1, & n = 0, \\ \frac{1}{2}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$x[n] = \delta[n+2] - \tfrac{1}{2}\delta[n+1] - \delta[n] + \tfrac{1}{2}\delta[n-1].$$

Inverse Transform by Power Series Expansion

Consider the z-transform

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|.$$

Using the power series expansion for $\log(1+x)$, with $|x| < 1$, we obtain

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}a^n z^{-n}}{n}. \quad \Rightarrow \quad x[n] = \begin{cases} (-1)^{n+1}\frac{a^n}{n}, & n \geq 1, \\ 0, & n \leq 0. \end{cases}$$

Sequence	Transform	ROC
$x[n]$	$X(z)$	R_x
$x_1[n]$	$X_1(z)$	R_{x_1}
$x_2[n]$	$X_2(z)$	R_{x_2}
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
$x^*[n]$	$X^*(z^*)$	R_x
$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
Initial-value theorem:		
$x[n] = 0, \quad n < 0$	$\lim_{z\to\infty} X(z) = x[0]$	

Shifted Exponential Sequence

Consider the z-transform

$$X(z) = \frac{1}{z - \frac{1}{4}}, \qquad |z| > \tfrac{1}{4}.$$

From the ROC, we identify this as corresponding to a right-sided sequence. We can first rewrite $X(z)$ in the form

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \tfrac{1}{4}. \tag{3.55}$$

$X(z)$ can be written as

$$X(z) = z^{-1}\left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right), \qquad |z| > \tfrac{1}{4}. \tag{3.58}$$

From the time-shifting property, we recognize the factor z^{-1} in Eq. (3.58) as being associated with a time shift of one sample to the right of the sequence $\left(\frac{1}{4}\right)^n u[n]$; i.e.,

$$x[n] = \left(\tfrac{1}{4}\right)^{n-1} u[n-1]. \tag{3.59}$$

Exponential Multiplication

Starting with the transform pair

$$u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \qquad |z| > 1, \tag{3.60}$$

we can use the exponential multiplication property to determine the z-transform of

$$x[n] = r^n \cos(\omega_0 n) u[n]. \tag{3.61}$$

First, $x[n]$ is expressed as

$$x[n] = \tfrac{1}{2}(re^{j\omega_0})^n u[n] + \tfrac{1}{2}(re^{-j\omega_0})^n u[n].$$

Then, using Eq. (3.60) and the exponential multiplication property, we see that

$$\tfrac{1}{2}(re^{j\omega_0})^n u[n] \stackrel{z}{\longleftrightarrow} \frac{\frac{1}{2}}{1 - re^{j\omega_0}z^{-1}}, \qquad |z| > r,$$

$$\tfrac{1}{2}(re^{-j\omega_0})^n u[n] \stackrel{z}{\longleftrightarrow} \frac{\frac{1}{2}}{1 - re^{-j\omega_0}z^{-1}}, \qquad |z| > r.$$

From the linearity property, it follows that

$$\begin{aligned} X(z) &= \frac{\frac{1}{2}}{1 - re^{j\omega_0}z^{-1}} + \frac{\frac{1}{2}}{1 - re^{-j\omega_0}z^{-1}}, \qquad |z| > r \\ &= \frac{(1 - r\cos\omega_0 z^{-1})}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}, \qquad |z| > r. \end{aligned} \tag{3.62}$$

Inverse of Non-Rational z-Transform

In this example, we use the differentiation property together with the time-shifting property to determine the inverse z-transform

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|,$$

we first differentiate to obtain a rational expression:

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}}.$$

From the differentiation property,

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}}, \qquad |z| > |a|. \qquad (3.63)$$

Specifically, we can express $nx[n]$ as

$$nx[n] = a(-a)^{n-1}u[n-1].$$

Therefore,

$$x[n] = (-1)^{n+1}\frac{a^n}{n}u[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \log(1+az^{-1}), \qquad |z| > |a|.$$

Time-Reversed Exponential Sequence

If the sequence $x[n]$ is real or we do not conjugate a complex sequence, the result becomes

$$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(1/z), \qquad \text{ROC} = \frac{1}{R_x}.$$

As an example of the use of the property of time reversal, consider the sequence

$$x[n] = a^{-n}u[-n],$$

which is a time-reversed version of $a^n u[n]$. From the time-reversal property, it follows that

$$X(z) = \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, \qquad |z| < |a^{-1}|.$$

Evaluating a Convolution Using the z-Transform

Let $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$. The corresponding z-transforms are

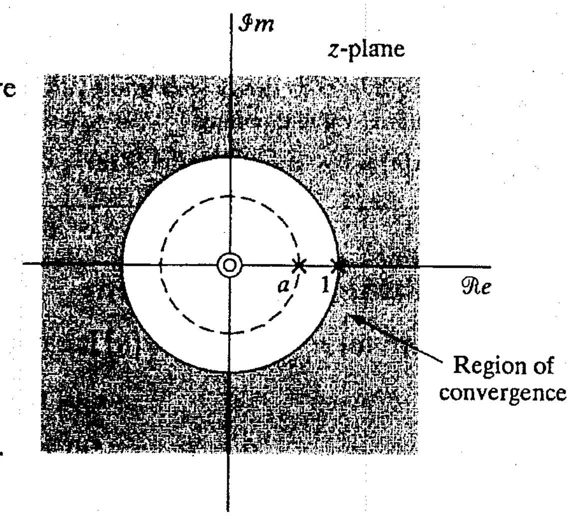
$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}, \qquad |z| > |a|,$$

and

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}, \qquad |z| > 1.$$

If $|a| < 1$, the z-transform of the convolution of $x_1[n]$ with $x_2[n]$ is then

$$Y(z) = \frac{1}{(1-az^{-1})(1-z^{-1})} = \frac{z^2}{(z-a)(z-1)}, \qquad |z| > 1.$$



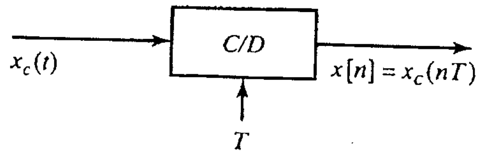
Expanding $Y(z)$ in Eq. (3.64) in a partial fraction expansion, we get

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right), \qquad |z| > 1.$$

Therefore,

$$y[n] = \frac{1}{1-a}(u[n] - a^{n+1}u[n]).$$

SAMPLING OF CONTINUOUS-TIME SIGNALS



$$x[n] = x_c(nT), \qquad -\infty < n < \infty. \tag{4.1}$$

In Eq. (4.1), T is the *sampling period*, and its reciprocal, $f_s = 1/T$, is the *sampling frequency*, in samples per second. We also express the sampling frequency as $\Omega_s = 2\pi/T$ when we want to use frequencies in radians per second.

In a practical setting, the operation of sampling is implemented by an analog-to-digital (A/D) converter. Such systems can be viewed as approximations to the ideal C/D converter.

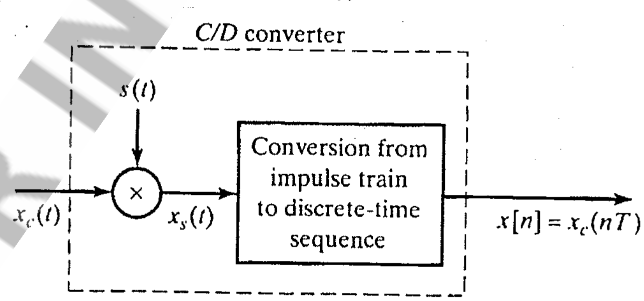
The sampling operation is generally not invertible; i.e., given the output $x[n]$, it is not possible in general to reconstruct $x_c(t)$, the input to the sampler, since many continuous-time signals can produce the same output sequence of samples. The inherent ambiguity in sampling is a fundamental issue in signal processing. Fortunately, it is possible to remove the ambiguity by restricting the input signals that go into the sampler.

To derive the frequency-domain relation between the input and output of an ideal C/D converter, let us first consider the conversion of $x_c(t)$ to $x_s(t)$ through modulation of the periodic impulse train

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \tag{4.2}$$

where $\delta(t)$ is the unit impulse function, or Dirac delta function. We modulate $s(t)$ with $x_c(t)$, obtaining

$$\begin{aligned} x_s(t) &= x_c(t)s(t) \\ &= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT). \end{aligned} \tag{4.3}$$



$x_s(t)$ can be expressed as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT). \tag{4.4}$$

The Fourier transform of a periodic impulse train is a periodic impulse train (Oppenheim and Willsky, 1997). Specifically,

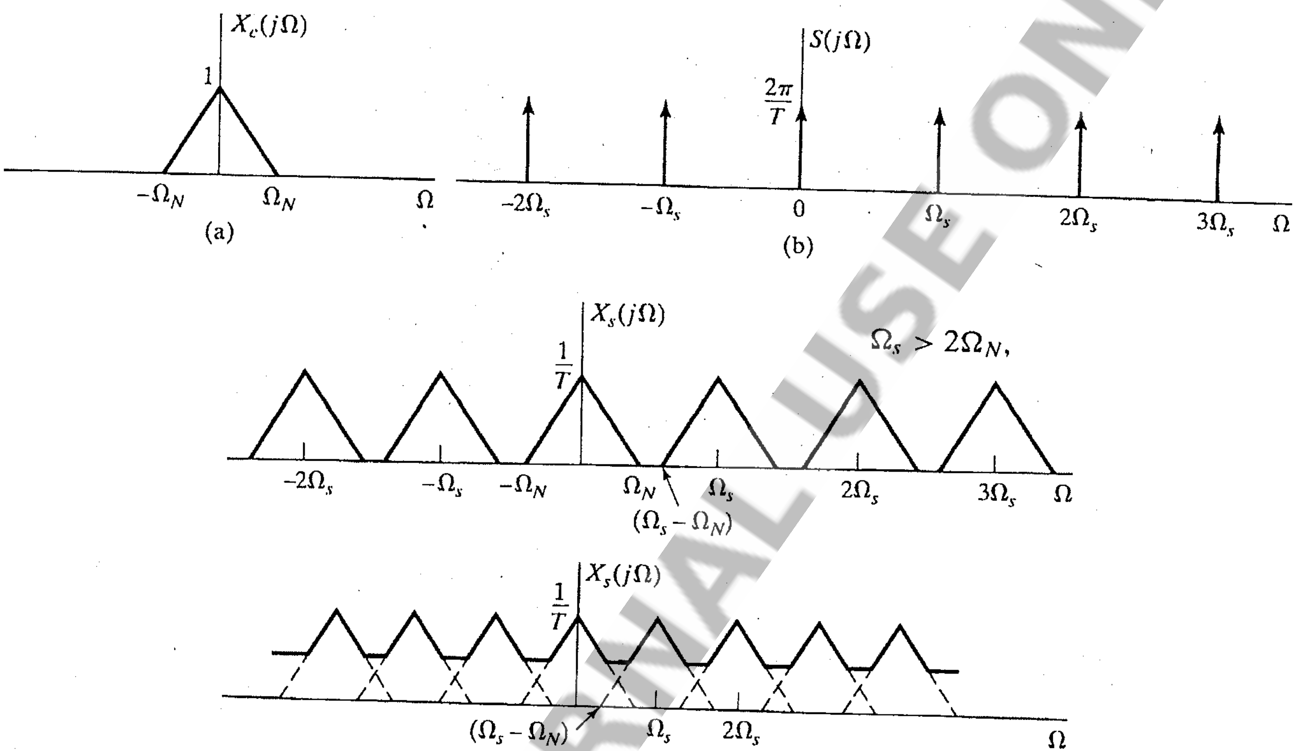
$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s), \tag{4.5}$$

where $\Omega_s = 2\pi/T$ is the sampling frequency in radians/s. Since

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega),$$

where $*$ denotes the operation of continuous-variable convolution, it follows that

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)). \tag{4.6}$$



Consequently, $x_c(t)$ can be recovered from $x_s(t)$ with an ideal lowpass filter. This is depicted in Figure 4.4(a), which shows the impulse train modulator followed by a linear time-invariant system with frequency response $H_r(j\Omega)$.

Since

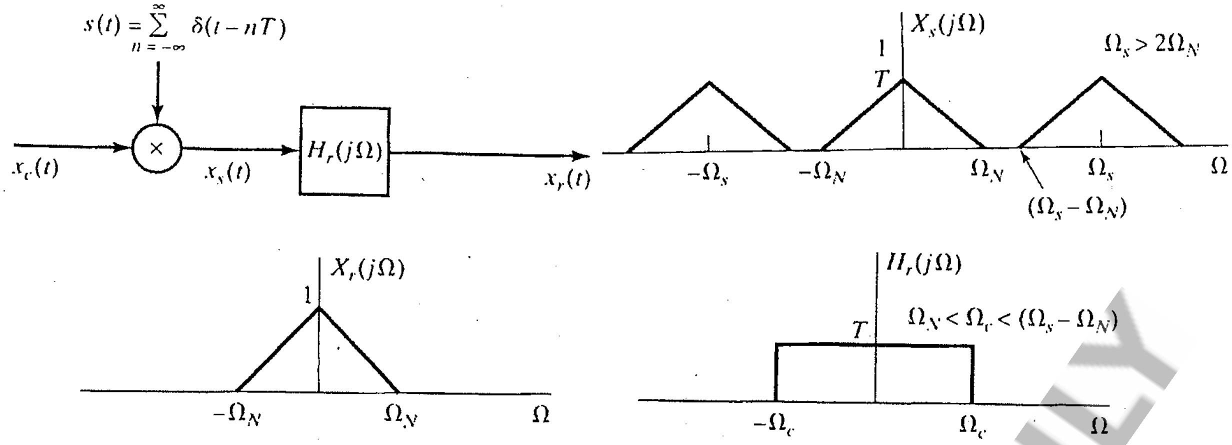
$$X_r(j\Omega) = H_r(j\Omega) X_s(j\Omega), \tag{4.8}$$

it follows that if $H_r(j\Omega)$ is an ideal lowpass filter with gain T and cutoff frequency Ω_c such that

$$\Omega_N < \Omega_c < (\Omega_s - \Omega_N), \tag{4.9}$$

then

$$X_r(j\Omega) = X_c(j\Omega), \tag{4.10}$$



Nyquist Sampling Theorem: Let $x_c(t)$ be a bandlimited signal with

$$X_c(j\Omega) = 0 \qquad \text{for } |\Omega| \geq \Omega_N. \tag{4.14a}$$

Then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N. \tag{4.14b}$$

The frequency Ω_N is commonly referred to as the *Nyquist frequency*, and the frequency $2\Omega_N$ that must be exceeded by the sampling frequency is called the *Nyquist rate*.

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\Omega T n}. \tag{4.15}$$

Since

$$x[n] = x_c(nT) \tag{4.16}$$

and

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \tag{4.17}$$

it follows that

$$X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T}). \tag{4.18}$$

Consequently,

$$X(e^{j\Omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)), \tag{4.19}$$

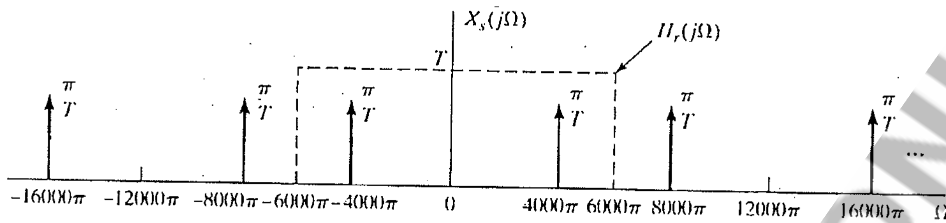
or equivalently,

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right). \tag{4.20}$$

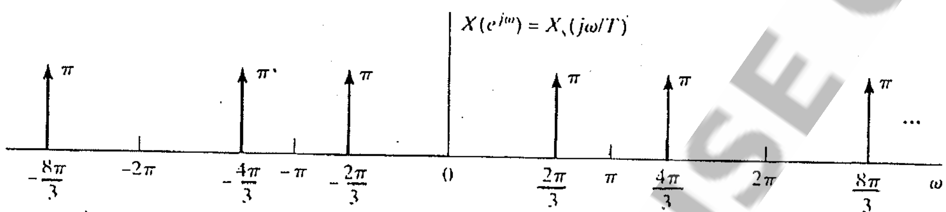
Sampling and Reconstruction of a Sinusoidal Signal

If we sample the continuous-time signal $x_c(t) = \cos(4000\pi t)$ with sampling period $T = 1/6000$, we obtain $x[n] = x_c(nT) = \cos(4000\pi Tn) = \cos(\omega_0 n)$, where $\omega_0 = 4000\pi T = 2\pi/3$. In this case, $\Omega_s = 2\pi/T = 12000\pi$, and the highest frequency of the signal is $\Omega_0 = 4000\pi$, so the conditions of the Nyquist sampling theorem are satisfied and there is no aliasing. The Fourier transform of $x_c(t)$ is

$$X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi).$$



(a)



(b)

From Eqs. (4.18)–(4.20), we see that $X(e^{j\omega})$ is simply a frequency-scaled version of $X_s(j\Omega)$ with the frequency scaling specified by $\omega = \Omega T$. This scaling can alternatively be thought of as a normalization of the frequency axis so that the frequency $\Omega = \Omega_s$ in $X_s(j\Omega)$ is normalized to $\omega = 2\pi$ for $X(e^{j\omega})$. The fact that there is a frequency scaling or normalization in the transformation from $X_s(j\Omega)$ to $X(e^{j\omega})$ is directly associated with the fact that there is a time normalization in the transformation from $x_s(t)$ to $x[n]$.

, where we have used the fact that scaling the independent variable of an impulse also scales its area, i.e., $\delta(\omega/T) = T\delta(\omega)$.

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

RECONSTRUCTION OF A BANDLIMITED SIGNAL FROM ITS SAMPLES

If we are given a sequence of samples, $x[n]$, we can form an impulse train $x_s(t)$ in which successive impulses are assigned an area equal to successive sequence values, i.e.,

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT). \tag{4.22}$$

The nth sample is associated with the impulse at $t = nT$, where T is the sampling period associated with the sequence $x[n]$. If this impulse train is the input to an ideal lowpass continuous-time filter with frequency response $H_r(j\Omega)$ and impulse response $h_r(t)$, then the output of the filter will be

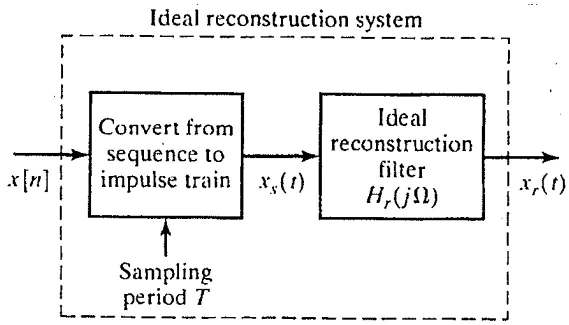
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT). \tag{4.23}$$

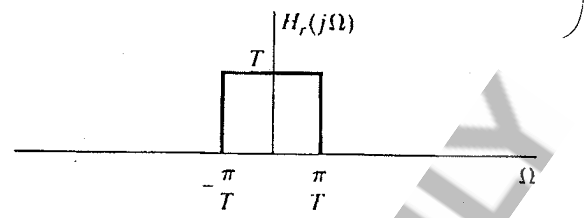
The corresponding impulse response, $h_r(t)$, is the inverse Fourier transform of $H_r(j\Omega)$, and for cutoff frequency π/T it is given by

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}. \tag{4.24}$$

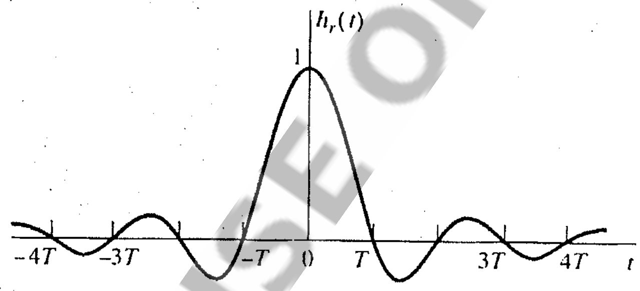
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}. \tag{4.25}$$

From the frequency-domain argument of Section 4.2, we saw that if $x[n] = x_c(nT)$, where $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, then $x_r(t)$ is equal to $x_c(t)$.

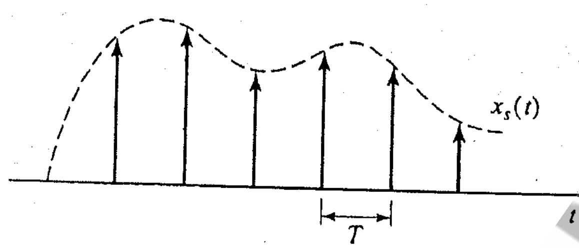


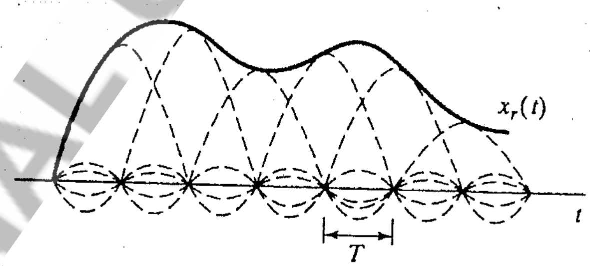


(b)

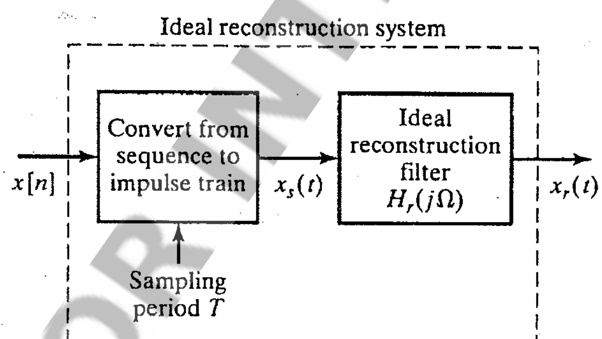


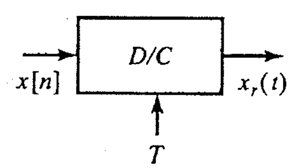
(c)





As suggested by this figure, the ideal lowpass filter *interpolates* between the impulses of $x_s(t)$ to construct a continuous-time signal $x_r(t)$.





It is useful to formalize the preceding discussion by defining an ideal system for reconstructing a bandlimited signal from a sequence of samples. We will call this system the *ideal discrete-to-continuous-time (D/C) converter*.

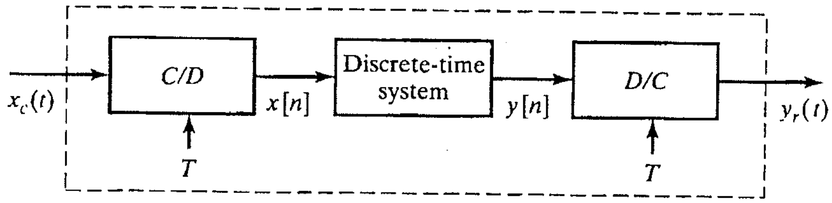
The properties of the ideal D/C converter are most easily seen in the frequency domain. To derive an input/output relation in this domain, consider the Fourier transform

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] H_r(j\Omega) e^{-j\Omega T n}.$$

By factoring $H_r(j\Omega)$ out of the sum, we can write

$$X_r(j\Omega) = H_r(j\Omega) X(e^{j\Omega T}). \tag{4.28}$$

Equation (4.28) provides a frequency-domain description of the ideal D/C converter. According to Eq. (4.28), $X(e^{j\omega})$ is frequency scaled (i.e., ω is replaced by ΩT). The ideal lowpass filter $H_r(j\Omega)$ selects the base period of the resulting periodic Fourier transform $X(e^{j\Omega T})$ and compensates for the $1/T$ scaling inherent in sampling. Thus, if the sequence $x[n]$ has been obtained by sampling a bandlimited signal at the Nyquist rate or higher, then the reconstructed signal $x_r(t)$ will be equal to the original bandlimited signal.



If the discrete-time system in Figure is linear and time invariant, we then have

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \tag{4.33}$$

where $H(e^{j\omega})$ is the frequency response of the system or, equivalently, the Fourier transform of the unit sample response, and $X(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier transforms of the input and output, respectively. Combining

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T}). \tag{4.34}$$

$$Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T})$$
$$= \begin{cases} TY(e^{j\Omega T}), & |\Omega| < \pi/T, \\ 0, & \text{otherwise.} \end{cases}$$

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right). \tag{4.35}$$

If $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, then the ideal lowpass reconstruction filter $H_r(j\Omega)$ cancels the factor $1/T$ and selects only the term in Eq. (4.35) for $k = 0$; i.e.,

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega), & |\Omega| < \pi/T, \\ 0, & |\Omega| \geq \pi/T. \end{cases} \tag{4.36}$$

Thus, if $X_c(j\Omega)$ is bandlimited and the sampling rate is above the Nyquist rate, the output is related to the input through an equation of the form

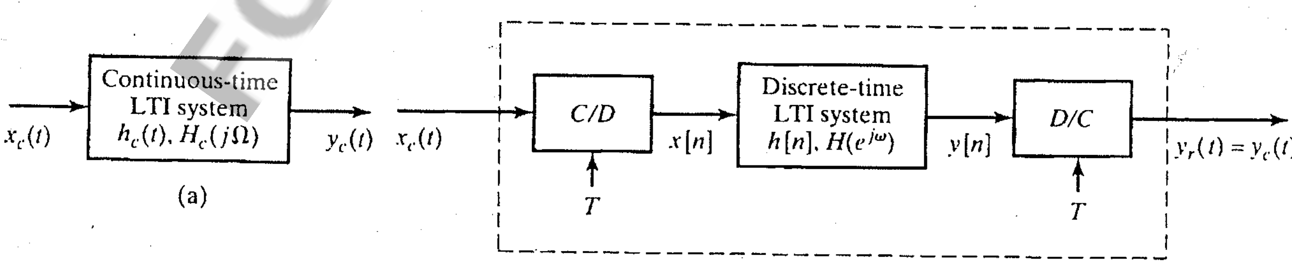
$$Y_r(j\Omega) = H_{\text{eff}}(j\Omega)X_c(j\Omega), \tag{4.37}$$

where

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T, \\ 0, & |\Omega| \geq \pi/T. \end{cases} \tag{4.38}$$

That is, the overall continuous-time system is equivalent to a linear time-invariant system whose *effective* frequency response is given by Eq. (4.38).

Impulse Invariance



(a)

$H_{\text{eff}}(j\Omega) = H_c(j\Omega)$

(b)

With $H_c(j\Omega)$ bandlimited, Eq. (4.38) specifies how to choose $H(e^{j\omega})$ so that $H_{\text{eff}}(j\Omega) = H_c(j\Omega)$. Specifically,

$$H(e^{j\omega}) = H_c(j\omega/T), \qquad |\omega| < \pi, \tag{4.49}$$

with the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \qquad |\Omega| \geq \pi/T. \tag{4.50}$$

Under the constraints of Eqs. (4.49) and (4.50), there is also a straightforward and useful relationship between the continuous-time impulse response $h_c(t)$ and the discrete-time impulse response $h[n]$. In particular, as we shall verify shortly,

$$h[n] = Th_c(nT); \tag{4.51}$$

i.e., the impulse response of the discrete-time system is a scaled, sampled version of $h_c(t)$. When $h[n]$ and $h_c(t)$ are related through Eq. (4.51), the discrete-time system is said to be an *impulse-invariant* version of the continuous-time system.

A Discrete-Time Lowpass Filter Obtained By Impulse Invariance

Suppose that we wish to obtain an ideal lowpass discrete-time filter with cutoff frequency $\omega_c < \pi$. We can do this by sampling a continuous-time ideal lowpass filter with cutoff frequency $\Omega_c = \omega_c/T < \pi/T$ defined by

$$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c, \\ 0, & |\Omega| \leq \Omega_c. \end{cases}$$

The impulse response of this continuous-time system is

$$h_c(t) = \frac{\sin(\Omega_c t)}{\pi t},$$

so we define the impulse response of the discrete-time system to be

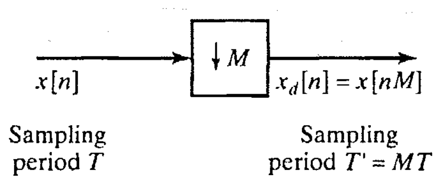
$$h[n] = Th_c(nT) = T\frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n},$$

where $\omega_c = \Omega_c T$. We have already shown that this sequence corresponds to the discrete-time Fourier transform

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi, \end{cases}$$

which is identical to $H_c(j\omega/T)$, as predicted by Eq. (4.56).

CHANGING THE SAMPLING RATE USING DISCRETE-TIME PROCESSING



Sampling Rate Reduction by an Integer Factor

The sampling rate of a sequence can be reduced by "sampling" it, i.e., by defining a new sequence

$$x_d[n] = x[nM] = x_c(nMT). \tag{4.71}$$

sampling with period $T' = MT$. Furthermore, if $X_c(j\Omega) = 0$ for $|\Omega| \geq \Omega_N$, then $x_d[n]$ is an exact representation of $x_c(t)$ if $\pi/T' = \pi/(MT) \geq \Omega_N$. That is, the sampling rate can be reduced by a factor of M without aliasing if the original sampling rate was at least M times the Nyquist rate or if the bandwidth of the sequence is first reduced by a factor of M by discrete-time filtering. In general, the operation of reducing the sampling rate (including any prefiltering) will be called *downsampling*.

First recall that the discrete-time Fourier transform of $x[n] = x_c(nT)$ is

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right). \tag{4.72}$$

Similarly, the discrete-time Fourier transform of $x_d[n] = x[nM] = x_c(nT')$ with $T' = MT$ is

$$X_d(e^{j\omega}) = \frac{1}{T'}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T'} - \frac{2\pi r}{T'}\right)\right). \tag{4.73}$$

Now, since $T' = MT$, we can write Eq. (4.73) as

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right). \tag{4.74}$$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi k}{T} - \frac{2\pi i}{MT}\right)\right)\right]. \tag{4.76}$$

The term inside the square brackets in Eq. (4.76) is recognized from Eq. (4.72) as

$$X(e^{j(\omega-2\pi i)/M}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi k}{T}\right)\right). \tag{4.77}$$

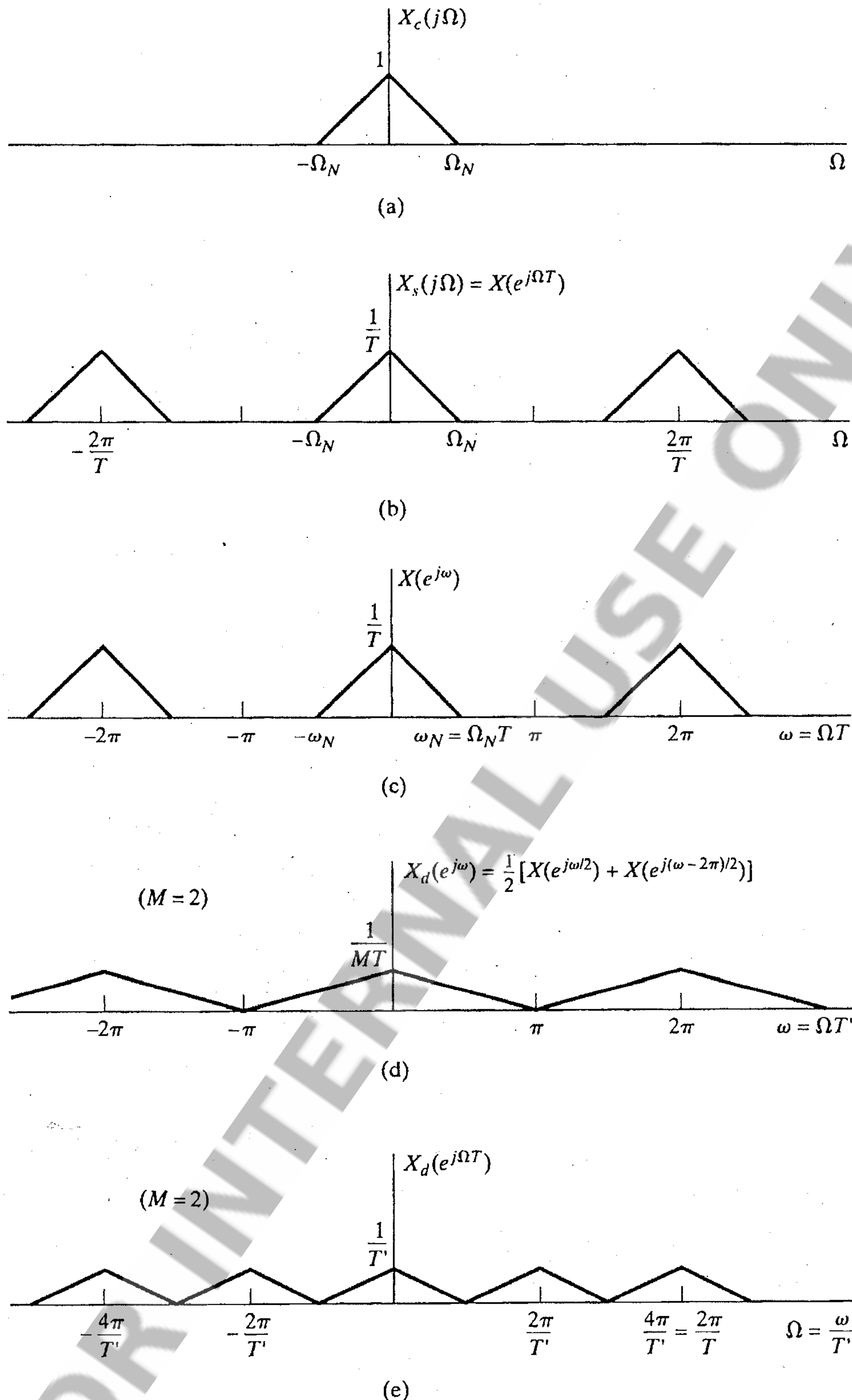
Thus, we can express Eq. (4.76) as

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\omega/M - 2\pi i/M)}). \tag{4.78}$$

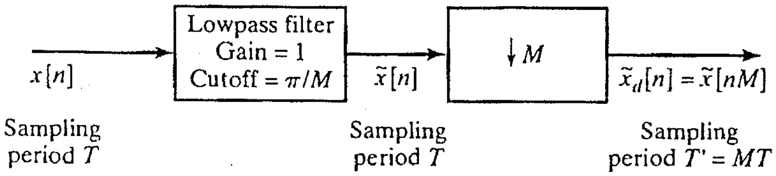
If we compare Eqs. (4.73) and (4.78), we see that $X_d(e^{j\omega})$ can be thought of as being composed of either an infinite set of copies of $X_c(j\Omega)$, frequency scaled through $\omega = \Omega T'$ and shifted by integer multiples of $2\pi/T'$ (Eq. (4.73)), or M copies of the periodic Fourier transform $X(e^{j\omega})$, frequency scaled by M and shifted by integer multiples of 2π (Eq. (4.78)). Either interpretation makes it clear that $X_d(e^{j\omega})$ is periodic with period 2π (as are all discrete-time Fourier transforms) and that aliasing can be avoided by ensuring that $X(e^{j\omega})$ is bandlimited, i.e.,

$$X(e^{j\omega}) = 0, \qquad \omega_N \leq |\omega| \leq \pi, \tag{4.79}$$

and $2\pi/M \geq 2\omega_N$.



In this example, $2\pi/T = 4\Omega_N$; i.e., the original sampling rate is exactly twice the minimum rate to avoid aliasing. Thus, when the original sampled sequence is down-sampled by a factor of $M = 2$, no aliasing results.



From the preceding discussion, we see that a general system for downsampling by a factor of M is the one shown in Figure . Such a system is called a *decimator,* and downsampling by lowpass filtering followed by compression has been termed *decimation* (Crochiere and Rabiner, 1983).

Increasing the Sampling Rate by an Integer Factor

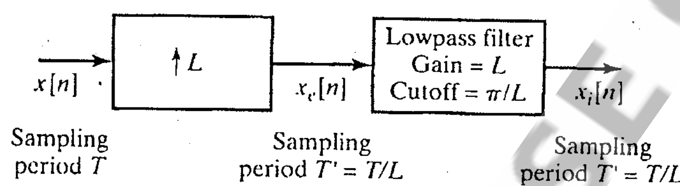


Figure shows a system for obtaining $x_i[n]$ from $x[n]$ using only discrete-time processing. The system on the left is called a *sampling rate expander* (see Crochiere and Rabiner, 1983) or simply an *expander*. Its output is

$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots, \\ 0, & \text{otherwise}, \end{cases} \tag{4.84}$$

or equivalently,

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]. \tag{4.85}$$

The Fourier transform of $x_e[n]$ can be expressed as

$$\begin{aligned} X_e(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-kL] \right) e^{-j\omega n} \\ &= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega Lk} = X(e^{j\omega L}). \end{aligned} \tag{4.86}$$

Thus, the Fourier transform of the output of the expander is a frequency-scaled version of the Fourier transform of the input; i.e., ω is replaced by ωL so that ω is now normalized by

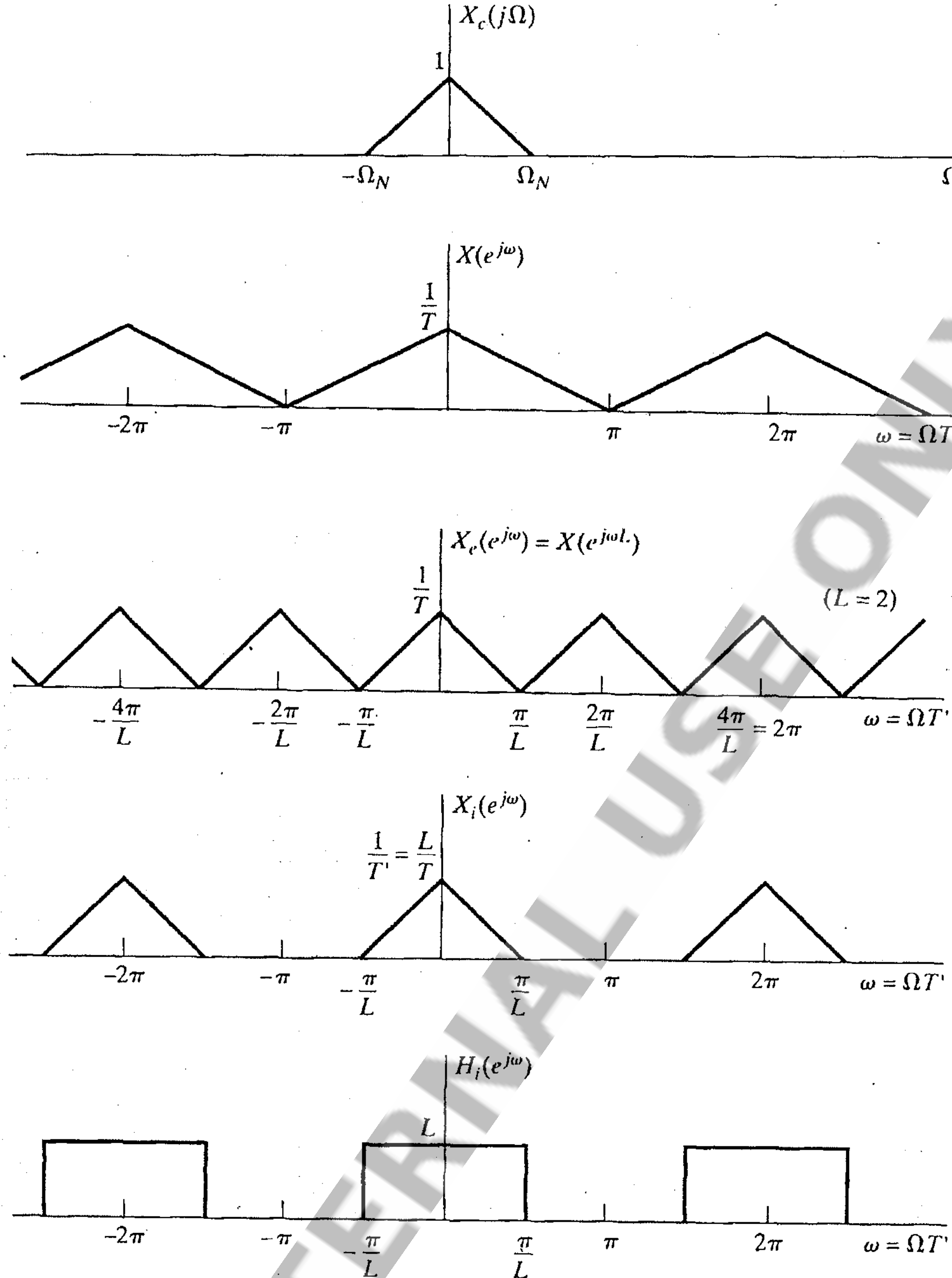
$$\omega = \Omega T'. \tag{4.87}$$

As in the case of the D/C converter, it is possible to obtain an interpolation formula for $x_i[n]$ in terms of $x[n]$. First note that the impulse response of the lowpass filter is

$$h_i[n] = \frac{\sin(\pi n/L)}{\pi n/L}. \tag{4.88}$$

Using Eq. (4.85), we obtain

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin[\pi(n-kL)/L]}{\pi(n-kL)/L}. \tag{4.89}$$

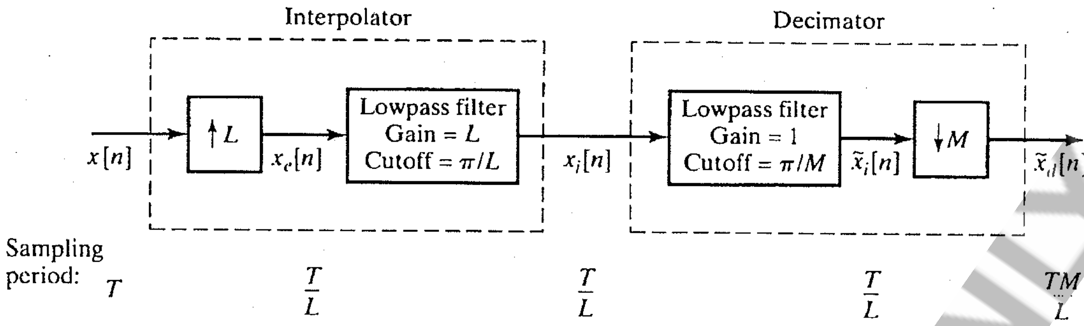


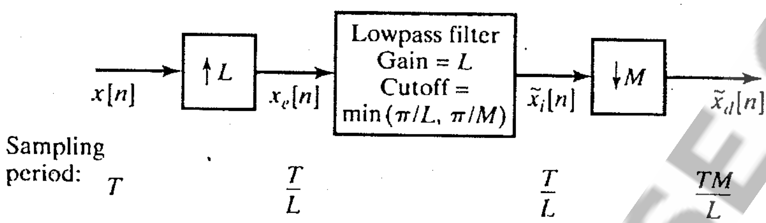
We see that $X_i(e^{j\omega})$ can be obtained from $X_e(e^{j\omega})$ by correcting the amplitude scale from $1/T$ to $1/T'$ and by removing all the frequency-scaled images of $X_c(j\Omega)$ except at integer multiples of 2π.

In general, the required gain would be L, since $L(1/T) = [1/(T/L)] = 1/T'$, and the cutoff frequency would be π/L.

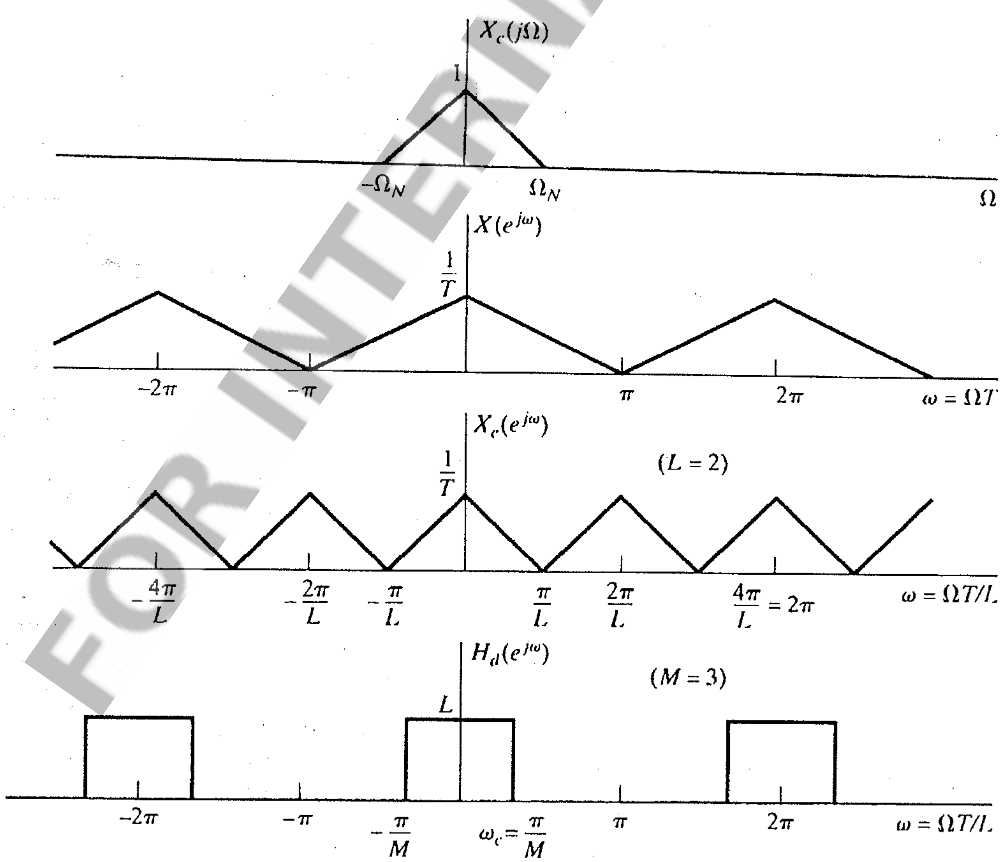
That system is therefore called an *interpolator*, since it fills in the missing samples, and the operation of upsampling is therefore considered to be synonymous with *interpolation*.

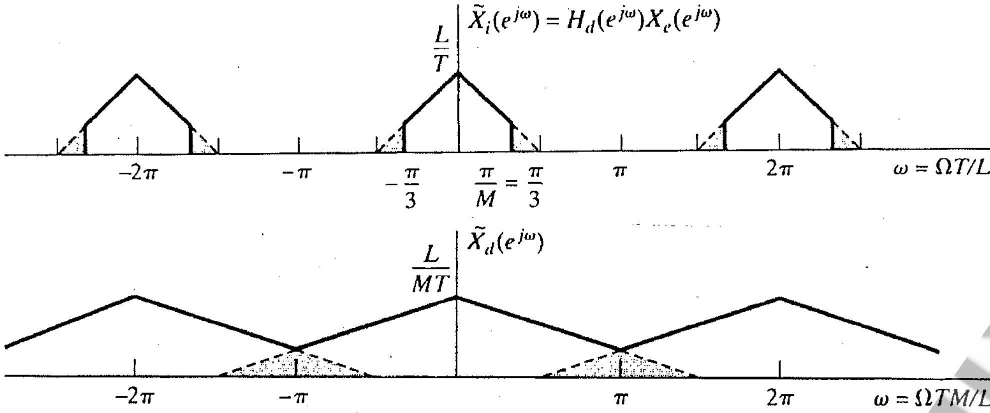
Changing the Sampling Rate by a Noninteger Factor





If we wish to change the sampling period to $T' = (3/2)T$, we must first interpolate by a factor $L = 2$ and then decimate by a factor of $M = 3$. Since this implies a net decrease in sampling rate, and the original signal was sampled at the Nyquist rate, we must incorporate additional lowpass filtering in order to avoid aliasing. If we were interested only in interpolating by a factor of 2, we could choose the lowpass filter to have a cutoff frequency of $\omega_c = \pi/2$ and a gain of $L = 2$. However, since the output of the filter will be decimated by $M = 3$, we must use a cutoff frequency of $\omega_c = \pi/3$, but the gain of the filter should still be 2 as in Figure

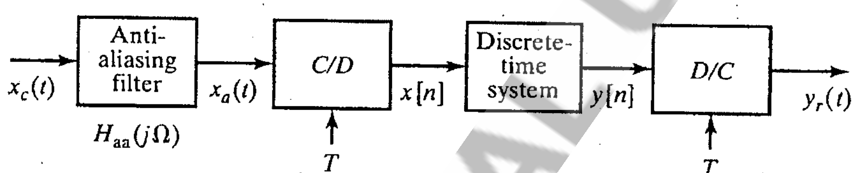




Note that the shaded regions show the aliasing that would have occurred if the cutoff frequency of the interpolation lowpass filter had been $\pi/2$ instead of $\pi/3$.

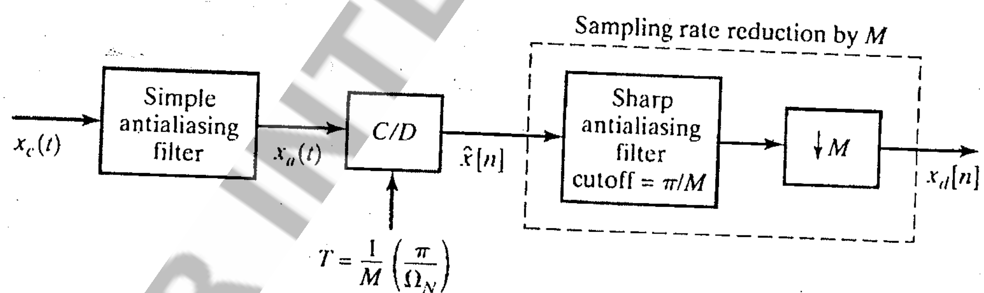
Prefiltering to Avoid Aliasing

In processing analog signals using discrete-time systems, it is generally desirable to minimize the sampling rate. This is because the amount of arithmetic processing required to implement the system is proportional to the number of samples to be processed. If the input is not bandlimited or if the Nyquist frequency of the input is too high, prefiltering may be necessary.



In this context, the lowpass filter that precedes the C/D converter is called an *antialiasing filter*. Ideally, the frequency response of the antialiasing filter would be

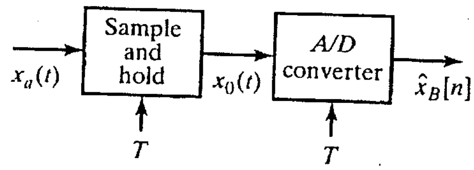
$$H_{aa}(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c < \pi/T, \\ 0, & |\Omega| > \Omega_c. \end{cases} \tag{4.108}$$



With Ω_N denoting the highest frequency component to eventually be retained after the antialiasing filtering is completed, we first apply a very simple antialiasing filter that has a gradual cutoff with significant attenuation at $M\Omega_N$. Next, implement the C/D conversion at a sampling rate much higher than $2\Omega_N$, e.g., at $2M\Omega_N$. After that, sampling rate reduction by a factor of M that includes sharp antialiasing filtering is implemented in the discrete-time domain. Subsequent discrete-time processing can then be done at the low sampling rate to minimize computation.

Analog-to-Digital (A/D) Conversion

An ideal C/D converter converts a continuous-time signal into a discrete-time signal, where each sample is known with infinite precision. As an approximation to this for digital signal processing, the system of Figure converts a continuous-time (analog) signal into a digital signal, i.e., a sequence of finite-precision or quantized samples.



However, the conversion is not instantaneous, and for this reason, a high-performance A/D system typically includes a sample-and-hold. The ideal sample-and-hold system is the system whose output is

$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t - nT), \tag{4.111}$$

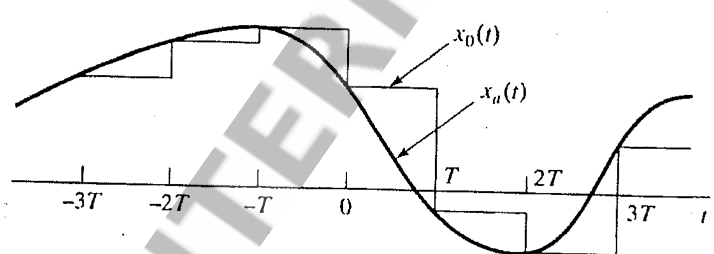
where $x[n] = x_a(nT)$ are the ideal samples of $x_a(t)$ and $h_0(t)$ is the impulse response of the zero-order-hold system, i.e.,

$$h_0(t) = \begin{cases} 1, & 0 < t < T, \\ 0, & \text{otherwise.} \end{cases} \tag{4.112}$$

If we note that Eq. (4.111) has the equivalent form

$$x_0(t) = h_0(t) * \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t - nT), \tag{4.113}$$

we see that the ideal sample-and-hold is equivalent to impulse train modulation followed by linear filtering with the zero-order-hold system.

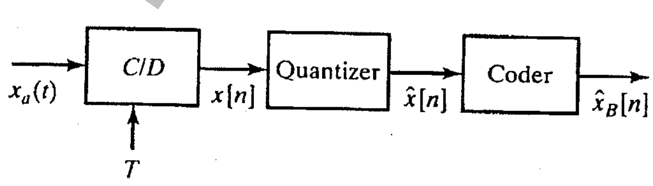


Analysis of Quantization Errors

The quantizer is a nonlinear system whose purpose is to transform the input sample $x[n]$ into one of a finite set of prescribed values. We represent this operation as

$$\hat{x}[n] = Q(x[n]) \tag{4.114}$$

and refer to $\hat{x}[n]$ as the quantized sample. Quantizers can be defined with either uniformly or nonuniformly spaced quantization levels; however, when numerical calculations are to be done on the samples, the quantization steps usually are uniform.



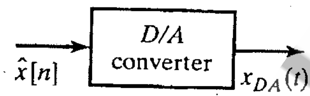
From Figure we see that the quantized sample $\hat{x}[n]$ will generally be different from the true sample value $x[n]$. The difference between them is the *quantization error*, defined as

$$e[n] = \hat{x}[n] - x[n]. \tag{4.117}$$

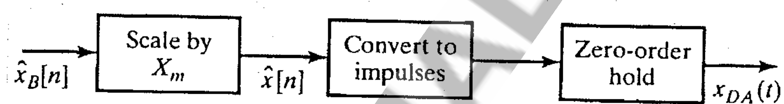
A simplified, but useful, model of the quantizer is depicted in Figure In this model, the quantization error samples are thought of as an additive noise signal. The model is exactly equivalent to the quantizer if we know $e[n]$. The statistical representation of quantization errors is based on the following assumptions:

1. The error sequence $e[n]$ is a sample sequence of a stationary random process.
2. The error sequence is uncorrelated with the sequence $x[n]$.
3. The random variables of the error process are uncorrelated; i.e., the error is a white-noise process.
4. The probability distribution of the error process is uniform over the range of quantization error.

D/A Conversion



(a)



(b)

In terms of Fourier transforms, the reconstruction is represented as

$$X_r(j\Omega) = X(e^{j\Omega T})H_r(j\Omega), \tag{4.127}$$

where $X(e^{j\omega})$ is the discrete-time Fourier transform of the sequence of samples and $X_r(j\Omega)$ is the Fourier transform of the reconstructed continuous-time signal. The ideal reconstruction filter is

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| < \pi/T, \\ 0, & |\Omega| > \pi/T. \end{cases} \tag{4.128}$$

$$\begin{aligned} x_{DA}(t) &= \sum_{n=-\infty}^{\infty} X_m \hat{x}_B[n] h_0(t - nT) \\ &= \sum_{n=-\infty}^{\infty} \hat{x}[n] h_0(t - nT), \end{aligned} \tag{4.130}$$

To simplify our discussion, we define

$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t-nT), \tag{4.132}$$

$$e_0(t) = \sum_{n=-\infty}^{\infty} e[n]h_0(t-nT), \tag{4.133}$$

so that Eq. (4.131) can be written as

$$x_{DA}(t) = x_0(t) + e_0(t). \tag{4.134}$$

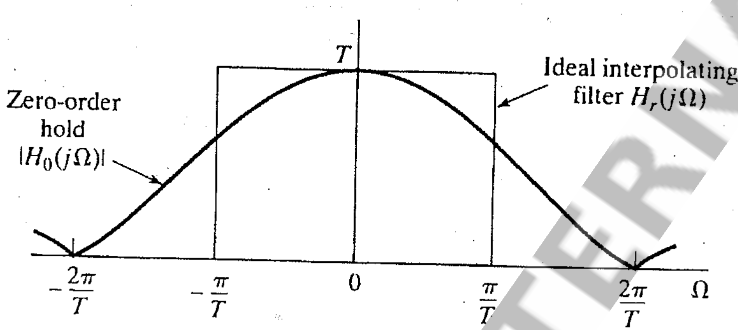
The signal component $x_0(t)$ is related to the input signal $x_a(t)$, since $x[n] = x_a(nT)$. The noise signal $e_0(t)$ depends on the quantization-noise samples $e[n]$ in the same way that $x_0(t)$ depends on the unquantized signal samples.

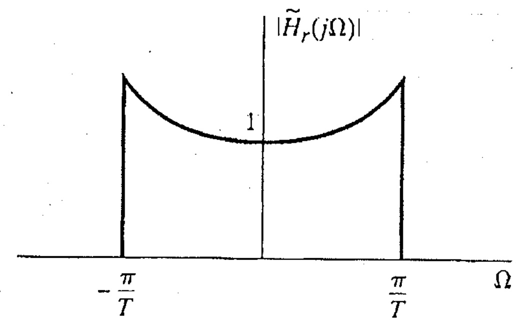
$$X_0(j\Omega) = \left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)\right] H_0(j\Omega). \tag{4.137}$$

if we define a compensated reconstruction filter as

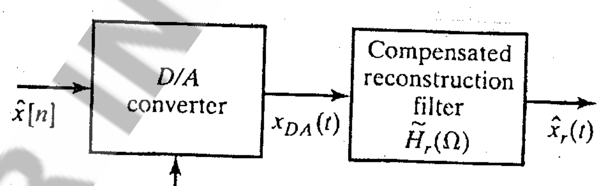
$$\tilde{H}_r(j\Omega) = \frac{H_r(j\Omega)}{H_0(j\Omega)}, \tag{4.138}$$

then the output of the filter will be $x_a(t)$ if the input is $x_0(t)$. The frequency response of the zero-order-hold filter is easily shown to be





$$\tilde{H}_r(j\Omega) = \begin{cases} \dfrac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2}, & |\Omega| < \pi/T, \\ 0, & |\Omega| > \pi/T. \end{cases}$$



In other words, the output would be

$$\hat{x}_r(t) = x_a(t) + e_a(t), \tag{4.142}$$

where $e_a(t)$ would be a bandlimited white-noise signal.

$H(e^{j\Omega T})$ is the frequency response of the discrete-time system. Similarly, assuming that the quantization noise introduced by the A/D converter is white noise with variance $\sigma_e^2 = \Delta^2/12$, it can be shown that the power spectrum of the output noise is

$$P_{e_a}(j\Omega) = |\tilde{H}_r(j\Omega)H_0(j\Omega)H(e^{j\Omega T})|^2\sigma_e^2, \tag{4.145}$$

i.e., the input quantization noise is changed by the successive stages of discrete- and continuous-time filtering.

TRANSFORM ANALYSIS OF LINEAR TIME-INVARIANT SYSTEMS

LTI system can be completely characterized in the time domain by its impulse response $h[n]$, with the output $y[n]$ due to a given input $x[n]$ specified through the convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \tag{5.1}$$

transform, and we showed that $Y(z)$, the z-transform of the output of an LTI system, is related to $X(z)$, the z-transform of the input, and $H(z)$, the z-transform of the system impulse response, by

$$Y(z) = H(z)X(z), \tag{5.2}$$

$H(z)$ is referred to as the *system function.* Since the z-transform and a sequence form a unique pair, it follows that any LTI system is completely characterized by its system function, again assuming convergence.

THE FREQUENCY RESPONSE OF LTI SYSTEMS

The frequency response $H(e^{j\omega})$ of an LTI system was defined as the complex gain (eigenvalue) that the system applies to the complex exponential input (eigenfunction) $e^{j\omega n}$. Furthermore,

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \tag{5.3}$$

where $X(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier transforms of the system input and output, respectively. With the frequency response expressed in polar form, the magnitude and phase of the Fourier transforms of the system input and output are related by

$$|Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})|, \tag{5.4a}$$

$$\sphericalangle Y(e^{j\omega}) = \sphericalangle H(e^{j\omega}) + \sphericalangle X(e^{j\omega}). \tag{5.4b}$$

$|H(e^{j\omega})|$ is referred to as the *magnitude* response or the *gain* of the system, and $\sphericalangle H(e^{j\omega})$ is referred to as the *phase response* or *phase shift* of the system. For example, the ideal lowpass filter was defined as the discrete-time linear time-invariant system whose frequency response is

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases} \tag{5.5}$$

The corresponding impulse response was shown in to be

$$h_{\text{lp}}[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty. \tag{5.6}$$

Analogously, the *ideal highpass filter* is defined as

$$H_{\text{hp}}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_c, \\ 1, & \omega_c < |\omega| \le \pi, \end{cases} \tag{5.7}$$

and since $H_{\text{hp}}(e^{j\omega}) = 1 - H_{\text{lp}}(e^{j\omega})$, its impulse response is

$$h_{\text{hp}}[n] = \delta[n] - h_{\text{lp}}[n] = \delta[n] - \frac{\sin \omega_c n}{\pi n}. \tag{5.8}$$

The ideal lowpass filters are noncausal, and their impulse responses extend from $-\infty$ to $+\infty$. Therefore, it is not possible to compute the output of either the ideal lowpass or the ideal highpass filter either recursively or nonrecursively; i.e., the systems are not *computationally realizable.*

Phase Distortion and Delay

To understand the effect of the phase of a linear system, let us first consider the ideal delay system.

$$|H_{\mathrm{id}}(e^{j\omega})| = 1, \tag{5.11a}$$

$$\sphericalangle H_{\mathrm{id}}(e^{j\omega}) = -\omega n_d, \qquad |\omega| < \pi, \tag{5.11b}$$

with periodicity 2π in ω assumed. For now, we will assume that n_d is an integer.

In many applications, delay distortion would be considered a rather mild form of phase distortion, since its effect is only to shift the sequence in time.

For example, an ideal lowpass filter with linear phase would be defined as

$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi. \end{cases} \tag{5.12}$$

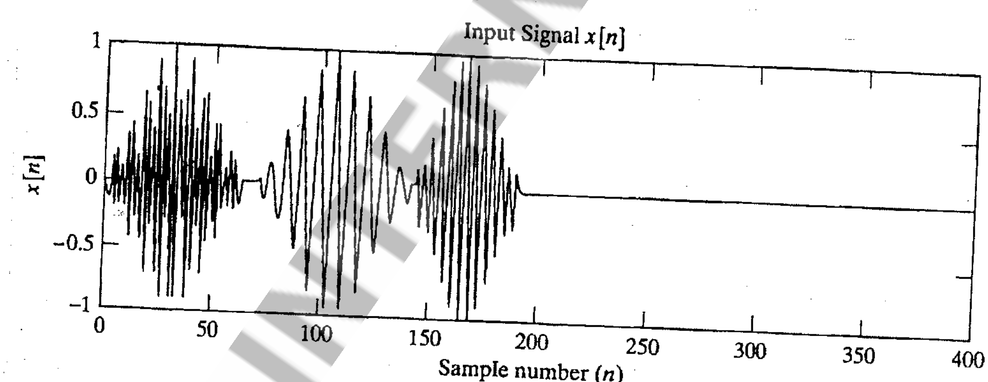
Its impulse response is

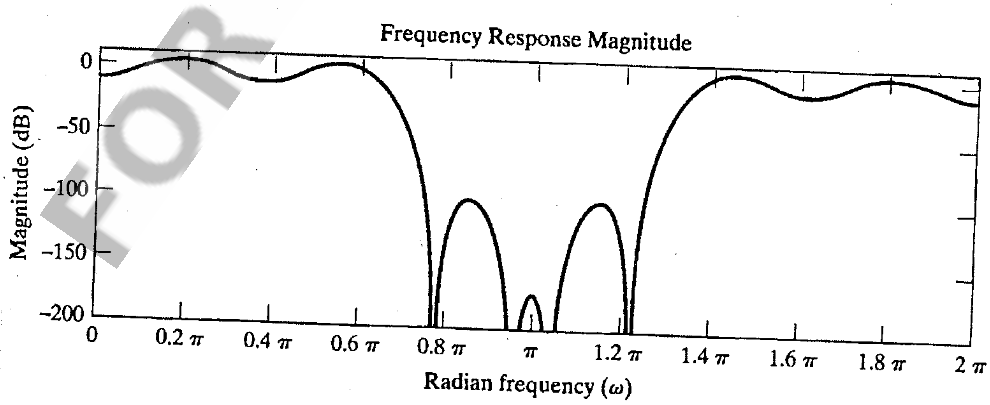
$$h_{\mathrm{lp}}[n] = \frac{\sin \omega_c(n - n_d)}{\pi(n - n_d)}, \qquad -\infty < n < \infty. \tag{5.13}$$

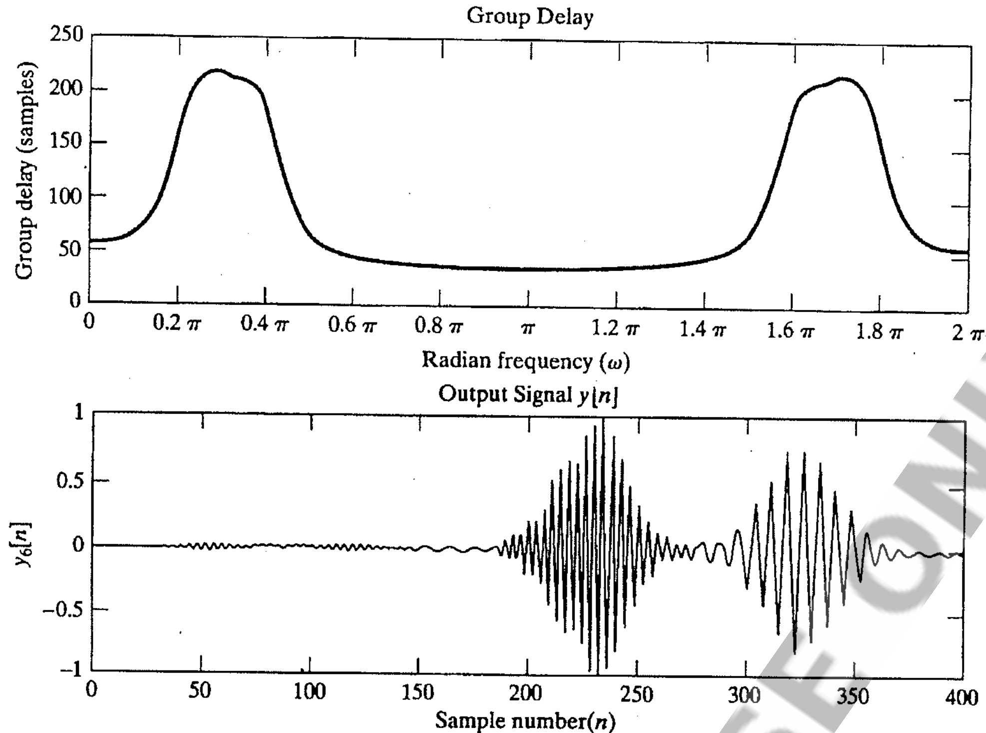
With phase specified as a continuous function of ω, the group delay of a system is defined as

$$\tau(\omega) = \mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}. \tag{5.15}$$

The deviation of the group delay from a constant indicates the degree of nonlinearity of the phase.







SYSTEM FUNCTIONS FOR SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

While ideal frequency-selective filters are useful conceptually, they cannot be implemented with finite computation. Therefore, it is of interest to consider a class of systems that can be implemented as approximations to ideal frequency-selective filters.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z),$$

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z).$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}. \tag{5.19}$$

Each of the factors $(1 - c_k z^{-1})$ in the numerator contributes a zero at $z = c_k$ and a pole at $z = 0$. Similarly, each of the factors $(1 - d_k z^{-1})$ in the denominator contributes a zero at $z = 0$ and a pole at $z = d_k$.

Suppose that the system function of a linear time-invariant system is

$$H(z) = \frac{(1+z^{-1})^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)}. \tag{5.20}$$

$$H(z) = \frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}} = \frac{Y(z)}{X(z)}.$$

Thus,

$$\left(1+\tfrac{1}{4}z^{-1}-\tfrac{3}{8}z^{-2}\right)Y(z) = (1+2z^{-1}+z^{-2})X(z),$$

and the difference equation is

$$y[n]+\tfrac{1}{4}y[n-1]-\tfrac{3}{8}y[n-2] = x[n]+2x[n-1]+x[n-2].$$

Stability and Causality

For a given ratio of polynomials, each possible choice for the region of convergence will lead to a different impulse response, but they will all correspond to the same difference equation. However, if we assume that the system is causal, it follows that $h[n]$ must be a right-sided sequence, and therefore, the region of convergence of $H(z)$ must be outside the outermost pole. Alternatively, if we assume that the system is stable, then

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \tag{5.23}$$

Since Eq. (5.23) is identical to the condition that

$$\sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty \tag{5.24}$$

for $|z| = 1$, the condition for stability is equivalent to the condition that the ROC of $H(z)$ include the unit circle.

In order for a linear time-invariant system whose input and output satisfy a difference equation of the form of Eq. (5.16) to be both causal and stable, the ROC of the corresponding system function must be outside the outermost pole *and* include the unit circle. Clearly, this requires that all the poles of the system function be inside the unit circle.

Inverse Systems

For a given linear time-invariant system with system function $H(z)$, the corresponding inverse system is defined to be the system with system function $H_i(z)$ such that if it is cascaded with $H(z)$, the overall effective system function is unity; i.e.,

$$G(z) = H(z)H_i(z) = 1. \tag{5.27}$$

Many systems do have inverses, and the class of systems with rational system functions provides a very useful and interesting example. Specifically, consider

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}, \tag{5.31}$$

with zeros at $z = c_k$ and poles at $z = d_k$, in addition to possible zeros and/or poles at $z = 0$ and $z = \infty$.

Then

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}; \tag{5.32}$$

i.e., the poles of $H_i(z)$ are the zeros of $H(z)$ and vice versa.

Let $H(z)$ be

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}$$

with ROC $|z| > 0.9$. Then $H_i(z)$ is

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}.$$

Since $H_i(z)$ has only one pole, there are only two possibilities for its ROC, and the only choice for the ROC of $H_i(z)$ that overlaps with $|z| > 0.9$ is $|z| > 0.5$. Therefore, the impulse response of the inverse system is

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1}u[n-1].$$

In this case, the inverse system is both causal and stable.

A generalization is that if $H(z)$ is a causal system with zeros at c_k, $k = 1, \ldots, M$, then its inverse system will be causal if and only if we associate the region of convergence,

$$|z| > \max_k |c_k|,$$

with $H_i(z)$. If we also require that the inverse system be stable, then the region of convergence of $H_i(z)$ must include the unit circle. Therefore, it must be true that

$$\max_k |c_k| < 1;$$

i.e., all the zeros of $H(z)$ must be inside the unit circle. Thus, a linear time-invariant system is stable and causal and also has a stable and causal inverse if and only if both the poles and the zeros of $H(z)$ are inside the unit circle. Such systems are referred to as *minimum-phase* systems.

Recall that any rational function of z^{-1} with only first-order poles can be expressed in the form

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}, \tag{5.34}$$

where the terms in the first summation would be obtained by long division of the denominator into the numerator and would be present only if $M \geq N$.

If the system is assumed to be causal, then the ROC is outside all of the poles in Eq. (5.34), and it follows that

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n], \tag{5.35}$$

where the first summation is included only if $M \geq N$.

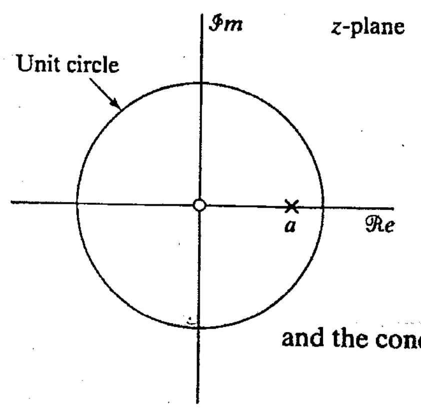
A First-Order IIR System

Consider a causal system whose input and output satisfy the difference equation

$$y[n] - ay[n-1] = x[n]. \tag{5.36}$$

The system function is (by inspection)

$$H(z) = \frac{1}{1 - az^{-1}}. \tag{5.37}$$



The region of convergence is $|z| > |a|$, and the condition for stability is $|a| < 1$. The inverse z-transform of $H(z)$ is

$$h[n] = a^n u[n]. \tag{5.38}$$

For the second class of systems, $H(z)$ has no poles except at $z = 0$; i.e., $N = 0$,

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}. \tag{5.39}$$

(We assume, without loss of generality, that $a_0 = 1$.) In this case, $H(z)$ is determined to within a constant multiplier by its zeros. From Eq. (5.39), $h[n]$ is seen by inspection to be

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n, & 0 \leq n \leq M, \\ 0, & \text{otherwise.} \end{cases} \tag{5.40}$$

In this case, the impulse response is finite in length; i.e., it is zero outside a finite interval. Consequently, these systems are called *finite impulse response* (FIR) systems. Note that for FIR systems, the difference equation of Eq. (5.16) is identical to the convolution sum, i.e.,

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]. \tag{5.41}$$

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}. \qquad H(e^{j\omega}) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})}.$$

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}|1 - c_k e^{-j\omega}|}{\prod_{k=1}^{N}|1 - d_k e^{-j\omega}|}.$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}),$$

$$|H(e^{j\omega})|^2 = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})(1 - c_k^* e^{j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})(1 - d_k^* e^{j\omega})}.$$

The function $20\log_{10}|H(e^{j\omega})|$ is referred to as the *log magnitude* of $H(e^{j\omega})$ and is expressed in *decibels* (dB). Sometimes this quantity is called the *gain in dB;* i.e.,

$$\text{Gain in dB} = 20\log_{10}|H(e^{j\omega})|. \tag{5.51}$$

When $|H(e^{j\omega})| < 1$, the quantity $20\log_{10}|H(e^{j\omega})|$ is negative. This would be the case, for example, in the stopband of a frequency-selective filter. It is common practice to define

$$\begin{aligned}\text{Attenuation in dB} &= -20\log_{10}|H(e^{j\omega})| \\ &= -\text{Gain in dB}.\end{aligned} \tag{5.52}$$

Another advantage to expressing the magnitude in decibels stems from Eq. (5.4a), which, after taking logarithms of both sides, becomes

$$20\log_{10}|Y(e^{j\omega})| = 20\log_{10}|H(e^{j\omega})| + 20\log_{10}|X(e^{j\omega})|. \tag{5.53}$$

The phase response for a rational system function has the form

$$\sphericalangle H(e^{j\omega}) = \sphericalangle\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M}\sphericalangle[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N}\sphericalangle[1 - d_k e^{-j\omega}]. \tag{5.54}$$

The corresponding group delay for a rational system function is

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{N}\frac{d}{d\omega}(\arg[1 - d_k e^{-j\omega}]) - \sum_{k=1}^{M}\frac{d}{d\omega}(\arg[1 - c_k e^{-j\omega}]), \tag{5.55}$$

where arg[ ] represents the continuous phase.

When the angle of a complex number is computed, with the use of an arctangent subroutine on a calculator or with a computer system subroutine, the principal value is obtained. The principal value of the phase of $H(e^{j\omega})$ is denoted as $\mathrm{ARG}[H(e^{j\omega})]$, where

$$-\pi < \mathrm{ARG}[H(e^{j\omega})] \leq \pi. \tag{5.57}$$

Any other angle that gives the correct complex value of the function $H(e^{j\omega})$ can be represented in terms of the principal value as

$$\sphericalangle H(e^{j\omega}) = \mathrm{ARG}[H(e^{j\omega})] + 2\pi r(\omega), \tag{5.58}$$

where $r(\omega)$ is a positive or negative integer that can be different at each value of ω.

If the principal value is used to compute the phase response as a function of ω, then $\mathrm{ARG}[H(e^{j\omega})]$ may be a discontinuous function. The discontinuities introduced by taking the principal value will be jumps of 2π radians.

Except at the discontinuities of $\mathrm{ARG}[H(e^{j\omega})]$ corresponding to jumps between $+\pi$ and $-\pi$,

$$\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\} = \frac{d}{d\omega}\{\mathrm{ARG}[H(e^{j\omega})]\}. \tag{5.62}$$

Consequently, the group delay can be obtained from the principal value by differentiating, except at the discontinuities. Similarly, we can express the group delay in terms of the ambiguous phase $\sphericalangle H(e^{j\omega})$ as

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}[\sphericalangle H(e^{j\omega})], \tag{5.63}$$

with the interpretation that impulses caused by discontinuities of size 2π in $\sphericalangle H(e^{j\omega})$ are ignored.

The square of the magnitude of such a factor is

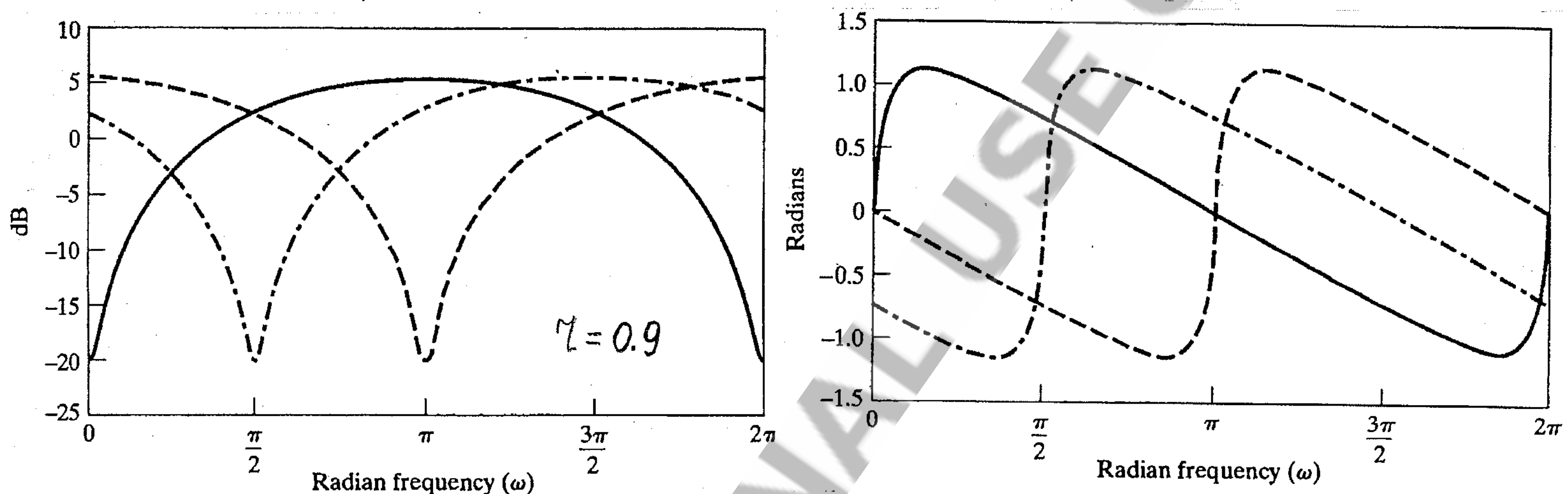
$$|1 - re^{j\theta}e^{-j\omega}|^2 = (1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{j\omega}) = 1 + r^2 - 2r\cos(\omega - \theta). \quad (5.64)$$

The principal value phase for such a factor is

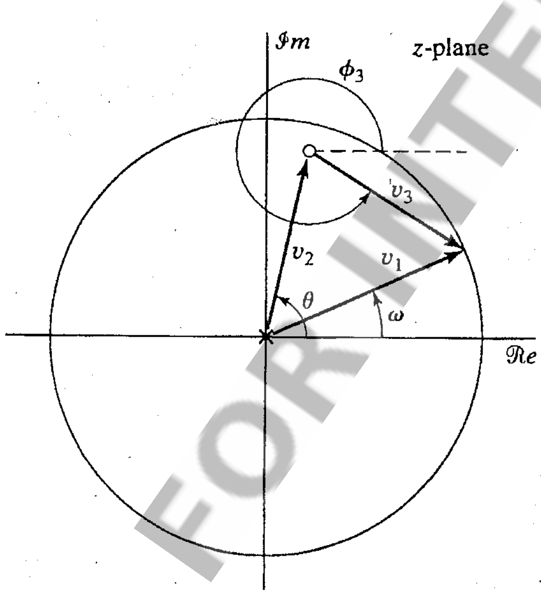
$$\text{ARG}[1 - re^{j\theta}e^{-j\omega}] = \arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]. \quad (5.66)$$

Differentiating the right-hand side of Eq. (5.66) (except at discontinuities) gives the group delay of the factor as

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega - \theta)}{1 + r^2 - 2r\cos(\omega - \theta)} = \frac{r^2 - r\cos(\omega - \theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}. \quad (5.67)$$



A simple geometric construction is often very useful in approximate sketching of frequency-response functions directly from the pole–zero plot.



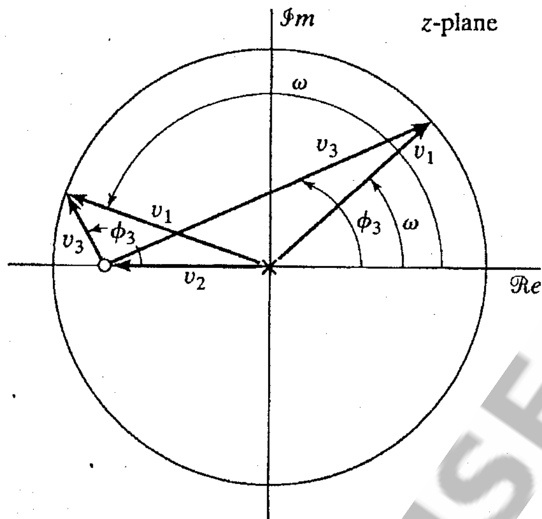
$$|1 - re^{j\theta}e^{-j\omega}| = \left|\frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}}\right| = \frac{|v_3|}{|v_1|}.$$

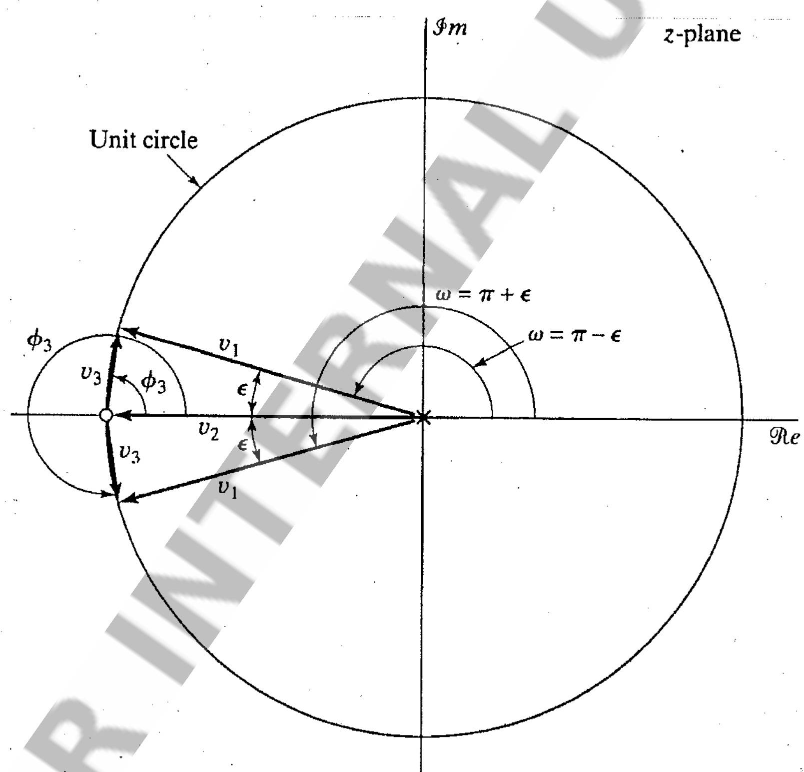
$$H(z) = (1 - re^{j\theta}z^{-1}) = \frac{(z - re^{j\theta})}{z}, \qquad r < 1.$$

The corresponding phase is

$$\begin{aligned}\sphericalangle(1 - re^{j\theta}e^{-j\omega}) &= \sphericalangle(e^{j\omega} - re^{j\theta}) - \sphericalangle(e^{j\omega}) = \sphericalangle(v_3) - \sphericalangle(v_1) \\ &= \phi_3 - \phi_1 = \phi_3 - \omega.\end{aligned} \quad (5.70)$$

Typically, a vector such as v_3 from a zero to the unit circle is referred to as a zero vector, and a vector from a pole to the unit circle is called a pole vector. Thus, the contribution of a single zero factor $(1 - re^{j\theta} z^{-1})$ to the magnitude function at frequency ω is the length of the zero vector v_3 from the zero to the point $z = e^{j\omega}$ on the unit circle. The vector has minimum length when $\omega = \theta$. This accounts for the sharp dip in the magnitude function at $\omega = \theta$. Note that the pole vector v_1 from the pole at $z = 0$ to $z = e^{j\omega}$ always has unit length. Thus, it does not have any effect on the magnitude response.

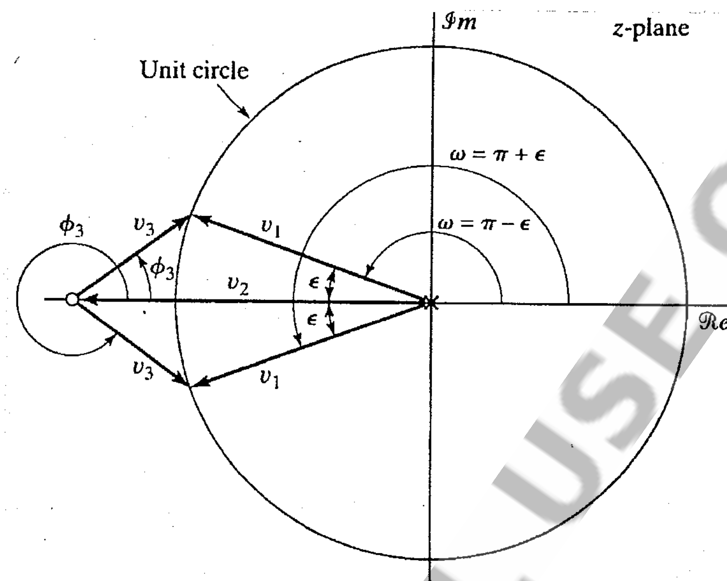




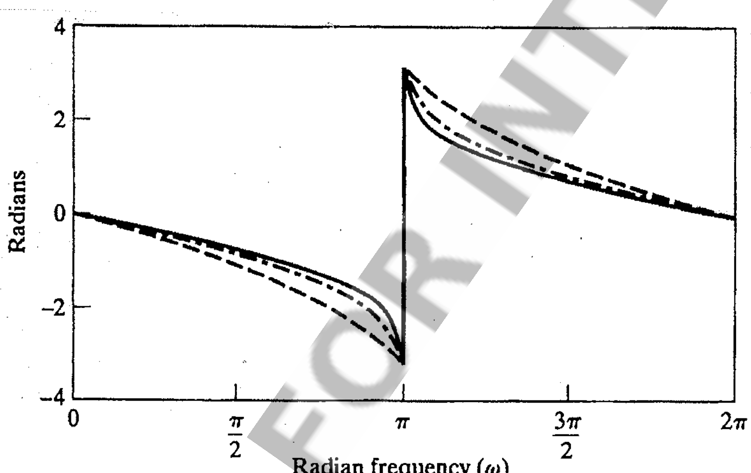
The geometric construction for a zero on the unit circle at $z = -1$ is shown. Indicated are vectors for two different frequencies, $\omega = (\pi - \varepsilon)$ and $\omega = (\pi + \varepsilon)$, where ε is small. Two observations can be made. First, the length of the vector v_3 approaches zero as ω approaches the angle of the zero vector ($\varepsilon \to 0$). Therefore, the multiplicative contribution to the frequency response is zero ($-\infty$ dB). Second, the vector v_3 changes its angle discontinuously by π radians as ω goes from $(\pi - \varepsilon)$ to $(\pi + \varepsilon)$.

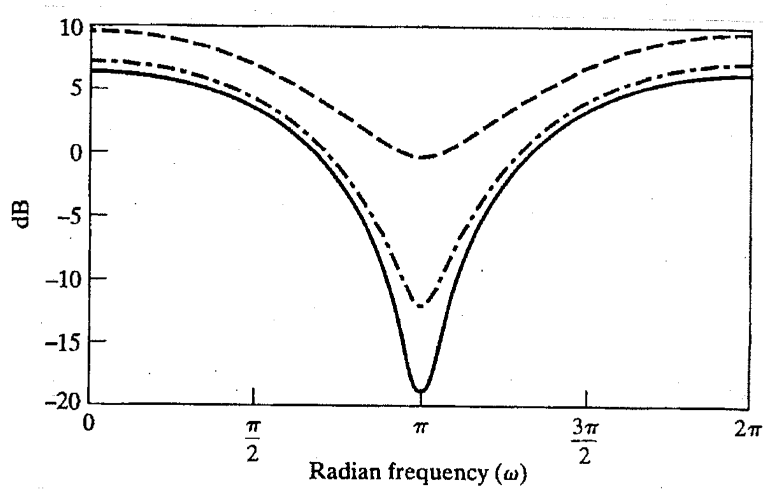
If $r > 1$, the log magnitude function behaves similarly to the case $r < 1$; i.e., it dips more sharply as $r \rightarrow 1$.

The phase function in Figure shows a discontinuity of 2π radians at $\omega = \theta$ for all values of $r > 1$. The source of this discontinuity can be seen from Figure 5.14, which shows vectors for $\omega = (\pi - \varepsilon)$ and $\omega = (\pi + \varepsilon)$. Note that the pole vector v_1 has an angle of ω, which varies continuously from $\omega = 0$ to $\omega = 2\pi$. The angle of the zero vector v_3 is labeled ϕ_3 in the figure. If this angle is measured positively in the counterclockwise direction, the figure shows that ϕ_3 jumps from zero to 2π radians as ω goes from $(\pi - \varepsilon)$ to $(\pi + \varepsilon)$.



If the factor represents a pole of $H(z)$, then all the contributions will enter with opposite sign. Thus, the contribution of a pole $z = re^{j\theta}$ would be the negative of the curves in Figures 5.8 and 5.11. Instead of dipping toward zero ($-\infty$ dB), the magnitude function would peak around $\omega = \theta$. The dependence on r would be the same as for a zero; i.e., the closer r is to 1, the more peaked will be the contribution to the magnitude function. For stable and causal systems, there will, of course, be no poles outside the unit circle; i.e., r will always be less than 1.





LINEAR SYSTEMS WITH GENERALIZED LINEAR PHASE

In designing filters and other signal-processing systems that pass some portion of the frequency band undistorted, it is desirable to have approximately constant frequency-response magnitude and zero phase in that band. For causal systems, zero phase is not attainable, and consequently, some phase distortion must be allowed.

For example, consider a more general frequency response with linear phase, i.e.,

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha}, \qquad |\omega| < \pi. \tag{5.126}$$

The signal $x[n]$ is filtered by the zero-phase frequency response $|H(e^{j\omega})|$, and the filtered output is then "time shifted" by the (integer or noninteger) amount α. Suppose, for example, that $H(e^{j\omega})$ is the linear-phase ideal lowpass filter

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi. \end{cases} \tag{5.127}$$

The corresponding impulse response is

$$h_{\text{lp}}[n] = \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)}. \tag{5.128}$$

Specifically, a system is referred to as a *generalized linear-phase system* if its frequency response can be expressed in the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega+j\beta}, \tag{5.135}$$

where α and β are constants and $A(e^{j\omega})$ is a real (possibly bipolar) function of ω.

However, if we ignore any discontinuities that result from the addition of constant phase over all or part of the band $|\omega| < \pi$, then such a system can be characterized by constant group delay. That is, the class of systems such that

$$\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\} = \alpha \tag{5.136}$$

have linear phase of the more general form

$$\arg[H(e^{j\omega})] = \beta - \omega\alpha, \qquad 0 < \omega < \pi, \tag{5.137}$$

where β and α are both real constants.

Type I FIR Linear-Phase Systems

A type I system is defined as a system that has a symmetric impulse response

$$h[n] = h[M-n], \qquad 0 \leq n \leq M, \tag{5.148}$$

with M an even integer. The delay $M/2$ is an integer. The frequency response is

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}. \tag{5.149}$$

By applying the symmetry condition, Eq. (5.148), the sum in Eq. (5.149) can be rewritten in the form

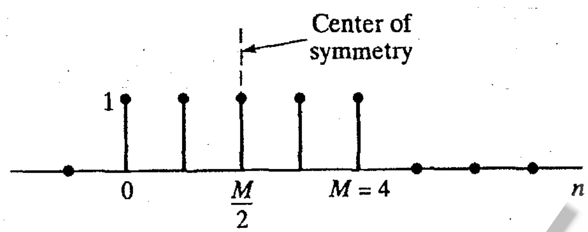
$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right), \tag{5.150a}$$

where

$\beta = 0,\ \tau = \frac{M}{2}$

$$a[0] = h[M/2], \tag{5.150b}$$

$$a[k] = 2h[(M/2)-k], \qquad k = 1, 2, \ldots, M/2. \tag{5.150c}$$



If the impulse response is

$$h[n] = \begin{cases} 1, & 0 \le n \le 4, \\ 0, & \text{otherwise}, \end{cases}$$

The frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{4} e^{-j\omega n} = \frac{1-e^{-j\omega 5}}{1-e^{-j\omega}} \\ &= e^{-j\omega 2}\frac{\sin(5\omega/2)}{\sin(\omega/2)}. \end{aligned} \tag{5.156}$$

Type II FIR Linear-Phase Systems

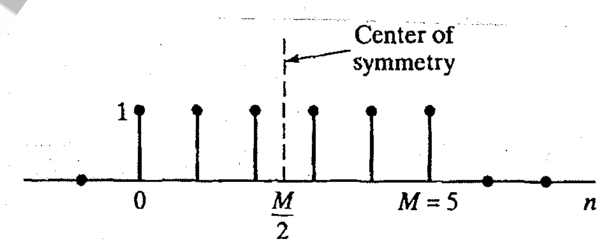
A type II system has a symmetric impulse response , with M an odd integer. $H(e^{j\omega})$ for this case can be expressed as

$$H(e^{j\omega}) = e^{-j\omega M/2}\left\{\sum_{k=1}^{(M+1)/2} b[k]\cos\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right\}, \tag{5.151a}$$

where

$\beta = 0,\ \tau = \frac{M}{2} = \text{"}N + \frac{1}{2}\text{"}$

$$b[k] = 2h[(M+1)/2-k], \qquad k = 1, 2, \ldots, (M+1)/2. \tag{5.151b}$$



If the length of the impulse response of the previous example is extended by one sample, we obtain the impulse response

$$H(e^{j\omega}) = e^{-j\omega 5/2}\frac{\sin(3\omega)}{\sin(\omega/2)}. \tag{5.157}$$

Type III FIR Linear-Phase Systems

If the system has an antisymmetric impulse response

$$h[n] = -h[M-n], \qquad 0 \le n \le M, \tag{5.152}$$

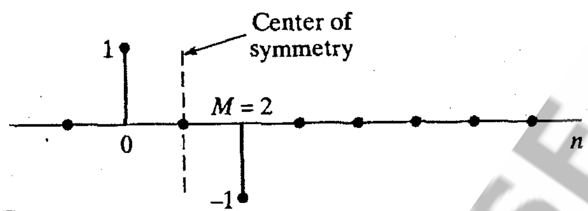
with M an even integer, then $H(e^{j\omega})$ has the form

$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{M/2} c[k]\sin \omega k\right], \tag{5.153a}$$

where

$\beta = \frac{\pi}{2},\ \tau = \frac{M}{2}$

$$c[k] = 2h[(M/2)-k], \qquad k = 1, 2, \ldots, M/2. \tag{5.153b}$$



If the impulse response is

$$h[n] = \delta[n] - \delta[n-2], \tag{5.158}$$

as in Figure 5.36(c), then

$$\begin{aligned} H(e^{j\omega}) &= 1 - e^{-j2\omega} \\ &= j[2\sin(\omega)]e^{-j\omega}. \end{aligned} \tag{5.159}$$

Type IV FIR Linear-Phase Systems

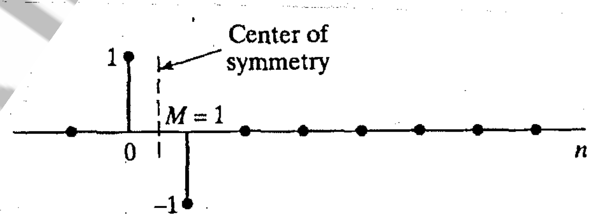
If the impulse response is antisymmetric as in Eq. (5.152) and M is odd, then

$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{(M+1)/2} d[k]\sin\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right], \tag{5.154a}$$

where

$\beta = \frac{\pi}{2},\ \tau = \frac{M}{2} = \text{"}N + \frac{1}{2}\text{"}$

$$d[k] = 2h[(M+1)/2-k], \qquad k = 1, 2, \ldots, (M+1)/2. \tag{5.154b}$$



In this case (Figure 5.36(d)), the impulse response is

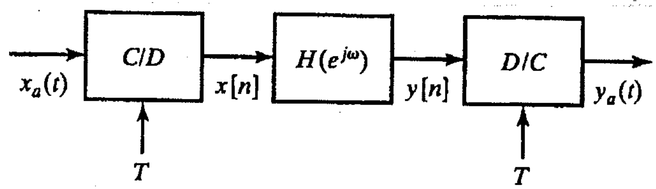
$$h[n] = \delta[n] - \delta[n-1], \tag{5.160}$$

for which the frequency response is

$$\begin{aligned} H(e^{j\omega}) &= 1 - e^{-j\omega} \\ &= j[2\sin(\omega/2)]e^{-j\omega/2}. \end{aligned} \tag{5.161}$$

Filter design techniques

Filters are a particularly important class of linear time-invariant systems. Strictly speaking, the term *frequency-selective filter* suggests a system that passes certain frequency components and totally rejects all others, but in a broader context any system that modifies certain frequencies relative to others is also called a filter.

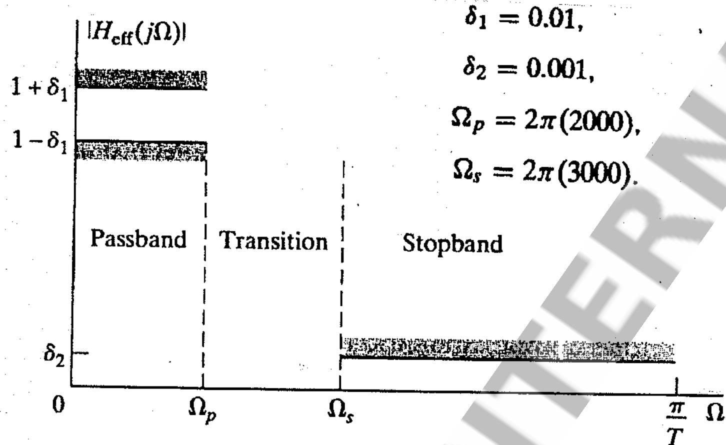


if a linear time-invariant discrete-time system is used as in Figure 7.1, and if the input is bandlimited and the sampling frequency is high enough to avoid aliasing, then the overall system behaves as a linear time-invariant continuous-time system with frequency response

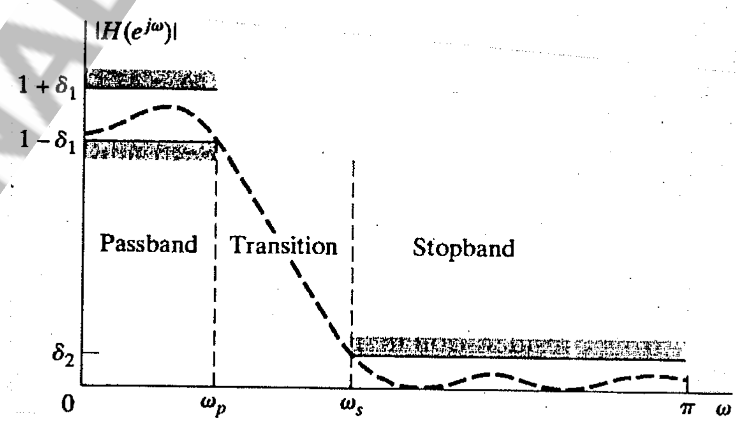
$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T, \\ 0, & |\Omega| > \pi/T. \end{cases} \tag{7.1a}$$

In such cases, it is straightforward to convert from specifications on the effective continuous-time filter to specifications on the discrete-time filter through the relation $\omega = \Omega T$. That is, $H(e^{j\omega})$ is specified over one period by the equation

$$H(e^{j\omega}) = H_{\text{eff}}\left(j\frac{\omega}{T}\right), \qquad |\omega| < \pi. \tag{7.1b}$$



$$\delta_1 = 0.01,$$
$$\delta_2 = 0.001,$$
$$\Omega_p = 2\pi(2000),$$
$$\Omega_s = 2\pi(3000).$$



ideal passband gain in decibels $= 20\log_{10}(1) = 0$ dB
maximum passband gain in decibels $= 20\log_{10}(1.01) = 0.086$ dB
maximum stopband gain in decibels $= 20\log_{10}(0.001) = -60$ dB

Filter Design by Impulse Invariance

In the impulse invariance design procedure for transforming continuous-time filters into discrete-time filters, the impulse response of the discrete-time filter is chosen proportional to equally spaced samples of the impulse response of the continuous-time filter; i.e.,

$$h[n] = T_d h_c(nT_d), \tag{7.4}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k\right). \tag{7.5}$$

If the continuous-time filter is bandlimited, so that

$$H_c(j\Omega) = 0, \qquad |\Omega| \geq \pi/T_d, \tag{7.6}$$

then

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right), \qquad |\omega| \leq \pi; \tag{7.7}$$

However, if the continuous-time filter approaches zero at high frequencies, the aliasing may be negligibly small, and a useful discrete-time filter can result from the sampling of the impulse response of a continuous-time filter.

In the impulse invariance design procedure, the discrete-time filter specifications are first transformed to continuous-time filter specifications through the use of Eq. (7.7). Assuming that the aliasing involved in the transformation from $H_c(j\Omega)$ to $H(e^{j\omega})$ will be negligible, we obtain the specifications on $H_c(j\Omega)$ by applying the relation

$$\Omega = \omega/T_d \tag{7.8}$$

While the impulse invariance transformation from continuous time to discrete time is defined in terms of time-domain sampling, it is easy to carry out as a transformation on the system function.

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}. \tag{7.9}$$

The corresponding impulse response is

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{7.10}$$

The impulse response of the discrete-time filter obtained by sampling $T_d h_c(t)$ is

$$\begin{aligned} h[n] = T_d h_c(nT_d) &= \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n] \\ &= \sum_{k=1}^{N} T_d A_k (e^{s_k T_d})^n u[n]. \end{aligned} \tag{7.11}$$

The system function of the discrete-time filter is therefore given by

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}. \tag{7.12}$$

If the continuous-time filter is stable, corresponding to the real part of s_k being less than zero, then the magnitude of $e^{s_k T_d}$ will be less than unity, so that the corresponding pole in the discrete-time filter is inside the unit circle.

Impulse Invariance with a Butterworth Filter

Let us consider the design of a lowpass discrete-time filter by applying impulse invariance to an appropriate Butterworth continuous-time filter.[2] The specifications for the discrete-time filter are

$$0.89125 \leq |H(e^{j\omega})| \leq 1, \qquad 0 \leq |\omega| \leq 0.2\pi, \tag{7.13a}$$

$$|H(e^{j\omega})| \leq 0.17783, \qquad 0.3\pi \leq |\omega| \leq \pi. \tag{7.13b}$$

Since the parameter T_d cancels in the impulse invariance procedure, we can choose

Specifically, the magnitude-squared function of a Butterworth filter is of the form

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}, \tag{7.16}$$

so that the filter design process consists of determining the parameters N and Ω_c to meet the desired specifications.

Because of the preceding considerations, we want to design a continuous-time Butterworth filter with magnitude function $|H_c(j\Omega)|$ for which

$$0.89125 \le |H_c(j\Omega)| \le 1, \qquad 0 \le |\Omega| \le 0.2\pi, \tag{7.14a}$$

$$|H_c(j\Omega)| \le 0.17783, \qquad 0.3\pi \le |\Omega| \le \pi. \tag{7.14b}$$

Since the magnitude response of an analog Butterworth filter is a monotonic function of frequency, Eqs. (7.14a) and (7.14b) will be satisfied if

$$|H_c(j0.2\pi)| \ge 0.89125 \tag{7.15a}$$

and

$$|H_c(j0.3\pi)| \le 0.17783. \tag{7.15b}$$

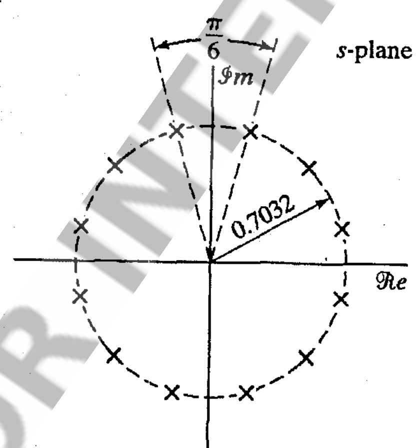
$$1+\left(\frac{0.2\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \tag{7.17a}$$

and

$$1+\left(\frac{0.3\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2. \tag{7.17b}$$

The solution of these two equations is $N = 5.8858$ and $\Omega_c = 0.70474$. The parameter N, however, must be an integer. Therefore, so that the specifications are met or exceeded, we must round N up to the nearest integer, $N = 6$.

With $\Omega_c = 0.7032$ and with $N = 6$, the 12 poles of the magnitude-squared function $H_c(s)H_c(-s) = 1/[1+(s/j\Omega_c)^{2N}]$ are uniformly distributed in angle on a circle of radius $\Omega_c = 0.7032$.



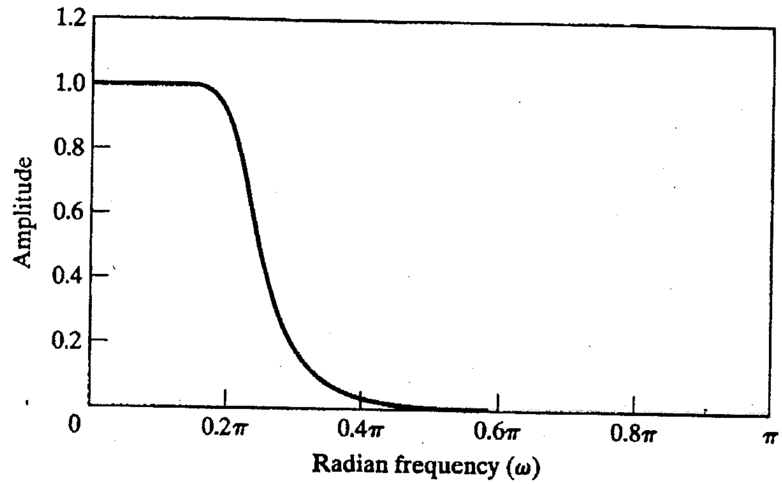
Pole pair 1: $-0.182 \pm j(0.679)$,

Pole pair 2: $-0.497 \pm j(0.497)$,

Pole pair 3: $-0.679 \pm j(0.182)$.

Therefore,

$$H_c(s) = \frac{0.12093}{(s^2+0.3640s+0.4945)(s^2+0.9945s+0.4945)(s^2+1.3585s+0.4945)}$$



$$H(z) = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.8557 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}}.$$

Bilinear Transformation

The technique discussed in this subsection avoids the problem of aliasing by using the bilinear transformation, an algebraic transformation between the variables s and z that maps the entire $j\Omega$-axis in the s-plane to one revolution of the unit circle in the z-plane.

With $H_c(s)$ denoting the continuous-time system function and $H(z)$ the discrete-time system function, the bilinear transformation corresponds to replacing s by

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right); \tag{7.20}$$

that is,

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right]. \tag{7.21}$$

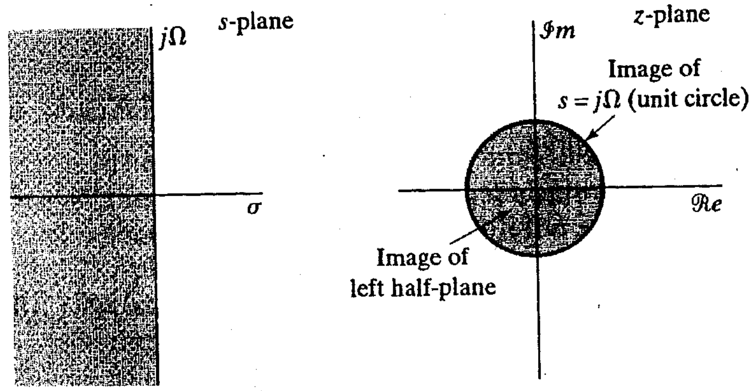
To develop the properties of the algebraic transformation specified in Eq. (7.20), we solve for z to obtain

$$z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s}, \tag{7.22}$$

and, substituting $s = \sigma + j\Omega$ into Eq. (7.22), we obtain

$$z = \frac{1 + \sigma T_d/2 + j\Omega T_d/2}{1 - \sigma T_d/2 - j\Omega T_d/2}. \tag{7.23}$$

If $\sigma < 0$, then, from Eq. (7.23), it follows that $|z| < 1$ for any value of Ω. Similarly, if $\sigma > 0$, then $|z| > 1$ for all Ω. That is, if a pole of $H_c(s)$ is in the left-half s-plane, its image in the z-plane will be inside the unit circle. Therefore, causal stable continuous-time filters map into causal stable discrete-time filters.



Next, to show that the $j\Omega$-axis of the s-plane maps onto the unit circle, we substitute $s = j\Omega$ into Eq. (7.22), obtaining

$$z = \frac{1 + j\Omega T_d/2}{1 - j\Omega T_d/2}. \tag{7.24}$$

From Eq. (7.24), it is clear that $|z| = 1$ for all values of s on the $j\Omega$-axis. That is, the $j\Omega$-axis maps onto the unit circle, so Eq. (7.24) takes the form

$$e^{j\omega} = \frac{1 + j\Omega T_d/2}{1 - j\Omega T_d/2}. \tag{7.25}$$

$$s = \frac{2}{T_d}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right), \tag{7.26}$$

or, equivalently,

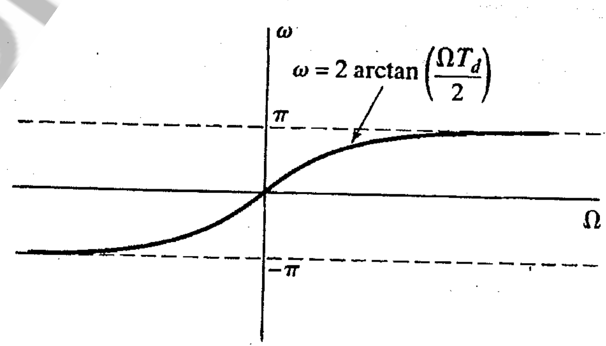
$$s = \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2}(j \sin \omega/2)}{2e^{-j\omega/2}(\cos \omega/2)}\right] = \frac{2j}{T_d}\tan(\omega/2). \tag{7.27}$$

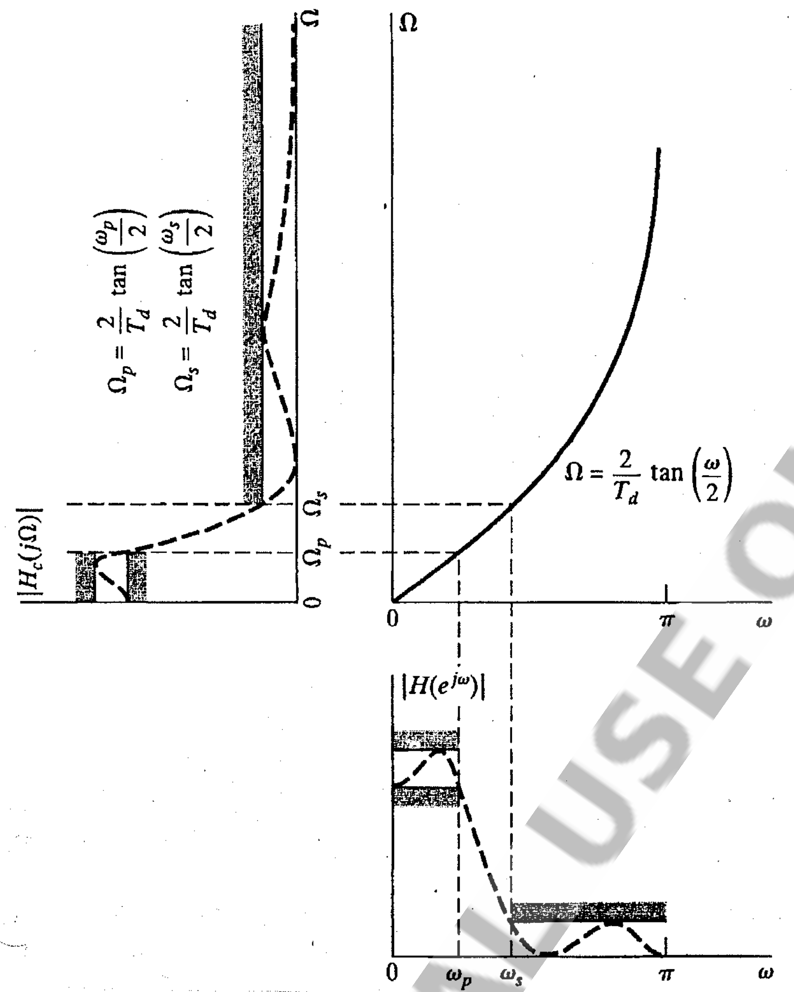
Equating real and imaginary parts on both sides of Eq. (7.27) leads to the relations $\sigma = 0$ and

$$\Omega = \frac{2}{T_d}\tan(\omega/2), \tag{7.28}$$

or

$$\omega = 2\arctan(\Omega T_d/2). \tag{7.29}$$





Bilinear Transformation of a Butterworth Filter

Consider the discrete-time filter specifications of Example 7.2, in which we illustrated the impulse invariance technique for the design of a discrete-time filter. The specifications on the discrete-time filter are

$$0.89125 \le |H(e^{j\omega})| \le 1, \qquad 0 \le \omega \le 0.2\pi, \tag{7.30a}$$

$$|H(e^{j\omega})| \le 0.17783, \qquad 0.3\pi \le \omega \le \pi. \tag{7.30b}$$

$$0.89125 \le |H_c(j\Omega)| \le 1, \qquad 0 \le \Omega \le \frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right), \tag{7.31a}$$

$$|H_c(j\Omega)| \le 0.17783, \qquad \frac{2}{T_d}\tan\left(\frac{0.3\pi}{2}\right) \le \Omega \le \infty. \tag{7.31b}$$

For convenience, we choose $T_d = 1$.

and
$$|H_c(j2\tan(0.1\pi))| \ge 0.89125$$

$$|H_c(j2\tan(0.15\pi))| \le 0.17783. \tag{7.32b}$$

The form of the magnitude-squared function for the Butterworth filter is

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}. \tag{7.33}$$

Solving for N and Ω_c with the equality sign in Eqs. (7.32a) and (7.32b), we obtain

$$1+\left(\frac{2\tan(0.1\pi)}{\Omega_c}\right)^{2N}=\left(\frac{1}{0.89}\right)^2 \tag{7.34a}$$

and

$$1+\left(\frac{2\tan(0.15\pi)}{\Omega_c}\right)^{2N}=\left(\frac{1}{0.178}\right)^2, \tag{7.34b}$$

and solving for N in Eqs. (7.34a) and (7.34b) gives

$$\begin{aligned} N &= \frac{\log\left[\left(\left(\frac{1}{0.178}\right)^2-1\right)\Big/\left(\left(\frac{1}{0.89}\right)^2-1\right)\right]}{2\log[\tan(0.15\pi)/\tan(0.1\pi)]} \\ &= 5.305. \end{aligned} \tag{7.35}$$

Since N must be an integer, we choose $N = 6$. Substituting $N = 6$ into Eq. (7.34b), we obtain $\Omega_c = 0.766$. For this value of Ω_c, the passband specifications are exceeded and the stopband specifications are met exactly. This is reasonable for the bilinear transformation, since we do not have to be concerned with aliasing. That is, with proper prewarping, we can be certain that the resulting discrete-time filter will meet the specifications exactly at the desired stopband edge.

$$H_c(s)=\frac{0.20238}{(s^2+0.3996s+0.5871)(s^2+1.0836s+0.5871)(s^2+1.4802s+0.5871)}. \tag{7.36}$$

The system function for the discrete-time filter is then obtained by applying the bilinear transformation to $H_c(s)$ with $T_d = 1$. The result is

$$\begin{aligned} H(z) &= \frac{0.0007378(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})} \\ &\quad \times \frac{1}{(1-0.9044z^{-1}+0.2155z^{-2})}. \end{aligned} \tag{7.37}$$

The simplest method of FIR filter design is called the *window method*. This method generally begins with an ideal desired frequency response that can be represented as

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]e^{-j\omega n}, \tag{7.40}$$

where $h_d[n]$ is the corresponding impulse response sequence, which can be expressed in terms of $H_d(e^{j\omega})$ as

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega. \tag{7.41}$$

The simplest way to obtain a causal FIR filter from $h_d[n]$ is to define a new system with impulse response $h[n]$ given by[5]

$$h[n] = \begin{cases} h_d[n], & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases} \tag{7.42}$$

More generally, we can represent $h[n]$ as the product of the desired impulse response and a finite-duration "window" $w[n]$; i.e.,

$$h[n] = h_d[n]w[n], \tag{7.43}$$

where, for simple truncation as in Eq. (7.42), the window is the *rectangular window*

$$w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases} \tag{7.44}$$

It follows from the modulation, or windowing, theorem (Section 2.9.7) that

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta. \tag{7.45}$$

If $w[n] = 1$ for all n (i.e., if we do not truncate at all), $W(e^{j\omega})$ is a periodic impulse train with period 2π, and therefore, $H(e^{j\omega}) = H_d(e^{j\omega})$.

Consequently, the choice of window is governed by the desire to have $w[n]$ as short as possible in duration, so as to minimize computation in the implementation of the filter, while having $W(e^{j\omega})$ approximate an impulse.

$$W(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}} = e^{-j\omega M/2}\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}. \tag{7.46}$$

As M increases, the width of the "main lobe" decreases. The main lobe is usually defined as the region between the first zero-crossings on either side of the origin. For the rectangular window, the width of the main lobe is $\Delta_{\omega_m} = 4\pi/(M+1)$. However, for the rectangular window, the side lobes are large, and in fact, as M increases, the peak amplitudes of the main lobe and the side lobes grow in a manner such that the area under each lobe is a constant while the width of each lobe decreases with M. Consequently, as $W(e^{j(\omega-\theta)})$ "slides by" a discontinuity of $H_d(e^{j\theta})$ with increasing ω, the integral of $W(e^{j(\omega-\theta)})H_d(e^{j\theta})$ will oscillate as each side lobe of $W(e^{j(\omega-\theta)})$ moves past the discontinuity.

In the theory of Fourier series, it is well known that this nonuniform convergence, the *Gibbs phenomenon*, can be moderated through the use of a less abrupt truncation of the Fourier series. By tapering the window smoothly to zero at each end, the height of the side lobes can be diminished; however, this is achieved at the expense of a wider main lobe and thus a wider transition at the discontinuity.

Rectangular

$$w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{7.47a}$$

Bartlett (triangular)

$$w[n] = \begin{cases} 2n/M, & 0 \le n \le M/2, \\ 2 - 2n/M, & M/2 < n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{7.47b}$$

Hanning

$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{7.47c}$$

Hamming

$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{7.47d}$$

Blackman

$$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases} \tag{7.47e}$$

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hanning	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

Incorporation of Generalized Linear Phase

In designing many types of FIR filters, it is desirable to obtain causal systems with a generalized linear phase response.

Specifically, note that all the windows have the property that

$$w[n] = \begin{cases} w[M-n], & 0 \le n \le M, \\ 0, & \text{otherwise;} \end{cases} \tag{7.48}$$

i.e., they are symmetric about the point $M/2$. As a result, their Fourier transforms are of the form

$$W(e^{j\omega}) = W_e(e^{j\omega})e^{-j\omega M/2}, \tag{7.49}$$

where $W_e(e^{j\omega})$ is a real, even function of ω.

if $h_d[M-n] = h_d[n]$ $\qquad H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$, (7.50)

where $A_e(e^{j\omega})$ is real and is an even function of ω.

if $h_d[M-n] = -h_d[n]$ $\qquad H(e^{j\omega}) = jA_o(e^{j\omega})e^{-j\omega M/2}$, (7.51)

where $A_o(e^{j\omega})$ is real and is an odd function of ω.

Linear-Phase Lowpass Filter

The desired frequency response is defined as

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases} \tag{7.56}$$

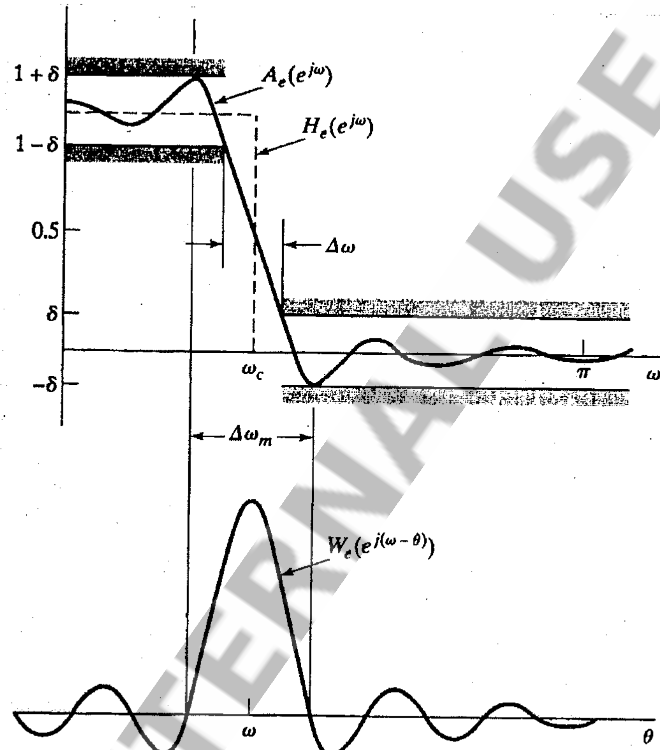
where the generalized linear phase factor has been incorporated into the definition of the ideal lowpass filter. The corresponding ideal impulse response is

$$h_{\text{lp}}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega M/2} e^{j\omega n} d\omega = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} \tag{7.57}$$

for $-\infty < n < \infty$. It is easily shown that $h_{\text{lp}}[M-n] = h_{\text{lp}}[n]$, so if we use a symmetric window in the equation

$$h[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} w[n], \tag{7.58}$$

then a linear-phase system will result.



Clearly, the windows with the smaller side lobes yield better approximations of the ideal response at a discontinuity . Also, the third column, which shows the width of the main lobe, suggests that narrower transition regions can be achieved by increasing M. Thus, through the choice of the shape and duration of the window, we can control the properties of the resulting FIR filter.

The Kaiser Window Filter Design Method

The trade-off between the main-lobe width and side-lobe area can be quantified by seeking the window function that is maximally concentrated around $\omega = 0$ in the frequency domain.

The Kaiser window is defined as

$$w[n] = \begin{cases} \dfrac{I_0[\beta(1 - [(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M, \\ 0, & \text{otherwise}, \end{cases} \tag{7.59}$$

where $\alpha = M/2$, and $I_0(\cdot)$ represents the zeroth-order modified Bessel function of the first kind.

Given that δ is fixed, the passband cutoff frequency ω_p of the lowpass filter is defined to be the highest frequency such that $|H(e^{j\omega})| \geq 1-\delta$. The stopband cutoff frequency ω_s is defined to be the lowest frequency such that $|H(e^{j\omega})| \leq \delta$. Therefore, the transition region has width

$$\Delta\omega = \omega_s - \omega_p \tag{7.60}$$

Defining

$$A = -20\log_{10}\delta, \tag{7.61}$$

Kaiser determined empirically that the value of β needed to achieve a specified value of A is given by

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50, \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \leq A \leq 50, \\ 0.0, & A < 21. \end{cases} \tag{7.62}$$

Furthermore, Kaiser found that to achieve prescribed values of A and $\Delta\omega$, M must satisfy

$$M = \frac{A-8}{2.285\Delta\omega}. \tag{7.63}$$

Equation (7.63) predicts M to within ± 2 over a wide range of values of $\Delta\omega$ and A. Thus, with these formulas, the Kaiser window design method requires almost no iteration or trial and error.

Kaiser Window Design of a Lowpass Filter

For this example, we use the same specifications as in Examples 7.4, 7.5, and 7.6, i.e., $\omega_p = 0.4\pi$, $\omega_s = 0.6\pi$, $\delta_1 = 0.01$, and $\delta_2 = 0.001$. Since filters designed by the window method inherently have $\delta_1 = \delta_2$, we must set $\delta = 0.001$. The cutoff frequency of the underlying ideal lowpass filter must be found. Due to the symmetry of the approximation at the discontinuity of $H_d(e^{j\omega})$, we would set

$$\omega_c = \frac{\omega_p + \omega_s}{2} = 0.5\pi.$$

To determine the parameters of the Kaiser window, we first compute

$$\Delta\omega = \omega_s - \omega_p = 0.2\pi, \qquad A = -20\log_{10}\delta = 60.$$

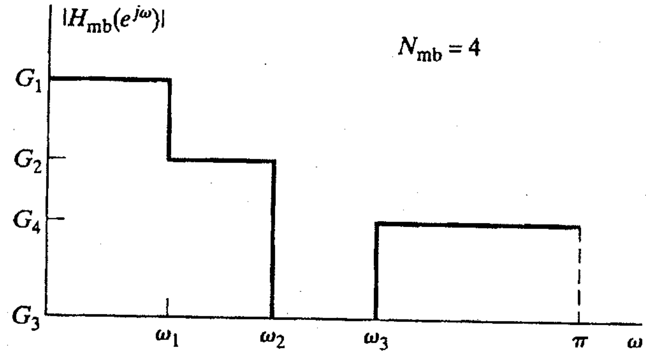
We substitute these two quantities into Eqs. (7.62) and (7.63) to obtain the required values of β and M. For this example the formulas predict

$$\beta = 5.653, \qquad M = 37.$$

The impulse response of the filter is computed using Eqs. (7.58) and (7.59). We obtain

$$h[n] = \begin{cases} \dfrac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)} \cdot \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \leq n \leq M, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha = M/2 = 37/2 = 18.5$. Since $M = 37$ is an odd integer, the resulting linear-phase system would be of type II.



This generalized multiband filter includes lowpass, highpass, bandpass, and bandstop filters as special cases. If such a magnitude function is multiplied by a linear phase factor $e^{-j\omega M/2}$, the corresponding ideal impulse response is

$$h_{mb}[n] = \sum_{k=1}^{N_{mb}} (G_k - G_{k+1}) \frac{\sin \omega_k (n - M/2)}{\pi (n - M/2)}, \tag{7.68}$$

where N_{mb} is the number of bands and $G_{N_{mb}+1} = 0$. If $h_{mb}[n]$ is multiplied by a Kaiser window, the type of approximations that we have observed at the single discontinuity of the lowpass and highpass systems will occur at *each* of the discontinuities.

OPTIMUM APPROXIMATIONS OF FIR FILTERS

That is,

$$h[n] = \begin{cases} h_d[n], & 0 \le n \le M, \\ 0, & \text{otherwise,} \end{cases} \tag{7.72}$$

minimizes the expression

$$\varepsilon^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_d(e^{j\omega}) - H(e^{j\omega})|^2 d\omega. \tag{7.73}$$

However, as we have seen, this approximation criterion leads to adverse behavior at discontinuities of $H_d(e^{j\omega})$.

In designing a causal type I linear-phase FIR filter, it is convenient first to consider the design of a zero-phase filter, i.e., one for which

$$h_e[n] = h_e[-n], \tag{7.74}$$

and then to insert a delay sufficient to make it causal.

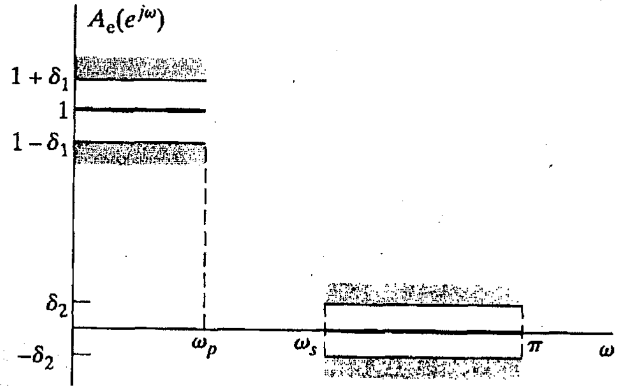
The corresponding frequency response is given by

$$A_e(e^{j\omega}) = \sum_{n=-L}^{L} h_e[n] e^{-j\omega n}, \tag{7.75}$$

with $L = M/2$ an integer, or

$$A_e(e^{j\omega}) = h_e[0] + \sum_{n=1}^{L} 2h_e[n] \cos(\omega n). \tag{7.76}$$

Note that $A_e(e^{j\omega})$ is a real, even, and periodic function of ω. A causal system can be obtained from $h_e[n]$ by delaying it by $L = M/2$ samples.



The Parks–McClellan algorithm is based on reformulating the filter design problem as a problem in polynomial approximation. Specifically, the terms $\cos(\omega n)$ in Eq. (7.76) can be expressed as a sum of powers of $\cos\omega$ in the form

$$\cos(\omega n) = T_n(\cos\omega), \tag{7.79}$$

$T_n(x)$ is the nth-order Chebyshev polynomial, defined as $T_n(x) = \cos(n\cos^{-1}x)$.

Consequently, Eq. (7.76) can be rewritten as an Lth-order polynomial in $\cos\omega$, namely,

$$A_e(e^{j\omega}) = \sum_{k=0}^{L} a_k(\cos\omega)^k, \tag{7.80}$$

where the a_k's are constants that are related to $h_e[n]$, the values of the impulse response. With the substitution $x = \cos\omega$, we can express Eq. (7.80) as

$$A_e(e^{j\omega}) = P(x)|_{x=\cos\omega}, \tag{7.81}$$

where $P(x)$ is the Lth-order polynomial

$$P(x) = \sum_{k=0}^{L} a_k x^k. \tag{7.82}$$

To formalize the approximation problem in this case, let us define an approximation error function

$$E(\omega) = W(\omega)[H_d(e^{j\omega}) - A_e(e^{j\omega})], \tag{7.83}$$

where the weighting function $W(\omega)$ incorporates the approximation error parameters into the design process. In this design method, the error function $E(\omega)$, the weighting function $W(\omega)$, and the desired frequency response $H_d(e^{j\omega})$ are defined only over closed subintervals of $0 \le \omega \le \pi$.

For example, suppose that we wish to obtain an approximation as in Figure 7.31, where L, ω_p, and ω_s are fixed design parameters. For this case,

$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le \omega \le \omega_p, \\ 0, & \omega_s \le \omega \le \pi. \end{cases} \tag{7.84}$$

The weighting function $W(\omega)$ allows us to weight the approximation errors differently in the different approximation intervals.

$$W(\omega) = \begin{cases} \dfrac{1}{K} & 0 \le \omega \le \omega_p, \\ 1, & \omega_s \le \omega \le \pi, \end{cases} \tag{7.85}$$

where $K = \delta_1/\delta_2$.

The particular criterion used in this design procedure is the so-called minimax or Chebyshev criterion, where

$$\min_{\{h_e[n]:0\le n\le L\}} \left(\max_{\omega\in F} |E(\omega)| \right),$$

where F is the closed subset of $0 \le \omega \le \pi$ such that $0 \le \omega \le \omega_p$ or $\omega_s \le \omega \le \pi$.

Alternation Theorem: Let F_P denote the closed subset consisting of the disjoint union of closed subsets of the real axis x. Then

$$P(x) = \sum_{k=0}^{r} a_k x^k$$

is an rth-order polynomial. Also, $D_P(x)$ denotes a given desired function of x that is continuous on F_P; $W_P(x)$ is a positive function, continuous on F_P, and

$$E_P(x) = W_P(x)[D_P(x) - P(x)].$$

is the weighted error. The maximum error is defined as

$$\|E\| = \max_{x\in F_P} |E_P(x)|.$$

A necessary and sufficient condition that $P(x)$ be the unique rth-order polynomial that minimizes $\|E\|$ is that $E_p(x)$ exhibit *at least* $(r+2)$ alternations; i.e., there must exist at least $(r+2)$ values x_i in F_p such that $x_1 < x_2 < \cdots < x_{r+2}$ and such that $E_p(x_i) = -E_P(x_{i+1}) = \pm\|E\|$ for $i = 1, 2, \ldots, (r+1)$.

The Parks–McClellan Algorithm

The alternation theorem gives necessary and sufficient conditions on the error for optimality in the Chebyshev or minimax sense. Although the theorem does not state explicitly how to find the optimum filter, the conditions that are presented serve as the basis for an efficient algorithm for finding it.

From the alternation theorem, we know that the optimum filter $A_e(e^{j\omega})$ will satisfy the set of equations

$$W(\omega_i)[H_d(e^{j\omega_i}) - A_e(e^{j\omega_i})] = (-1)^{i+1}\delta, \qquad i = 1, 2, \ldots, (L+2), \tag{7.99}$$

where δ is the optimum error .

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^L & \dfrac{1}{W(\omega_1)} \\ 1 & x_2 & x_2^2 & \cdots & x_2^L & \dfrac{-1}{W(\omega_2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{L+2} & x_{L+2}^2 & \cdots & x_{L+2}^L & \dfrac{(-1)^{L+2}}{W(\omega_{L+2})} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \delta \end{bmatrix} = \begin{bmatrix} H_d(e^{j\omega_1}) \\ H_d(e^{j\omega_2}) \\ \vdots \\ H_d(e^{j\omega_{L+2}}) \end{bmatrix}, \tag{7.100}$$

where $x_i = \cos\omega_i$. This set of equations serves as the basis for an iterative algorithm for finding the optimum $A_e(e^{j\omega})$. The procedure begins by guessing a set of alternation frequencies ω_i for $i = 1, 2, \ldots, (L+2)$. Note that ω_p and ω_s are fixed and are necessarily members of the set of alternation frequencies. Specifically, if $\omega_\ell = \omega_p$, then $\omega_{\ell+1} = \omega_s$.

Parks and McClellan (1972a, 1972b) found that, for the given set of the extremal frequencies,

$$\delta = \frac{\sum_{k=1}^{L+2} b_k H_d(e^{j\omega_k})}{\sum_{k=1}^{L+2} \frac{b_k(-1)^{k+1}}{W(\omega_k)}}, \tag{7.101}$$

where

$$b_k = \prod_{\substack{i=1\\ i\neq k}}^{L+2} \frac{1}{(x_k - x_i)} \tag{7.102}$$

and, as before, $x_i = \cos\omega_i$.

Now, since $A_e(e^{j\omega})$ is known to be an Lth-order trigonometric polynomial, we can interpolate a trigonometric polynomial through $(L+1)$ of the $(L+2)$ known values $E(\omega_i)$ (or equivalently, $A_e(e^{j\omega_i})$). Parks and McClellan used the Lagrange interpolation formula to obtain

$$A_e(e^{j\omega}) = P(\cos\omega) = \frac{\sum_{k=1}^{L+1} [d_k/(x - x_k)]C_k}{\sum_{k=1}^{L+1} [d_k/(x - x_k)]}, \tag{7.103a}$$

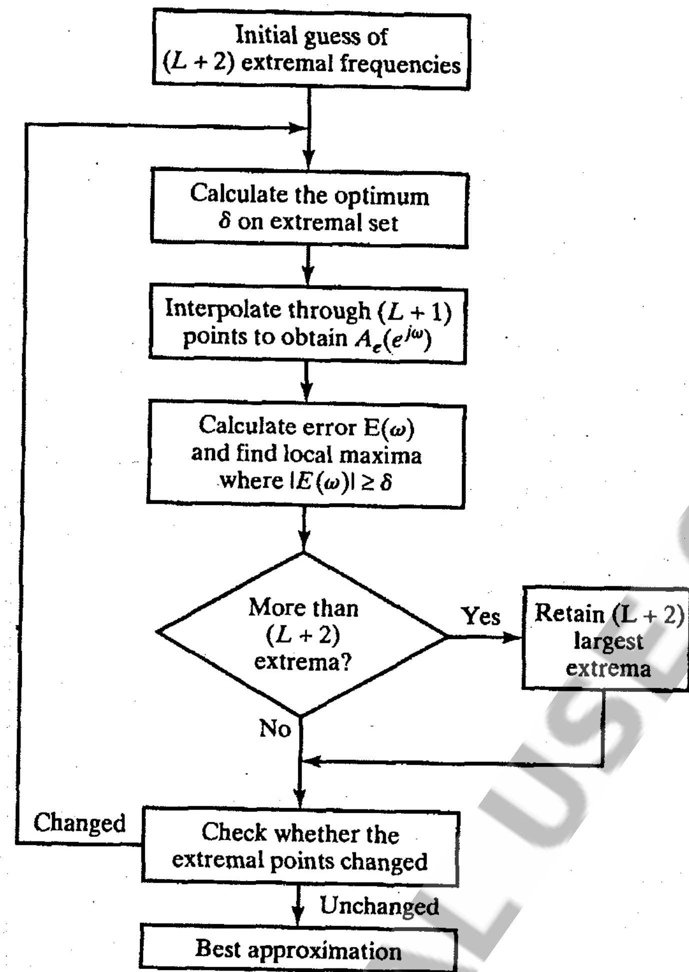
where $x = \cos\omega$, $x_i = \cos\omega_i$,

$$C_k = H_d(e^{j\omega_k}) - \frac{(-1)^{k+1}\delta}{W(\omega_k)}, \tag{7.103b}$$

and

$$d_k = \prod_{\substack{i=1\\ i\neq k}}^{L+1} \frac{1}{(x_k - x_i)} = b_k(x_k - x_{L+2}). \tag{7.103c}$$

The polynomial of Eq. (7.103a) can be used to evaluate $A_e(e^{j\omega})$ and also $E(\omega)$ on a dense set of frequencies in the passband and stopband. If $|E(\omega)| \le \delta$ for all ω in the passband and stopband, then the optimum approximation has been found. Otherwise we must find a new set of extremal frequencies.



After the algorithm has converged, the impulse response can be computed from samples of the polynomial representation using the discrete Fourier transform.

THE DISCRETE FOURIER TRANSFORM

REPRESENTATION OF PERIODIC SEQUENCES: THE DISCRETE FOURIER SERIES

Consider a sequence $\tilde{x}[n]$ that is periodic[1] with period N, so that $\tilde{x}[n] = \tilde{x}[n + rN]$ for any integer values of n and r.

$$\tilde{x}[n] = \frac{1}{N}\sum_{k} \tilde{X}[k]e^{j(2\pi/N)kn}.$$

$e_k[n]$ in Eq. (8.1) are identical for values of k separated by N; i.e., $e_0[n] = e_N[n]$, $e_1[n] = e_{N+1}[n]$, and, in general,

$$e_{k+\ell N}[n] = e^{j(2\pi/N)(k+\ell N)n} = e^{j(2\pi/N)kn}e^{j2\pi\ell n} = e^{j(2\pi/N)kn} = e_k[n], \tag{8.3}$$

where ℓ is an integer. Consequently, the set of N periodic complex exponentials $e_0[n]$, $e_1[n], \ldots, e_{N-1}[n]$ defines all the distinct periodic complex exponentials with frequencies that are integer multiples of $(2\pi/N)$.

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j(2\pi/N)kn}.$$

Thus, the Fourier series coefficients $\tilde{X}[k]$ in Eq. (8.4) are obtained from $\tilde{x}[n]$ by the relation

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)kn}. \tag{8.9}$$

Note that the sequence $\tilde{X}[k]$ is periodic with period N; i.e., $\tilde{X}[0] = \tilde{X}[N]$, $\tilde{X}[1] = \tilde{X}[N+1]$.

For convenience in notation, these equations are often written in terms of the complex quantity

$$W_N = e^{-j(2\pi/N)}. \tag{8.10}$$

With this notation, the DFS analysis–synthesis pair is expressed as follows:

$$\text{Analysis equation:} \quad \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]W_N^{kn}. \tag{8.11}$$

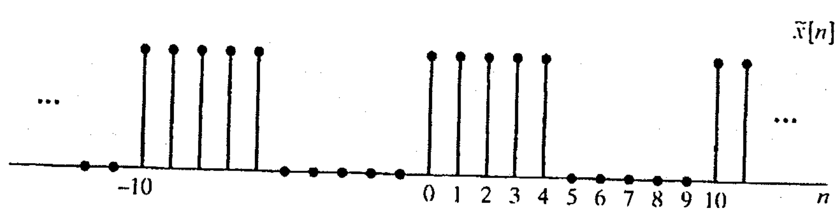
$$\text{Synthesis equation:} \quad \tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]W_N^{-kn}. \tag{8.12}$$

In both of these equations, $\tilde{X}[k]$ and $\tilde{x}[n]$ are periodic sequences. We will sometimes find it convenient to use the notation

$$\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k] \tag{8.13}$$

The Discrete Fourier Series of a Periodic Rectangular Pulse Train

For this example, $\tilde{x}[n]$ is the sequence shown in Figure 8.1, whose period is $N = 10$.

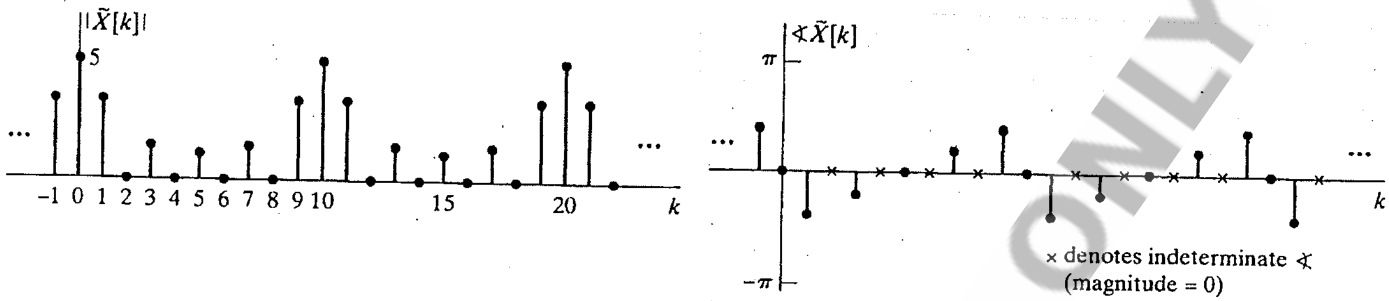


$$\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn} = \sum_{n=0}^{4} e^{-j(2\pi/10)kn}. \tag{8.17}$$

This finite sum has the closed form

$$\tilde{X}[k] = \frac{1 - W_{10}^{5k}}{1 - W_{10}^{k}} = e^{-j(4\pi k/10)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)}. \tag{8.18}$$

The magnitude and phase of the periodic sequence $\tilde{X}[k]$ are



PROPERTIES OF THE DISCRETE FOURIER SERIES

Linearity

Consider two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, both with period N, such that

$$\tilde{x}_1[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k] \tag{8.19a}$$

and

$$\tilde{x}_2[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_2[k] \tag{8.19b}$$

Then

$$a\tilde{x}_1[n] + b\tilde{x}_2[n] \overset{\mathcal{DFS}}{\longleftrightarrow} a\tilde{X}_1[k] + b\tilde{X}_2[k]. \tag{8.20}$$

Shift of a Sequence

If a periodic sequence $\tilde{x}[n]$ has Fourier coefficients $\tilde{X}[k]$, then $\tilde{x}[n-m]$ is a shifted version of $\tilde{x}[n]$, and

$$\tilde{x}[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} W_N^{km}\tilde{X}[k]. \tag{8.21}$$

$$W_N^{-n\ell}\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k-\ell].$$

Duality

Because of the strong similarity between the Fourier analysis and synthesis equations in continuous time, there is a duality between the time domain and frequency domain.

If

$$\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k], \tag{8.25a}$$

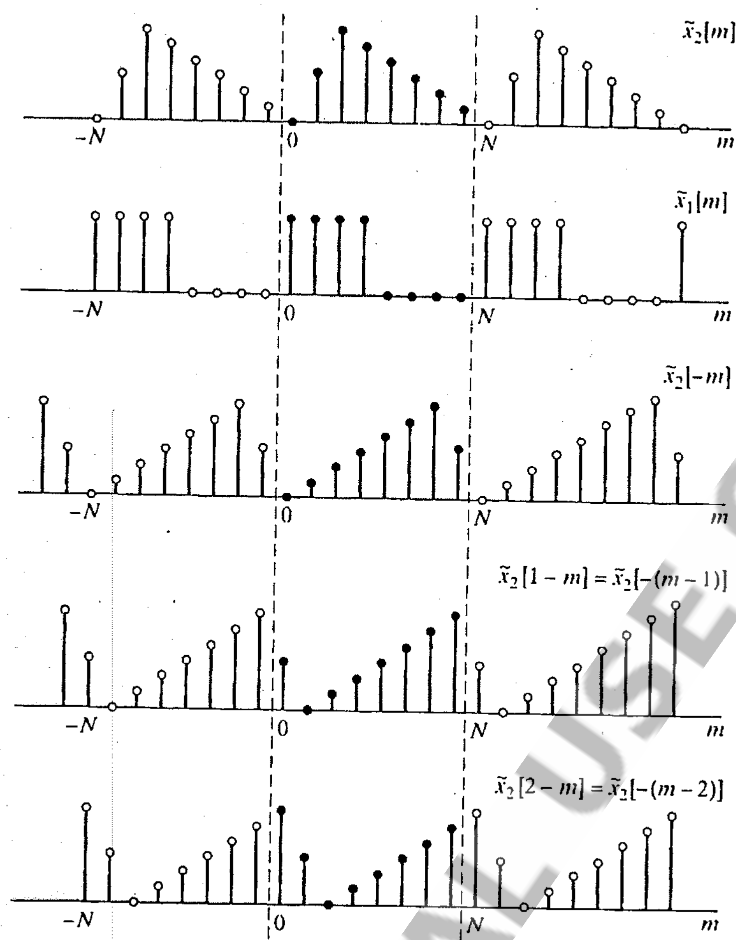
then

$$\tilde{X}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} N\tilde{x}[-k]. \tag{8.25b}$$

Periodic Convolution

In summary,

$$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k]\tilde{X}_2[k].$$



The duality theorem (Section 8.2.3) suggests that if

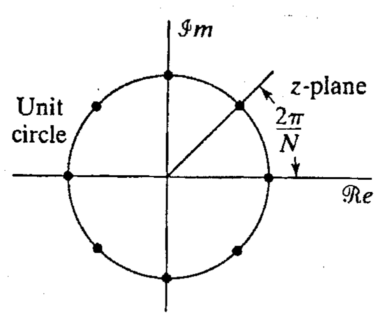
$$\tilde{x}_3[n] = \tilde{x}_1[n]\tilde{x}_2[n], \tag{8.33}$$

where $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ are periodic sequences, each with period N, has the discrete Fourier series coefficients given by

$$\tilde{X}_3[k] = \frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell], \tag{8.34}$$

SAMPLING THE FOURIER TRANSFORM

In this section, we discuss with more generality the relationship between an aperiodic sequence with Fourier transform $X(e^{j\omega})$ and the periodic sequence for which the DFS coefficients correspond to samples of $X(e^{j\omega})$ equally spaced in frequency.



Consider an aperiodic sequence $x[n]$ with Fourier transform $X(e^{j\omega})$, and assume that a sequence $\tilde{X}[k]$ is obtained by sampling $X(e^{j\omega})$ at frequencies $\omega_k = 2\pi k/N$; i.e.,

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=(2\pi/N)k} = X(e^{j(2\pi/N)k}). \tag{8.49}$$

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Since the Fourier transform is periodic in ω with period 2π, the resulting sequence is periodic in k with period N.

Note that the sequence of samples $\tilde{X}[k]$, being periodic with period N, *could* be the sequence of discrete Fourier series coefficients of a sequence $\tilde{x}[n]$. To obtain that sequence, we can simply substitute $\tilde{X}[k]$ obtained by sampling into Eq. (8.12):

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}. \tag{8.51}$$

Substituting Eq. (8.52) into Eq. (8.49) and then substituting the resulting expression for $\tilde{X}[k]$ into Eq. (8.51) gives

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x[m]e^{-j(2\pi/N)km}\right] W_N^{-kn}, \tag{8.53}$$

which, after we interchange the order of summation, becomes

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)}\right] = \sum_{m=-\infty}^{\infty} x[m]\tilde{p}[n-m]. \tag{8.54}$$

$$\tilde{p}[n-m] = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)} = \sum_{r=-\infty}^{\infty} \delta[n-m-rN] \tag{8.55}$$

and therefore,

$$\tilde{x}[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta[n-rN] = \sum_{r=-\infty}^{\infty} x[n-rN], \tag{8.56}$$

If $x[n]$ has finite length and we take a sufficient number of equally spaced samples of its Fourier transform (specifically, a number greater than or equal to the length of $x[n]$), then the Fourier transform is recoverable from these samples, and, equivalently, $x[n]$ is recoverable from the corresponding periodic sequence $\tilde{x}[n]$ through the relation

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases} \tag{8.57}$$

FOURIER REPRESENTATION OF FINITE-DURATION SEQUENCES: THE DISCRETE FOURIER TRANSFORM

We begin by considering a finite-length sequence $x[n]$ of length N samples such that $x[n] = 0$ outside the range $0 \le n \le N-1$. In many instances, we will want to assume that a sequence has length N even if its length is $M \le N$. In such cases, we simply recognize that the last $(N-M)$ samples are zero. To each finite-length sequence of length N, we can always associate a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]. \tag{8.58a}$$

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases} \tag{8.58b}$$

Recall from Section 8.4 that the DFS coefficients of $\tilde{x}[n]$ are samples (spaced in frequency by $2\pi/N$) of the Fourier transform of $x[n]$. Since $x[n]$ is assumed to have finite length N, there is no overlap between the terms $x[n - rN]$ for different values of r. Thus,

$$\tilde{x}[n] = x[((n))_N].$$

Thus, the DFT, $X[k]$, is related to the DFS coefficients, $\tilde{X}[k]$, by

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{8.61}$$

and

$$\tilde{X}[k] = X[(k \text{ modulo } N)] = X[((k))_N]. \tag{8.62}$$

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{kn}, & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{8.65}$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases} \tag{8.66}$$

$$\text{Analysis equation:} \quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}. \tag{8.67}$$

$$\text{Synthesis equation:} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}. \tag{8.68}$$

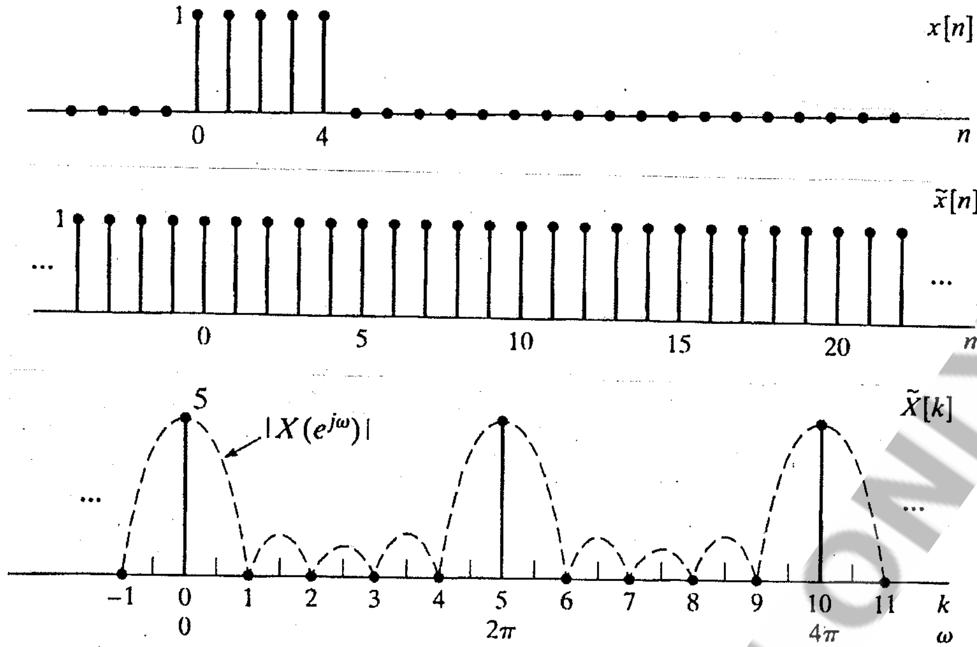
That is, the fact that $X[k] = 0$ for k outside the interval $0 \le k \le N-1$ and that $x[n] = 0$ for n outside the interval $0 \le n \le N-1$ is implied, but not always stated explicitly.

In defining the DFT representation, we are simply recognizing that we are *interested* in values of $x[n]$ only in the interval $0 \le n \le N-1$ because $x[n]$ is really zero outside that interval, and we are *interested* in values of $X[k]$ only in the interval $0 \le k \le N-1$ because these are the only values needed in Eq. (8.68).

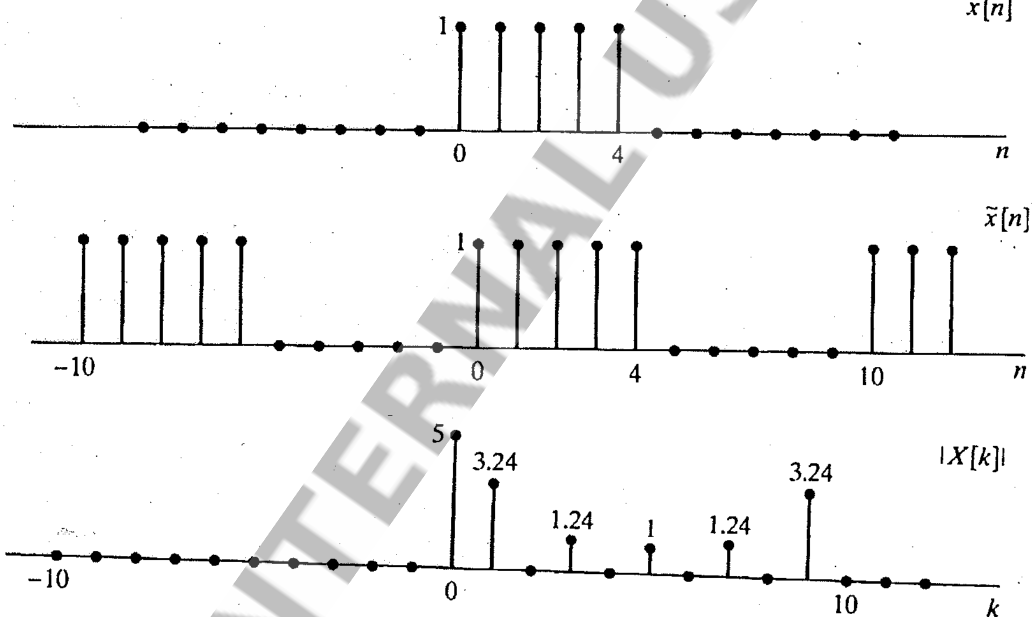
The DFT of a Rectangular Pulse

$$\tilde{X}[k] = \sum_{n=0}^{4} e^{-j(2\pi k/5)n} = \frac{1 - e^{-j2\pi k}}{1 - e^{-j(2\pi k/5)}}$$

$$= \begin{cases} 5, & k = 0, \pm 5, \pm 10, \ldots, \\ 0, & \text{otherwise}; \end{cases}$$



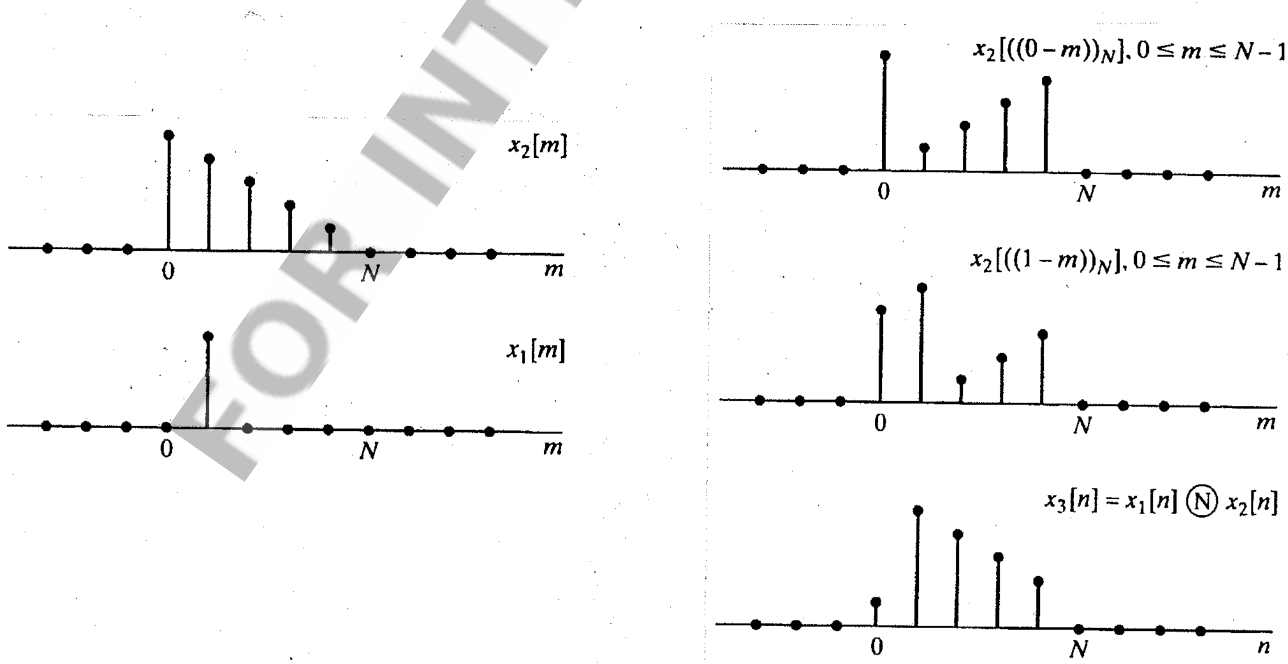
If, instead, we consider $x[n]$ to be of length $N = 10$, then the underlying periodic sequence is that shown in



PROPERTIES OF THE DISCRETE FOURIER TRANSFORM

Finite-Length Sequence (Length N)	N-point DFT (Length N)
1. $x[n]$	$X[k]$
2. $x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3. $ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4. $X[n]$	$Nx[((-k))_N]$
5. $x[((n-m))_N]$	$W_N^{km} X[k]$
6. $W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7. $\sum_{m=0}^{N-1} x_1(m) x_2[((n-m))_N]$	$X_1[k] X_2[k]$
8. $x_1[n] x_2[n]$	$\frac{1}{N} \sum_{\ell=0}^{N-1} X_1(\ell) X_2[((k-\ell))_N]$
9. $x^*[n]$	$X^*[((-k))_N]$
10. $x^*[((-n))_N]$	$X^*[k]$
11. $\mathcal{R}e\{x[n]\}$	$X_{\text{ep}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12. $j\mathcal{J}m\{x[n]\}$	$X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{J}m\{X[k]\} = -\mathcal{J}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$
16. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$

Circular Convolution with a Delayed Impulse



Circular Convolution of Two Rectangular Pulses

As another example of circular convolution, let

$$x_1[n] = x_2[n] = \begin{cases} 1, & 0 \le n \le L-1, \\ 0, & \text{otherwise,} \end{cases} \tag{8.122}$$

If we let N denote the DFT length, then, for $N = L$, the N-point DFTs are

$$X_1[k] = X_2[k] = \sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} N, & k = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{8.123}$$

If we explicitly multiply $X_1[k]$ and $X_2[k]$, we obtain

$$X_3[k] = X_1[k]X_2[k] = \begin{cases} N^2, & k = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{8.124}$$

from which it follows that

$$x_3[n] = N, \qquad 0 \le n \le N-1. \tag{8.125}$$

It is, of course, possible to consider $x_1[n]$ and $x_2[n]$ as $2L$-point sequences by augmenting them with L zeros.

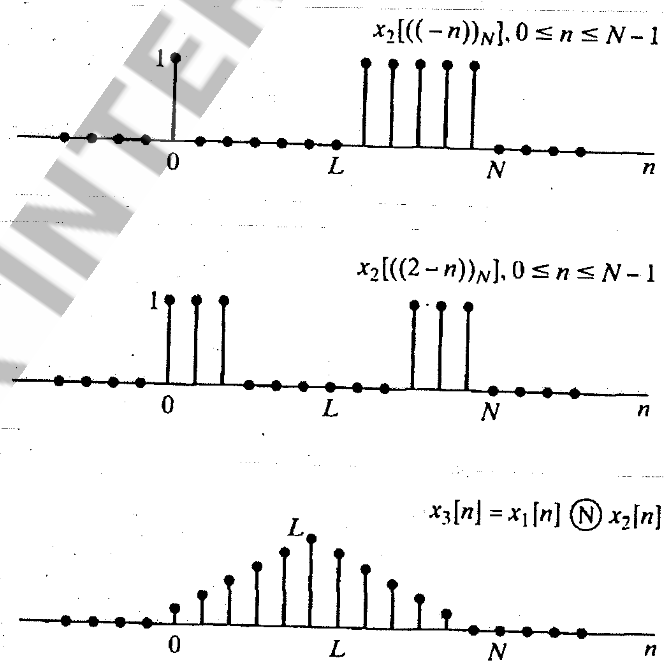
Note that for $N = 2L$,

$$X_1[k] = X_2[k] = \frac{1 - W_N^{Lk}}{1 - W_N^k},$$

so the DFT of the triangular-shaped sequence $x_3[n]$ in Figure 8.16(e) is

$$X_3[k] = \left(\frac{1 - W_N^{Lk}}{1 - W_N^k}\right)^2,$$

with $N = 2L$.



LINEAR CONVOLUTION USING THE DISCRETE FOURIER TRANSFORM

(a) Compute the N-point discrete Fourier transforms $X_1[k]$ and $X_2[k]$ of the two sequences $x_1[n]$ and $x_2[n]$, respectively.

(b) Compute the product $X_3[k] = X_1[k]X_2[k]$ for $0 \le k \le N-1$.

(c) Compute the sequence $x_3[n] = x_1[n] \circledN x_2[n]$ as the inverse DFT of $X_3[k]$.

In most applications, we are interested in implementing a linear convolution of two sequences; i.e., we wish to implement a linear time-invariant system.

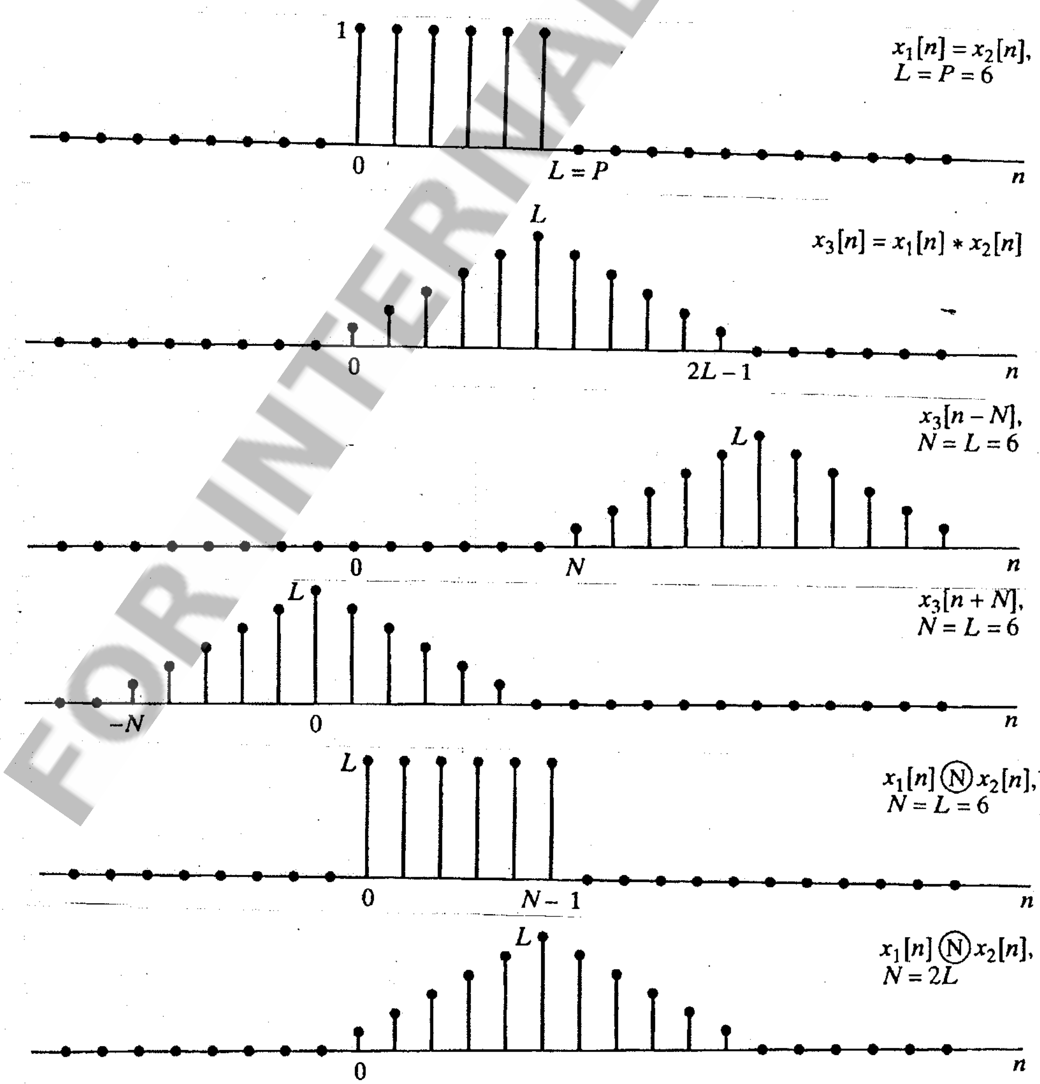
To obtain a linear convolution, we must ensure that circular convolution has the effect of linear convolution.

Consider a sequence $x_1[n]$ whose length is L points and a sequence $x_2[n]$ whose length is P points, and suppose that we wish to combine these two sequences by linear convolution to obtain a third sequence

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]. \tag{8.129}$$

Therefore, $(L+P-1)$ is the maximum length of the sequence $x_3[n]$ resulting from the linear convolution of a sequence of length L with a sequence of length P.

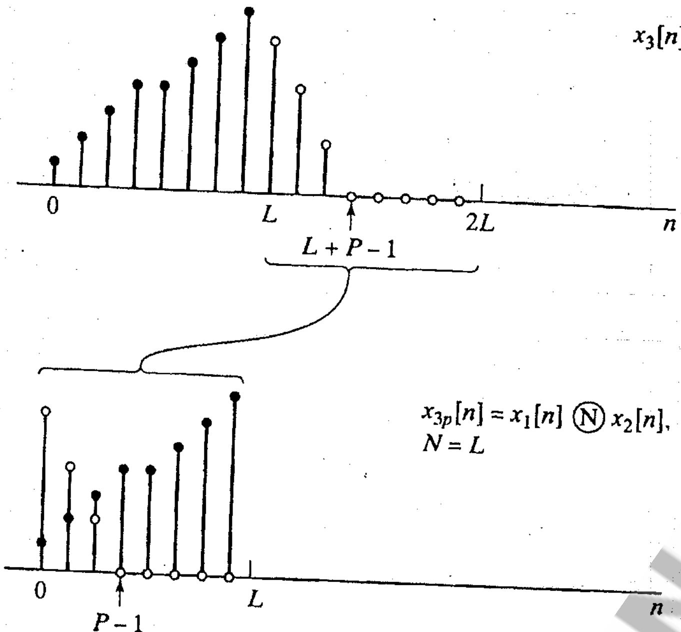
As we showed, if $x_1[n]$ has length L and $x_2[n]$ has length P, then $x_3[n]$ has maximum length $(L+P-1)$. Therefore, the circular convolution corresponding to $X_1[k]X_2[k]$ is identical to the linear convolution corresponding to $X_1(e^{j\omega})X_2(e^{j\omega})$ if N, the length of the DFTs, satisfies $N \ge L+P-1$.

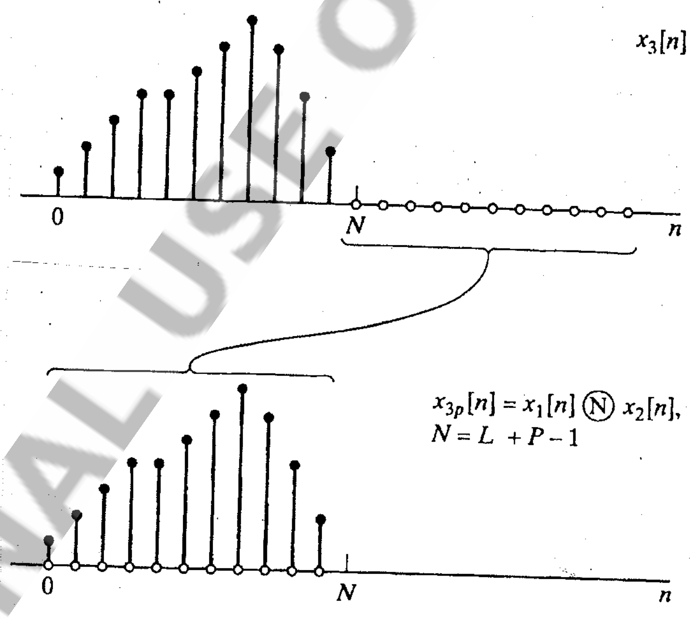


As Example points out, time aliasing in the circular convolution of two finite-length sequences can be avoided if $N \geq L + P - 1$. Also, it is clear that if $N = L = P$, all of the sequence values of the circular convolution may be different from those of the linear convolution. However, if $P < L$, some of the sequence values in an L-point circular convolution will be equal to the corresponding sequence values of the linear convolution. The time-aliasing interpretation is useful for showing this.

Implementing Linear Time-Invariant Systems Using the DFT

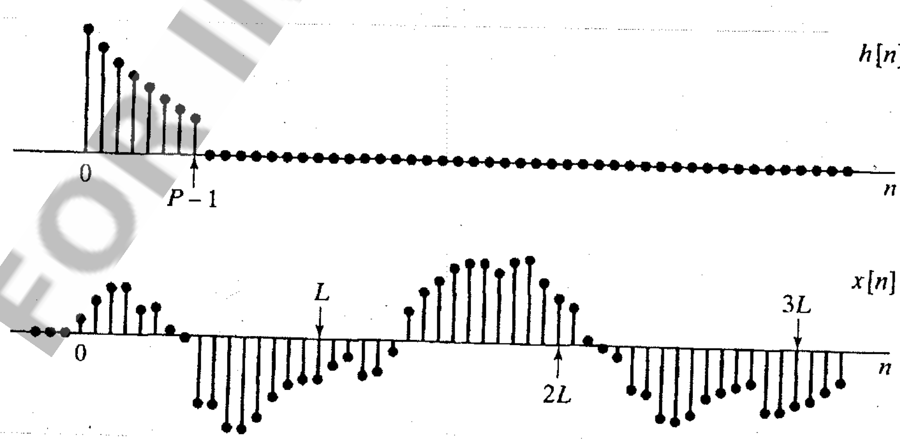
multiplying the DFTs of $x[n]$ and $h[n]$. Since we want the product to represent the DFT of the linear convolution of $x[n]$ and $h[n]$, which has length $(L+P-1)$, the DFTs that we compute must also be of at least that length, i.e., both $x[n]$ and $h[n]$ must be augmented with sequence values of zero amplitude. This process is often referred to as *zero-padding*.





This procedure permits the computation of the linear convolution of two finite-length sequences using the discrete Fourier transform; i.e., the output of an FIR system whose input also has finite length can be computed with the DFT.

block convolution



Henceforth, we will assume that $x[n] = 0$ for $n < 0$ and that the length of $x[n]$ is much greater than P. The sequence $x[n]$ can be represented as a sum of shifted finite-length segments of length L; i.e.,

$$x[n] = \sum_{r=0}^{\infty} x_r[n - rL], \tag{8.140}$$

where

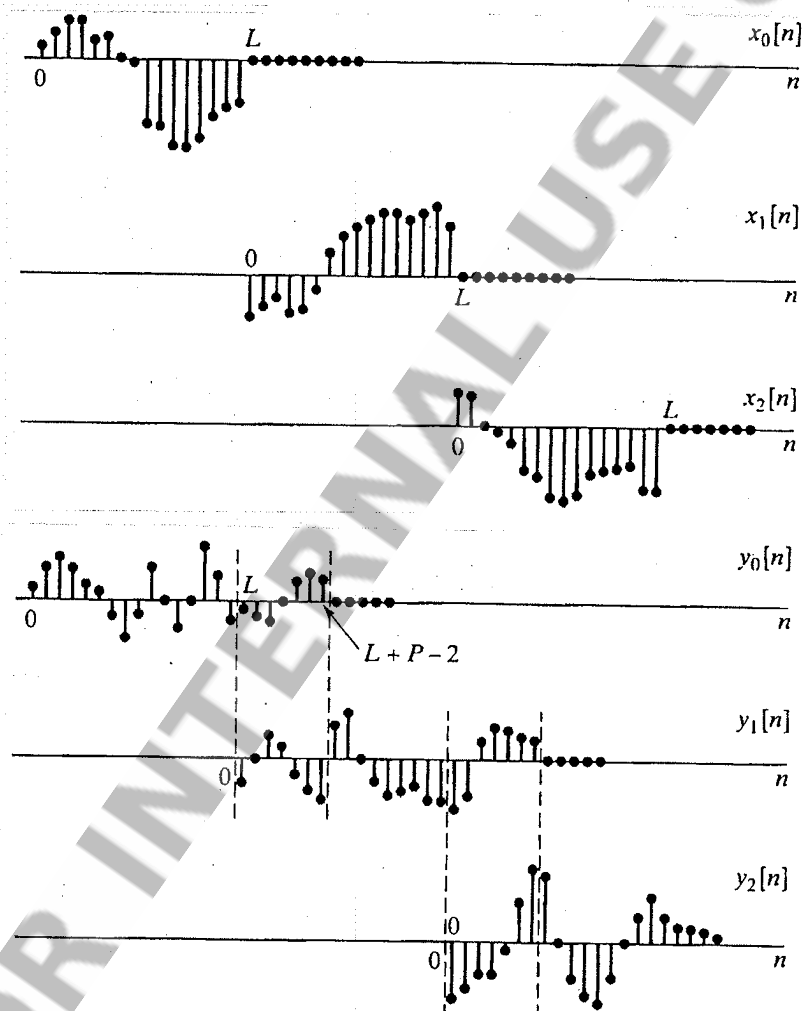
$$x_r[n] = \begin{cases} x[n + rL], & 0 \le n \le L-1, \\ 0, & \text{otherwise}. \end{cases} \tag{8.141}$$

Because convolution is a linear time-invariant operation, it follows from Eq. (8.140) that

$$y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} y_r[n - rL], \tag{8.142}$$

where

$$y_r[n] = x_r[n] * h[n]. \tag{8.143}$$



Since the sequences $x_r[n]$ have only L nonzero points and $h[n]$ is of length P, each of the terms $y_r[n] = x_r[n] * h[n]$ has length $(L + P - 1)$. Thus, the linear convolution $x_r[n] * h[n]$ can be obtained by the procedure described earlier using N-point DFTs, where $N \ge L + P - 1$. Since the beginning of each input section is separated from its neighbors by L points and each filtered section has length $(L + P - 1)$, the nonzero points in the filtered sections will overlap by $(P - 1)$ points, and these overlap samples must be added in carrying out the sum required

Computation of the Discrete Fourier Transform

Consequently, computation of the N-point DFT corresponds to the computation of N samples of the Fourier transform at N equally spaced frequencies $\omega_k = 2\pi k/N$, i.e., at N points on the unit circle in the z-plane. In this chapter, we discuss several methods for computing values of the DFT. The major focus of the chapter is a particularly efficient class of algorithms for the digital computation of the N-point DFT. Collectively, these efficient algorithms are called *fast Fourier transform* (FFT) algorithms.

the DFT of a finite-length sequence of length N is

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0, 1, \ldots, N-1, \tag{9.1}$$

where $W_N = e^{-j(2\pi/N)}$. The inverse discrete Fourier transform is given by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1. \tag{9.2}$$

To compute all N values therefore requires a total of N^2 complex multiplications and $N(N-1)$ complex additions.

Most approaches to improving the efficiency of the computation of the DFT exploit the symmetry and periodicity properties of W_N^{kn}; specifically,

1. $W_N^{k[N-n]} = W_N^{-kn} = (W_N^{kn})^*$ (complex conjugate symmetry);
2. $W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}$ (periodicity in n and k).

FFT algorithms are based on the fundamental principle of decomposing the computation of the discrete Fourier transform of a sequence of length N into successively smaller discrete Fourier transforms. The manner in which this principle is implemented leads to a variety of different algorithms, all with comparable improvements in computational speed.

DECIMATION-IN-TIME FFT ALGORITHMS

In computing the DFT, dramatic efficiency results from decomposing the computation into successively smaller DFT computations. In this process, we exploit both the symmetry and the periodicity of the complex exponential $W_N^{kn} = e^{-j(2\pi/N)kn}$. Algorithms in which the decomposition is based on decomposing the sequence $x[n]$ into successively smaller subsequences are called *decimation-in-time algorithms*.

With $X[k]$ given by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}, \qquad k = 0, 1, \ldots, N-1, \tag{9.10}$$

and separating $x[n]$ into its even- and odd-numbered points, we obtain

$$X[k] = \sum_{n \text{ even}} x[n] W_N^{nk} + \sum_{n \text{ odd}} x[n] W_N^{nk}, \tag{9.11}$$

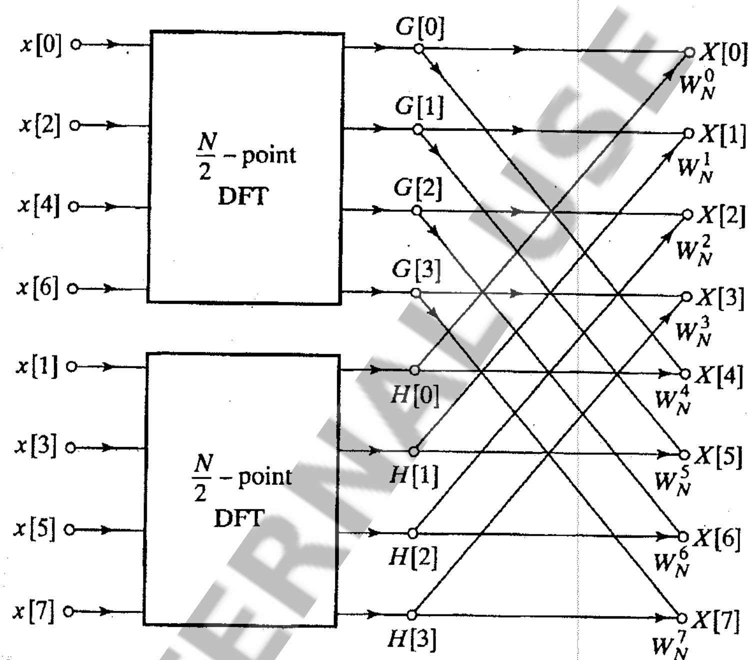
or, with the substitution of variables $n = 2r$ for n even and $n = 2r + 1$ for n odd,

$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r]W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1]W_N^{(2r+1)k}$$
$$= \sum_{r=0}^{(N/2)-1} x[2r](W_N^2)^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1](W_N^2)^{rk}. \qquad (9.12)$$

$$W_N^2 = e^{-2j(2\pi/N)} = e^{-j2\pi/(N/2)} = W_{N/2}. \qquad (9.13)$$

$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r]W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1]W_{N/2}^{rk}$$
$$= G[k] + W_N^k H[k], \qquad k = 0, 1, \ldots, N-1. \qquad (9.14)$$

Although the index k ranges over N values, $k = 0, 1, \ldots, N-1$, each of the sums must be computed only for k between 0 and $(N/2)-1$, since $G[k]$ and $H[k]$ are each periodic in k with period $N/2$.



Eq. (9.14) requires the computation of two $(N/2)$-point DFTs, which in turn requires $2(N/2)^2$ complex multiplications and approximately $2(N/2)^2$ complex additions if we do the $(N/2)$-point DFTs by the direct method. Then the two $(N/2)$-point DFTs must be combined, requiring N complex multiplications, corresponding to multiplying the second sum by W_N^k, and N complex additions, corresponding to adding the product obtained to the first sum. Consequently, the computation of Eq. (9.14) for all values of k requires at most $N + 2(N/2)^2$ or $N + N^2/2$ complex multiplications and complex additions. It is easy to verify that for $N > 2$, the total $N + N^2/2$ will be less than N^2.

If $N/2$ is even,

$$G[k] = \sum_{r=0}^{(N/2)-1} g[r]W_{N/2}^{rk} = \sum_{\ell=0}^{(N/4)-1} g[2\ell]W_{N/2}^{2\ell k} + \sum_{\ell=0}^{(N/4)-1} g[2\ell+1]W_{N/2}^{(2\ell+1)k}, \qquad (9.15)$$

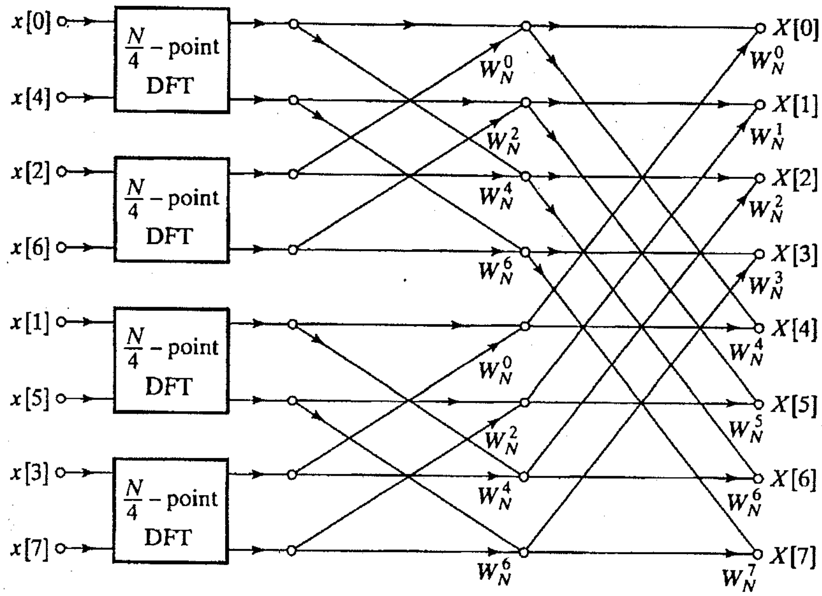
or

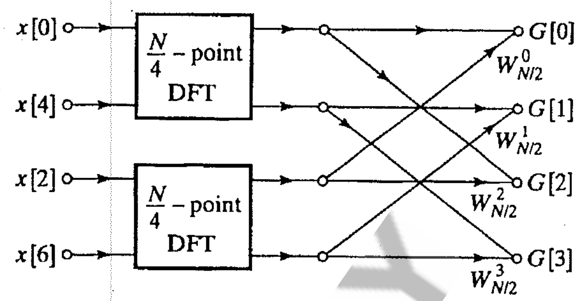
$$G[k] = \sum_{\ell=0}^{(N/4)-1} g[2\ell]W_{N/4}^{\ell k} + W_{N/2}^k \sum_{\ell=0}^{(N/4)-1} g[2\ell+1]W_{N/4}^{\ell k}. \qquad (9.16)$$

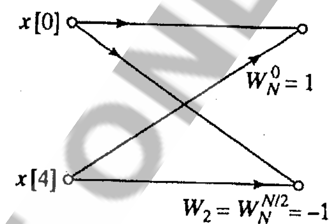
Similarly, $H[k]$ would be represented as

$$H[k] = \sum_{\ell=0}^{(N/4)-1} h[2\ell]W_{N/4}^{\ell k} + W_{N/2}^k \sum_{\ell=0}^{(N/4)-1} h[2\ell+1]W_{N/4}^{\ell k}. \qquad (9.17)$$

$$W_{N/2} = W_N^2.$$







Because of the shape of the flow graph, this elementary computation is called a *butterfly*. Since

$$W_N^{N/2} = e^{-j(2\pi/N)N/2} = e^{-j\pi} = -1, \tag{9.18}$$

the factor $W_N^{r+N/2}$ can be written as

$$W_N^{r+N/2} = W_N^{N/2} W_N^r = -W_N^r. \tag{9.19}$$

When the $(N/2)$-point transforms are decomposed into $(N/4)$-point transforms, the factor of $(N/2)^2$ is replaced by $N/2 + 2(N/4)^2$, so the overall computation then requires $N + N + 4(N/4)^2$ complex multiplications and additions. If $N = 2^\nu$, this can be done at most $\nu = \log_2 N$ times, so that after carrying out this decomposition as many times as possible, the number of complex multiplications and additions is equal to $N\nu = N \log_2 N$.

For example, if $N = 2^{10} = 1024$, then $N^2 = 2^{20} = 1{,}048{,}576$, and $N \log_2 N = 10{,}240$, a reduction of more than two orders of magnitude!

DECIMATION-IN-FREQUENCY FFT ALGORITHMS

The decimation-in-time FFT algorithms are all based on structuring the DFT computation by forming smaller and smaller subsequences of the input sequence $x[n]$. Alternatively, we can consider dividing the output sequence $X[k]$ into smaller and smaller subsequences in the same manner. FFT algorithms based on this procedure are commonly called *decimation-in-frequency* algorithms.

Since

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}, \qquad k = 0, 1, \ldots, N-1, \tag{9.23}$$

the even-numbered frequency samples are

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{n(2r)}, \qquad r = 0, 1, \ldots, (N/2) - 1, \tag{9.24}$$

which can be expressed as

$$X[2r] = \sum_{n=0}^{(N/2)-1} x[n]W_N^{2nr} + \sum_{n=N/2}^{N-1} x[n]W_N^{2nr}. \tag{9.25}$$

With a substitution of variables in the second summation

$$X[2r] = \sum_{n=0}^{(N/2)-1} x[n]W_N^{2nr} + \sum_{n=0}^{(N/2)-1} x[n+(N/2)]W_N^{2r[n+(N/2)]}. \tag{9.26}$$

$$W_N^{2r[n+(N/2)]} = W_N^{2rn}W_N^{rN} = W_N^{2rn}, \tag{9.27}$$

$$X[2r] = \sum_{n=0}^{(N/2)-1} (x[n]+x[n+(N/2)])W_{N/2}^{rn}, \qquad r = 0, 1, \ldots, (N/2)-1. \tag{9.28}$$

We can now consider obtaining the odd-numbered frequency points, given by

$$X[2r+1] = \sum_{n=0}^{N-1} x[n]W_N^{n(2r+1)}, \qquad r = 0, 1, \ldots, (N/2)-1. \tag{9.29}$$

$$X[2r+1] = \sum_{n=0}^{(N/2)-1} x[n]W_N^{n(2r+1)} + \sum_{n=N/2}^{N-1} x[n]W_N^{n(2r+1)}. \tag{9.30}$$

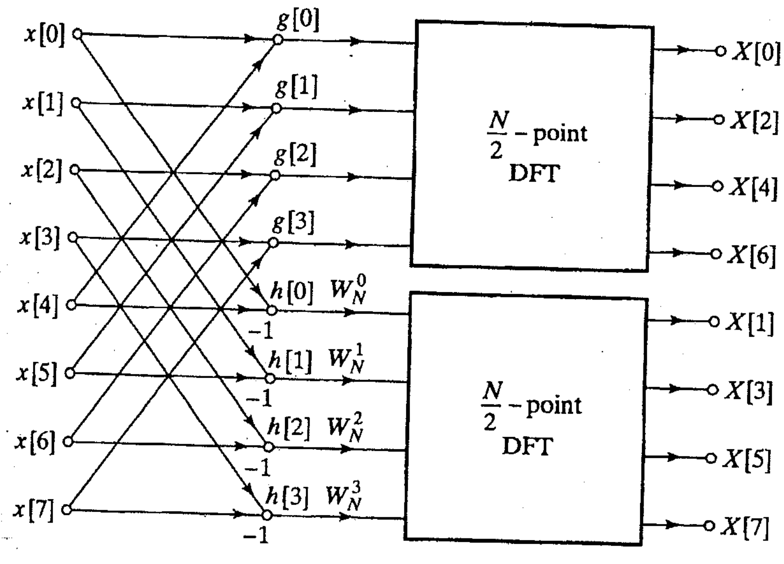
$$\begin{aligned}
\sum_{n=N/2}^{N-1} x[n]W_N^{n(2r+1)} &= \sum_{n=0}^{(N/2)-1} x[n+(N/2)]W_N^{[n+(N/2)](2r+1)} \\
&= W_N^{(N/2)(2r+1)} \sum_{n=0}^{(N/2)-1} x[n+(N/2)]W_N^{n(2r+1)} \\
&= -\sum_{n=0}^{(N/2)-1} x[n+(N/2)]W_N^{n(2r+1)},
\end{aligned} \tag{9.31}$$

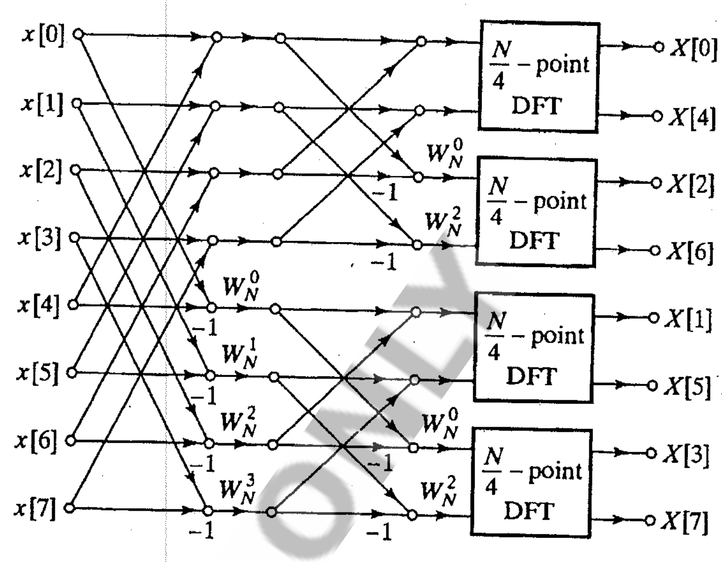
where we have used the fact that $W_N^{(N/2)2r} = 1$ and $W_N^{(N/2)} = -1$.

$$X[2r+1] = \sum_{n=0}^{(N/2)-1} (x[n]-x[n+(N/2)])W_N^{n(2r+1)}, \tag{9.32}$$

or, since $W_N^2 = W_{N/2}$,

$$X[2r+1] = \sum_{n=0}^{(N/2)-1} (x[n]-x[n+(N/2)])W_N^{n}W_{N/2}^{nr}, \qquad r = 0, 1, \ldots, (N/2)-1. \tag{9.33}$$





Thus, the total number of computations is the same for the decimation-in-frequency and the decimation-in-time algorithms.

Algorithms for More General Values of N

Although the special case of N a power of 2 leads to algorithms that have a simple structure, this is not the only restriction on N that can lead to a reduction in computation in the DFT. Indeed, in many cases it is desirable to evaluate the DFT efficiently for other values of N, and the same principles that were applied in the power of 2 decimation-in-time and decimation-in-frequency algorithms can be employed when N is a composite integer, i.e., the product of two or more integer factors. For example, if $N = RQ$, it is possible to express an N-point DFT as either the sum of R Q-point DFTs or as the sum of Q R-point DFTs and thereby obtain reductions in the number of computations. If N has many factors, the process can be repeated for each of the factors. Algorithms for general composite N involve more complicated indexing than the power of 2 case.

The Chirp Transform Algorithm

Another algorithm based on expressing the DFT as a convolution is referred to as the chirp transform algorithm (CTA).

To derive the CTA, we let $x[n]$ denote an N-point sequence and $X(e^{j\omega})$ its Fourier transform. We consider the evaluation of M samples of $X(e^{j\omega})$ that are equally spaced in angle on the unit circle

$$\omega_k = \omega_0 + k\Delta\omega, \qquad k = 0, 1, \ldots, M-1, \tag{9.36}$$

where the starting frequency ω_0 and the frequency increment $\Delta\omega$ can be chosen arbitrarily.

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n}, \qquad k = 0, 1, \ldots, M-1, \tag{9.37}$$

or, with W defined as

$$W = e^{-j\Delta\omega} \tag{9.38}$$

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega_0 n}W^{nk}. \tag{9.39}$$

To express $X(e^{j\omega_k})$ as a convolution, we use the identity

$$nk = \tfrac{1}{2}[n^2 + k^2 - (k-n)^2] \tag{9.40}$$

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega_0 n}W^{n^2/2}W^{k^2/2}W^{-(k-n)^2/2}. \tag{9.41}$$

Letting

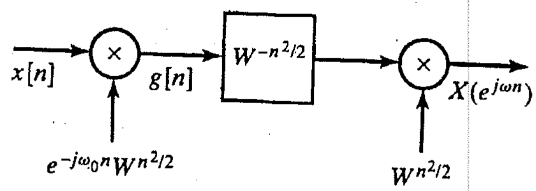
$$g[n] = x[n]e^{-j\omega_0 n}W^{n^2/2}, \tag{9.42}$$

we can then write

$$X(e^{j\omega_k}) = W^{k^2/2}\left(\sum_{n=0}^{N-1} g[n]W^{-(k-n)^2/2}\right), \qquad k = 0, 1, \ldots, M-1. \tag{9.43}$$

$$X(e^{j\omega_n}) = W^{n^2/2}\left(\sum_{k=0}^{N-1} g[k]W^{-(n-k)^2/2}\right), \qquad n = 0, 1, \ldots, M-1. \tag{9.44}$$

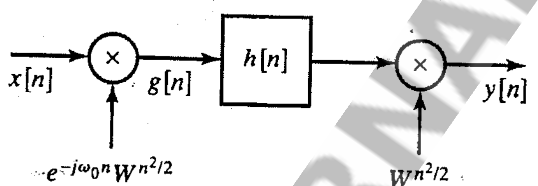
The sequence $W^{-n^2/2}$ can be thought of as a complex exponential sequence with linearly increasing frequency $n\Delta w$. In radar systems, such signals are called chirp



Since $g[n]$ is of finite duration, only a finite portion of the sequence $W^{-n^2/2}$ is used in obtaining $g[n] * W^{-n^2/2}$ over the interval $n = 0, 1, \ldots, M-1$, specifically, that portion from $n = -(N-1)$ to $n = M-1$. Let us define

$$h[n] = \begin{cases} W^{-n^2/2}, & -(N-1) \le n \le M-1, \\ 0, & \text{otherwise.} \end{cases} \tag{9.45}$$

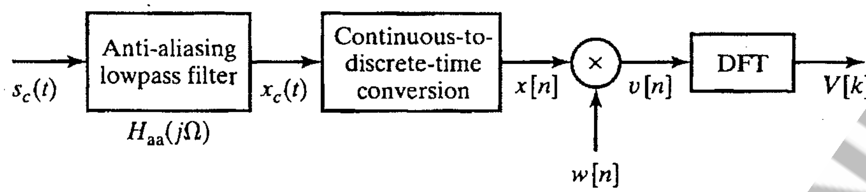
$$g[n] * W^{-n^2/2} = g[n] * h[n], \qquad n = 0, 1, \ldots, M-1. \tag{9.46}$$



$$X(e^{j\omega_n}) = y[n], \qquad n = 0, 1, \ldots, M-1.$$

Evaluation of frequency samples using the procedure has a number of potential advantages. In general, we do not require $N = M$ as in the FFT algorithms, and neither N nor M need be composite numbers. In fact, they may be prime numbers if desired. Furthermore, the parameter ω_0 is arbitrary. This increased flexibility over the FFT does not preclude efficient computation, since the convolution can be implemented efficiently using an FFT algorithm

FOURIER ANALYSIS OF SIGNALS USING THE DISCRETE FOURIER TRANSFORM



The conversion of $x_c(t)$ to the sequence of samples $x[n]$ is represented in the frequency domain by periodic replication and frequency normalization, i.e.,

$$X(e^{j\omega}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} + j\frac{2\pi r}{T}\right). \qquad (10.1)$$

As indicated, the sequence $x[n]$ is typically multiplied by a finite-duration window $w[n]$, since the input to the DFT must be of finite duration. This produces the finite-length sequence $v[n] = w[n]x[n]$. The effect in the frequency domain is a periodic convolution, i.e.,

$$V(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta. \qquad (10.2)$$

If $w[n]$ is constant over the range of n for which it is nonzero, it is referred to as a *rectangular window*. However, as we will see, there are good reasons to taper the window at its edges.

At this point, it is sufficient to observe that convolution of $W(e^{j\omega})$ with $X(e^{j\omega})$ will tend to smooth sharp peaks and discontinuities in $X(e^{j\omega})$.

The DFT of the windowed sequence $v[n] = w[n]x[n]$ is

$$V[k] = \sum_{n=0}^{N-1} v[n]e^{-j(2\pi/N)kn}, \qquad k = 0, 1, \ldots, N-1, \qquad (10.3)$$

where we assume that the window length L is less than or equal to the DFT length N. $V[k]$, the DFT of the finite-length sequence $v[n]$, corresponds to equally spaced samples of the Fourier transform of $v[n]$; i.e.,

$$V[k] = V(e^{j\omega})\big|_{\omega=2\pi k/N}. \qquad (10.4)$$

Since the spacing between DFT frequencies is $2\pi/N$, and the relationship between the normalized discrete-time frequency variable and the continuous-time frequency variable is $\omega = \Omega T$, the DFT frequencies correspond to the continuous-time frequencies

$$\Omega_k = \frac{2\pi k}{NT}. \qquad (10.5)$$

The discrete-time Fourier transform of a sinusoidal signal $A\cos(\omega_0 n+\phi)$ is a pair of impulses at $+\omega_0$ and $-\omega_0$ (repeating periodically with period 2π). In analyzing sinusoidal signals using the DFT, windowing and spectral sampling have an important effect.

Let us consider a continuous-time signal consisting of the sum of two sinusoidal components; i.e.,

$$s_c(t) = A_0\cos(\Omega_0 t+\theta_0) + A_1\cos(\Omega_1 t+\theta_1), \qquad -\infty < t < \infty. \tag{10.6}$$

Assuming ideal sampling with no aliasing and no quantization error, we obtain the discrete-time signal

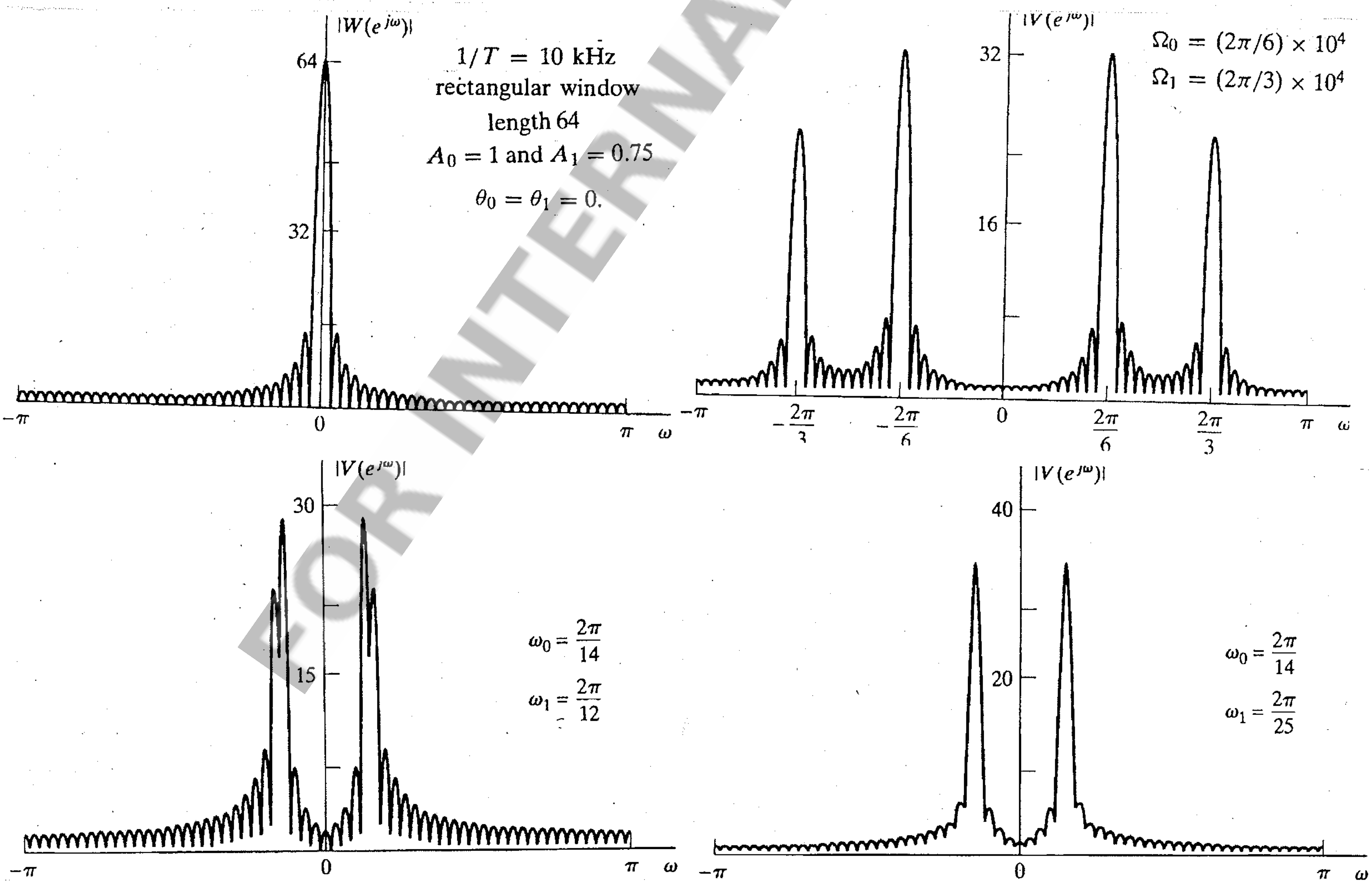
$$x[n] = A_0\cos(\omega_0 n+\theta_0) + A_1\cos(\omega_1 n+\theta_1), \qquad -\infty < n < \infty, \tag{10.7}$$

where $\omega_0 = \Omega_0 T$ and $\omega_1 = \Omega_1 T$. The windowed sequence $v[n]$

$$v[n] = A_0 w[n]\cos(\omega_0 n+\theta_0) + A_1 w[n]\cos(\omega_1 n+\theta_1). \tag{10.8}$$

$$\begin{aligned} v[n] = {} & \frac{A_0}{2}w[n]e^{j\theta_0}e^{j\omega_0 n} + \frac{A_0}{2}w[n]e^{-j\theta_0}e^{-j\omega_0 n} \\ & + \frac{A_1}{2}w[n]e^{j\theta_1}e^{j\omega_1 n} + \frac{A_1}{2}w[n]e^{-j\theta_1}e^{-j\omega_1 n}, \end{aligned} \tag{10.9}$$

$$\begin{aligned} V(e^{j\omega}) = {} & \frac{A_0}{2}e^{j\theta_0}W(e^{j(\omega-\omega_0)}) + \frac{A_0}{2}e^{-j\theta_0}W(e^{j(\omega+\omega_0)}) \\ & + \frac{A_1}{2}e^{j\theta_1}W(e^{j(\omega-\omega_1)}) + \frac{A_1}{2}e^{-j\theta_1}W(e^{j(\omega+\omega_1)}). \end{aligned} \tag{10.10}$$



Reduced resolution and leakage are the two primary effects on the spectrum as a result of applying a window to the signal. The resolution is influenced primarily by the width of the main lobe of $W(e^{j\omega})$, while the degree of leakage depends on the relative amplitude of the main lobe and the side lobes of $W(e^{j\omega})$.

The rectangular window, which has Fourier transform

$$W_r(e^{j\omega}) = \sum_{n=0}^{L-1} e^{-j\omega n} = e^{-j\omega(L-1)/2}\frac{\sin(\omega L/2)}{\sin(\omega/2)}, \qquad (10.11)$$

has the narrowest main lobe for a given length, but it has the largest side lobes of all the commonly used windows. As defined in Chapter 7, the Kaiser window is

$$w_K[n] = \begin{cases} \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le L-1, \\ 0, & \text{otherwise,} \end{cases} \qquad (10.12)$$

where $\alpha = (L-1)/2$ and $I_0(\cdot)$ is the zeroth-order modified Bessel function of the first kind.

We have already seen in the context of the filter design problem that this window has two parameters, β and L, which can be used to trade between main-lobe width and relative side-lobe amplitude. (Recall that the Kaiser window reduces to the rectangular window when $\beta = 0$.) The main-lobe width Δ_{ml} is defined as the symmetric distance between the central zero-crossings. The relative side-lobe level A_{sl} is defined as the ratio in dB of the amplitude of the main lobe to the amplitude of the largest side lobe.

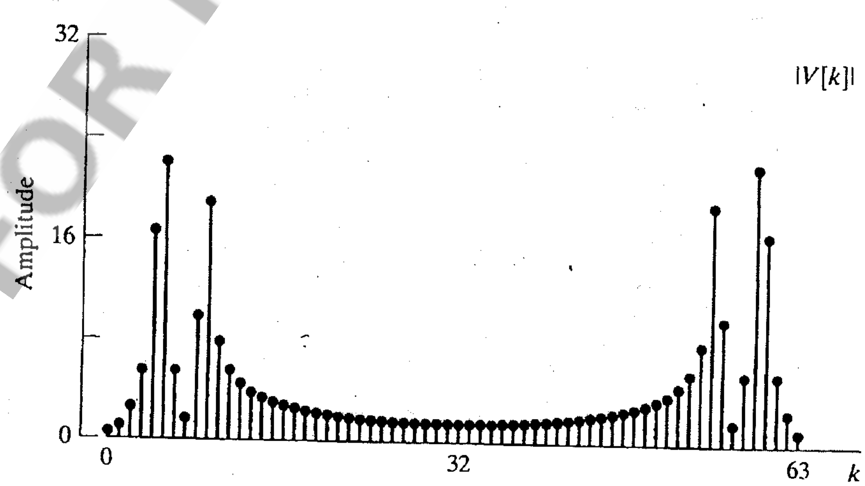
The trade-off between main-lobe width, relative side-lobe amplitude, and window length is displayed by the approximate relationship

$$L \simeq \frac{24\pi(A_{sl}+12)}{155\Delta_{ml}} + 1, \qquad (10.14)$$

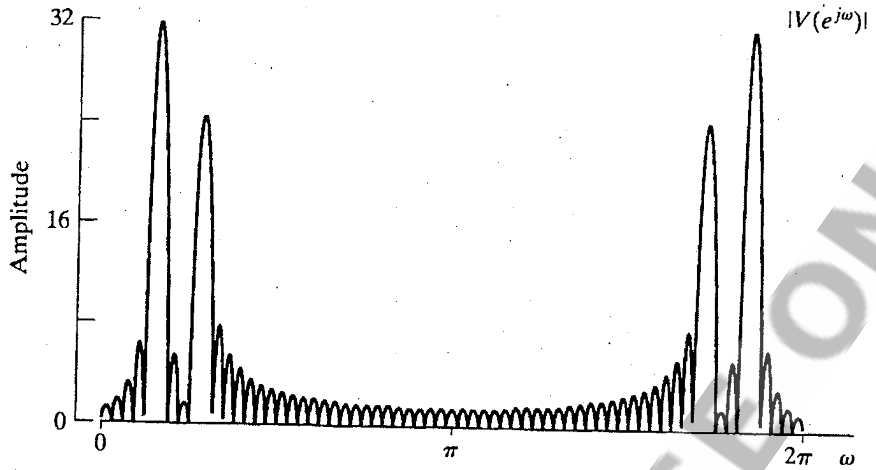
which was also given by Kaiser and Schafer (1980).

Let us consider the same parameters , i.e., $A_0 = 1$, $A_1 = 0.75$, $\omega_0 = 2\pi/14$, $\omega_1 = 4\pi/15$, and $\theta_1 = \theta_2 = 0$ in Eq. (10.8). $w[n]$ is a rectangular window of length 64. Then

$$v[n] = \begin{cases} \cos\left(\dfrac{2\pi}{14}n\right) + 0.75\cos\left(\dfrac{4\pi}{15}n\right), & 0 \le n \le 63, \\ 0, & \text{otherwise.} \end{cases} \qquad (10.15)$$

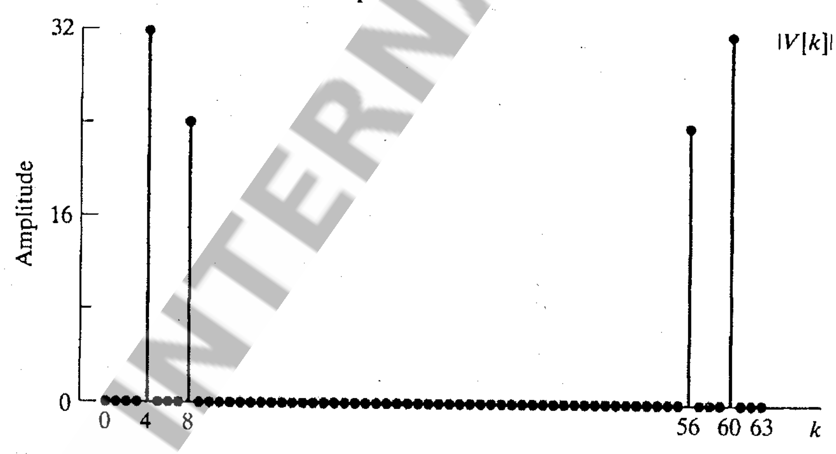


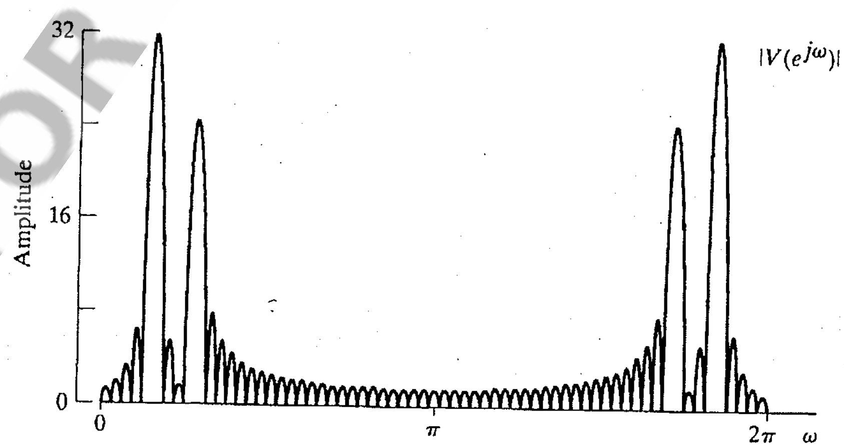
As is the usual convention in displaying the DFT of a time sequence, we display the DFT values in the range from $k = 0$ to $k = N - 1$, corresponding to displaying samples of the discrete-time Fourier transform in the frequency range 0 to 2π. Because of the inherent periodicity of the discrete-time Fourier transform, the first half of this range corresponds to the positive continuous-time frequencies, i.e., Ω between zero and π/T, and the second half of the range to the negative frequencies, i.e., Ω between $-\pi/T$ and zero.



$$v[n] = \begin{cases} \cos\left(\dfrac{2\pi}{16}n\right) + 0.75\cos\left(\dfrac{2\pi}{8}n\right), & 0 \le n \le 63, \\ 0, & \text{otherwise}, \end{cases} \tag{10.16}$$

Again, a rectangular window is used with $N = L = 64$. This is very similar to the previous example, except that in this case, the frequencies of the cosines coincide exactly with two of the DFT frequencies. Specifically, the frequency $\omega_1 = 2\pi/8 = 2\pi 8/64$ corresponds exactly to the DFT sample $k = 8$ and the frequency $\omega_0 = 2\pi/16 = 2\pi 4/64$ to the DFT sample $k = 4$.



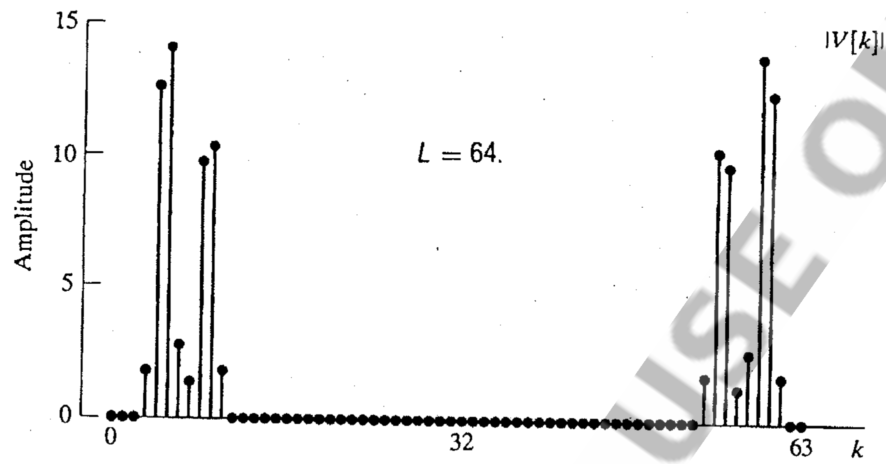


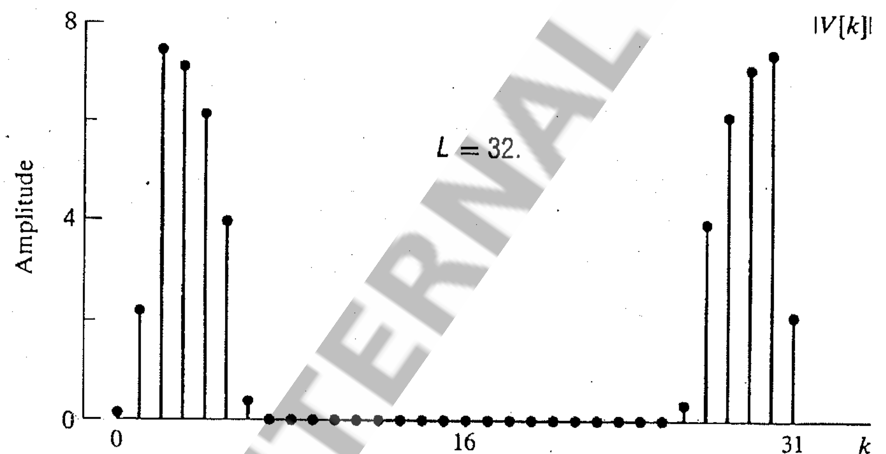
DFT Analysis of Sinusoidal Signals Using a Kaiser Window

Let us return to the frequency, amplitude, and phase parameters of Example 10.3, but now with a Kaiser window applied, so that

$$v[n] = w_K[n]\cos\left(\frac{2\pi}{14}n\right) + 0.75w_K[n]\cos\left(\frac{4\pi}{15}n\right), \tag{10.17}$$

where $w_K[n]$ is the Kaiser window as given by Eq. (10.12). We will select the Kaiser window parameter β to be equal to 5.48.





For a complete representation of a sequence of length L, the L-point DFT is sufficient, since the original sequence can be recovered exactly from it. However, as we saw in the preceding examples, simple examination of the L-point DFT can result in misleading interpretations. For this reason, it is common to apply zero-padding so that the spectrum is sufficiently oversampled and important features are therefore readily apparent. With a high degree of time-domain zero-padding or frequency-domain oversampling, simple interpolation (e.g., linear interpolation) between the DFT values provides a reasonably accurate picture of the Fourier spectrum, which can then be used, for example, to estimate the locations and amplitudes of spectral peaks.

THE TIME-DEPENDENT FOURIER TRANSFORM

The time-dependent Fourier transform of a signal $x[n]$ is defined as

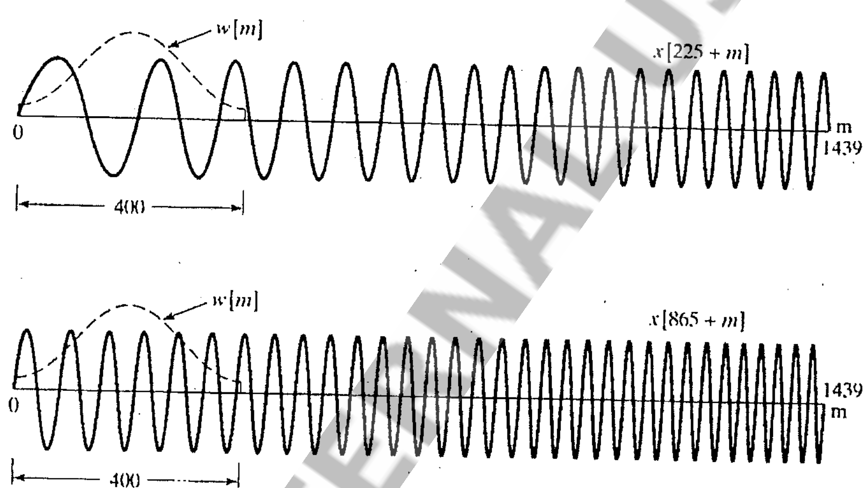
$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}, \tag{10.18}$$

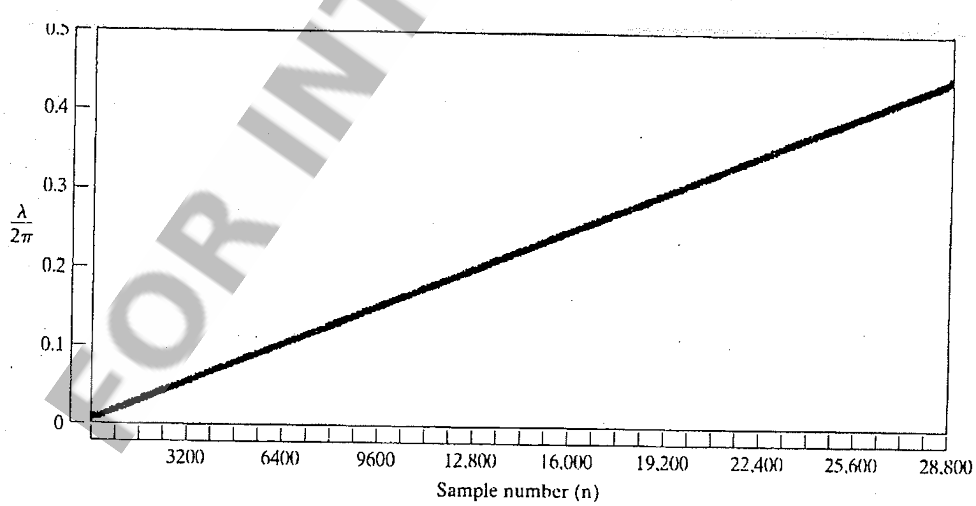
where $w[n]$ is a window sequence. In the time-dependent Fourier representation, the one-dimensional sequence $x[n]$, a function of a single discrete variable, is converted into a two-dimensional function of the time variable n, which is discrete, and the frequency variable λ, which is continuous.[2] Note that the time-dependent Fourier transform is periodic in λ with period 2π, and therefore, we need consider only values of λ for $0 \le \lambda < 2\pi$ or any other interval of length 2π.

$$x[n] = \cos(\omega_0 n^2), \qquad \omega_0 = 2\pi \times 7.5 \times 10^{-6}, \tag{10.19}$$

corresponding to a linear frequency modulation (i.e., the "instantaneous frequency" is $2\omega_0 n$).

Typically, $w[m]$ in Eq. (10.18) has finite length around $m = 0$, so that $X[n,\lambda)$ displays the frequency characteristics of the signal around time n.





The magnitude of the time-dependent Fourier transform of $x[n] = \cos(\omega_0 n^2)$ using a Hamming window of length 400.

Since $X[n, \lambda)$ is the discrete-time Fourier transform of $x[n+m]w[m]$, the time-dependent Fourier transform is invertible if the window has at least one nonzero sample.

$$x[n+m]w[m] = \frac{1}{2\pi}\int_0^{2\pi} X[n,\lambda)e^{j\lambda m}d\lambda, \qquad -\infty < m < \infty, \tag{10.20}$$

from which it follows that

$$x[n] = \frac{1}{2\pi w[0]}\int_0^{2\pi} X[n,\lambda)d\lambda \tag{10.21}$$

if $w[0] \neq 0$. $X[n, \lambda)$ can be written as

$$X[n,\lambda) = \sum_{m'=-\infty}^{\infty} x[m']w[-(n-m')]e^{j\lambda(n-m')}. \tag{10.22}$$

Equation (10.22) can be interpreted as the convolution

$$X[n,\lambda) = x[n] * h_\lambda[n], \tag{10.23a}$$

where

$$h_\lambda[n] = w[-n]e^{j\lambda n}. \tag{10.23b}$$

$$H_\lambda(e^{j\omega}) = W(e^{j(\lambda-\omega)}). \tag{10.24}$$

In general, a window that is nonzero for positive time will be called a *noncausal window,* since the computation of $X[n, \lambda)$ using Eq. (10.18) requires samples that *follow* sample n in the sequence. Equivalently, in the linear-filtering interpretation, the impulse response $h_\lambda[n] = w[-n]e^{j\lambda n}$ is noncausal.

Another possibility is to shift the window as n changes, keeping the time origin for Fourier analysis fixed

This leads to a definition for the time-dependent Fourier transform of the form

$$\check{X}[n,\lambda) = \sum_{m=-\infty}^{\infty} x[m]w[m-n]e^{-j\lambda m} = e^{-j\lambda n}X[n,\lambda) \tag{10.25}$$

The Effect of the Window

The primary purpose of the window in the time-dependent Fourier transform is to limit the extent of the sequence to be transformed so that the spectral characteristics are reasonably stationary over the duration of the window. The more rapidly the signal characteristics change, the shorter the window should be.

If we consider the time-dependent Fourier transform for fixed n, then it follows from the properties of Fourier transforms that

$$X[n,\lambda) = \frac{1}{2\pi}\int_0^{2\pi} e^{j\theta n}X(e^{j\theta})W(e^{j(\lambda-\theta)})d\theta; \tag{10.28}$$

i.e., the Fourier transform of the shifted signal is convolved with the Fourier transform of the window. In Section 10.2 we saw that the ability to resolve two narrowband signal components depends on the width of the main lobe of the Fourier transform of the window, while the degree of leakage of one component into the vicinity of the other depends on the relative side-lobe amplitude. The case of no window at all corresponds to $w[n] = 1$ for all n. In this case $W(e^{j\omega}) = 2\pi\delta(\omega)$ for $-\pi \le \omega \le \pi$, which gives precise frequency resolution, but no time resolution.

The preceding discussion suggests that if we are using the time-dependent Fourier transform to obtain a time-dependent estimate of the frequency spectrum of a signal, it is desirable to taper the window to lower the side lobes and to use as long a window as feasible to improve the frequency resolution.

Sampling in Time and Frequency

$$w[m] = 0 \qquad \text{outside the interval } 0 \le m \le L-1. \tag{10.29}$$

If we sample $X[n, \lambda)$ at N equally spaced frequencies $\lambda_k = 2\pi k/N$, with $N \ge L$, then we can still recover the original sequence from the sampled time-dependent Fourier transform. Specifically, if we define $X[n, k]$ to be

$$X[n, k] = X[n, 2\pi k/N) = \sum_{m=0}^{L-1} x[n+m]w[m]e^{-j(2\pi/N)km}, \qquad 0 \le k \le N-1, \tag{10.30}$$

then $X[n, k]$ is the DFT of the windowed sequence $x[n+m]w[m]$. Using the inverse DFT, we obtain

$$x[n+m]w[m] = \frac{1}{N}\sum_{k=0}^{N-1} X[n, k]e^{j(2\pi/N)km}, \qquad 0 \le m \le L-1. \tag{10.31}$$

Since we assume that the window $w[m] \ne 0$ for $0 \le m \le L-1$, the sequence values can be recovered in the interval from n through $(n+L-1)$ using the equation

$$x[n+m] = \frac{1}{Nw[m]}\sum_{k=0}^{N-1} X[n, k]e^{j(2\pi/N)km}, \qquad 0 \le m \le L-1, \tag{10.32}$$

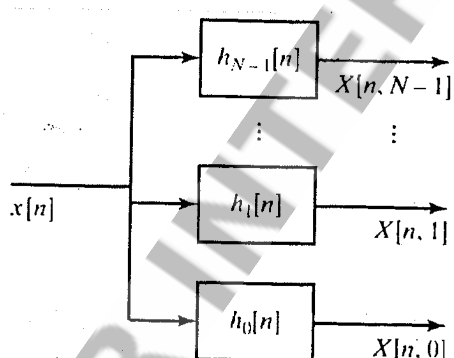
where it is assumed that $w[m] \ne 0$ for $0 \le m \le L-1$.

Eq. (10.30) can be rewritten as

$$X[n, k] = x[n] * h_k[n], \qquad 0 \le k \le N-1, \tag{10.33a}$$

where

$$h_k[n] = w[-n]e^{j(2\pi/N)kn}. \tag{10.33b}$$



Filter bank representation of the time-dependent Fourier transform.

$$H_k(e^{j\omega}) = W(e^{j[(2\pi k/N)-\omega]}). \tag{10.34}$$

Our discussion suggests that $x[n]$ for $-\infty < n < \infty$ can be reconstructed if $X[n, \lambda)$ or $X[n, k]$ is sampled in the time dimension as well. Specifically, using Eq. (10.32), we can reconstruct the signal in the interval $n_0 \le n \le n_0 + L - 1$ from $X[n_0, k]$, and we can reconstruct the signal in the interval $n_0 + L \le n \le n_0 + 2L - 1$ from $X[n_0 + L, k]$, etc. Thus, $x[n]$ can be reconstructed exactly from the time-dependent Fourier transform sampled in both the frequency and the time dimension.

we define this sampled time-dependent Fourier transform as

$$X[rR,k] = X[rR,2\pi k/N) = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j(2\pi/N)km}. \qquad (10.35)$$

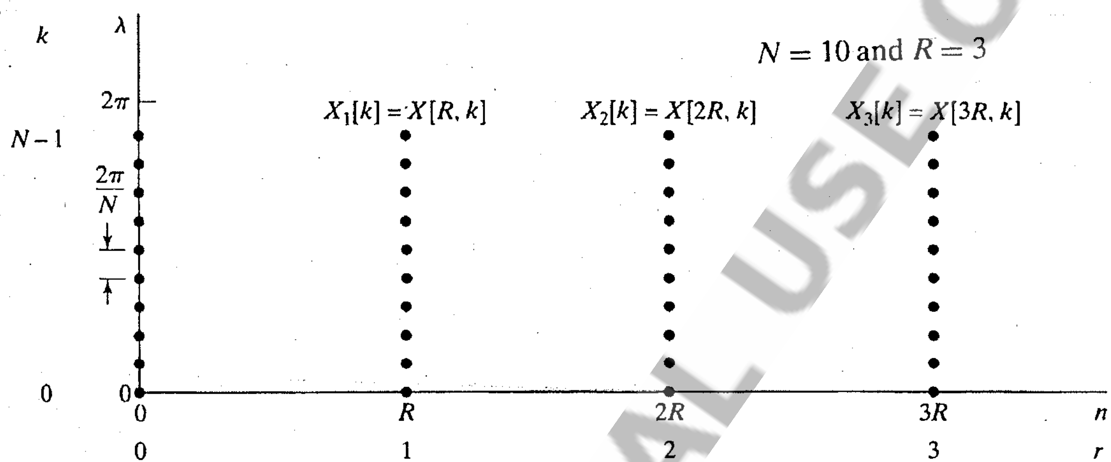
where r and k are integers such that $-\infty < r < \infty$ and $0 \le k \le N-1$. To further simplify our notation, we define

$$X_r[k] = X[rR,k] = X[rR,\lambda_k), \qquad -\infty < r < \infty, \quad 0 \le k \le N-1, \qquad (10.36)$$

where $\lambda_k = 2\pi k/N$. This notation denotes explicitly that the sampled time-dependent Fourier transform is simply a sequence of N-point DFTs of the windowed signal segments

$$x_r[m] = x[rR+m]w[m], \qquad -\infty < r < \infty, \quad 0 \le m \le L-1, \qquad (10.37)$$

with the window position moving in jumps of R samples in time.



Equation (10.35) involves the following integer parameters: the window length L; the number of samples in the frequency dimension, or the DFT length N; and the sampling interval in the time dimension, R. However, not all choices of these parameters will permit exact reconstruction of the signal. The choice $L \le N$ guarantees that we *can* reconstruct the windowed segments $x_r[m]$ from the block transforms $X_r[k]$. If $R < L$, the segments overlap, but if $R > L$, some of the samples of the signal are not used and therefore cannot be reconstructed from $X_r[k]$. Thus, in general, the three sampling parameters should satisfy the relation $N \ge L \ge R$.

BLOCK CONVOLUTION USING THE TIME-DEPENDENT FOURIER TRANSFORM

Assume that $x[n] = 0$ for $n < 0$, and suppose that we compute the time-dependent Fourier transform for $R = L$ and a rectangular window. In other words, the sampled time-dependent Fourier transform $X_r[k]$ consists of a set of N-point DFTs of segments of the input sequence

$$x_r[m] = x[rL+m], \qquad 0 \le m \le L-1. \qquad (10.38)$$

Since each sample of the signal $x[n]$ is included and the blocks do not overlap, it follows that

$$x[n] = \sum_{r=0}^{\infty} x_r[n-rL]. \qquad (10.39)$$

Now suppose that we define a new time-dependent Fourier transform

$$Y_r[k] = H[k]X_r[k], \qquad 0 \le k \le N-1, \tag{10.40}$$

where $H[k]$ is the N-point DFT of a finite-length unit sample sequence $h[n]$ such that $h[n] = 0$ for $n < 0$ and for $n > P-1$. If we compute the inverse DFT of $Y_r[k]$, we obtain

$$y_r[m] = \frac{1}{N}\sum_{k=0}^{N-1} Y_r[k]e^{j(2\pi/N)km} = \sum_{\ell=0}^{N-1} x_r[\ell]h[((m-\ell))_N]. \tag{10.41}$$

That is, $y_r[m]$ is the N-point circular convolution of $h[m]$ and $x_r[m]$. Since $h[m]$ has length P samples and $x_r[m]$ has length L samples, it follows from the discussion of Section 8.7 that if $N \ge L+P-1$, then $y_r[m]$ will be identical to the linear convolution of $h[m]$ with $x_r[m]$ in the interval $0 \le m \le L+P-2$, and it will be zero otherwise. Thus, it follows that if we construct an output signal

$$y[n] = \sum_{r=0}^{\infty} y_r[n-rL], \tag{10.42}$$

then $y[n]$ is the output of a linear time-invariant system with impulse response $h[n]$. The procedure just described corresponds exactly to the *overlap–add* method of block convolution.

FOURIER ANALYSIS OF STATIONARY RANDOM SIGNALS: THE PERIODOGRAM

Let us consider the problem of estimating the power density spectrum $P_{ss}(\Omega)$ of a continuous-time signal $s_c(t)$.

The antialiasing lowpass filter creates a new stationary random signal whose power spectrum is bandlimited, so that the signal can be sampled without aliasing. Then $x[n]$ is a stationary discrete-time random signal whose power density spectrum $P_{xx}(\omega)$ is proportional to $P_{ss}(\Omega)$ over the bandwidth of the antialiasing filter; i.e.,

$$P_{xx}(\omega) = \frac{1}{T}P_{ss}\left(\frac{\omega}{T}\right), \qquad |\omega| < \pi, \tag{10.50}$$

where we have assumed that the cutoff frequency of the antialiasing filter is π/T and that T is the sampling period.

Consequently, a reasonable estimate of $P_{xx}(\omega)$ will provide a reasonable estimate of $P_{ss}(\Omega)$. The window $w[n]$ in Figure 10.1 selects a finite-length segment (L samples) of $x[n]$, which we denote $v[n]$, the Fourier transform of which is

$$V(e^{j\omega}) = \sum_{n=0}^{L-1} w[n]x[n]e^{-j\omega n}. \tag{10.51}$$

Consider as an estimate of the power spectrum the quantity

$$I(\omega) = \frac{1}{LU}|V(e^{j\omega})|^2, \tag{10.52}$$

where the constant U anticipates a need for normalization to remove bias in the spectral estimate. When the window $w[n]$ is the rectangular window sequence, this estimator for the power spectrum is called the *periodogram*. If the window is not rectangular, $I(\omega)$ is called the *modified periodogram*. Clearly, the periodogram has some of the basic properties of the power spectrum. It is nonnegative, and for real signals, it is a real and even function of frequency.

Specifically, samples of the periodogram are given by

$$I(\omega_k) = \frac{1}{LU}|V[k]|^2, \qquad U = \frac{1}{L}\sum_{n=0}^{L-1}(w[n])^2 \tag{10.55}$$

where $V[k]$ is the N-point DFT of $w[n]x[n]$. If we want to choose N to be greater than the window length L, appropriate zero-padding would be applied to the sequence $w[n]x[n]$.

If a random signal has a nonzero mean, its power spectrum has an impulse at zero frequency. If the mean is relatively large, this component will dominate the spectrum estimate, causing low-amplitude, low-frequency components to be obscured by leakage. Therefore, in practice the mean is often estimated using Eq. (10.48), and the resulting estimate is subtracted from the random signal before computing the power spectrum estimate.

However, it has been shown (see Jenkins and Watts, 1968) that over a wide range of conditions, as the window length increases,

$$\text{var}[I(\omega)] \simeq P_{xx}^2(\omega). \tag{10.65}$$

That is, the variance of the periodogram estimate is approximately the same size as the square of the power spectrum that we are estimating. Therefore, since the variance does not asymptotically approach zero with increasing window length, the periodogram is not a consistent estimate.

Periodogram Averaging

The averaging of periodograms in spectrum estimation was first studied extensively by Bartlett (1953); later, after fast algorithms for computing the DFT were developed, Welch (1970) combined these computational algorithms with the use of a data window $w[n]$ to develop the method of averaging modified periodograms. In periodogram averaging, a data sequence $x[n], 0 \le n \le Q-1$, is divided into segments of length-L samples, with a window of length L applied to each; i.e., we form the segments

$$x_r[n] = x[rR+n]w[n], \qquad 0 \le n \le L-1. \tag{10.67}$$

The periodogram of the rth segment is

$$I_r(\omega) = \frac{1}{LU}|X_r(e^{j\omega})|^2, \tag{10.68}$$

where $X_r(e^{j\omega})$ is the discrete-time Fourier transform of $x_r[n]$. Each $I_r(\omega)$ has the properties of a periodogram, as described previously. Periodogram averaging consists of averaging together the K periodogram estimates $I_r(\omega)$; i.e., we form the time-averaged periodogram defined as

$$\bar{I}(\omega) = \frac{1}{K}\sum_{r=0}^{K-1} I_r(\omega). \tag{10.69}$$

To examine the variance, we use the fact that, in general, the variance of the average of K independent identically distributed random variables is $1/K$ times the variance of each individual random variable. (See Papoulis, 1991.) Therefore,

$$\text{var}[\bar{I}(\omega)] \simeq \frac{1}{K}P_{xx}^2(\omega). \tag{10.76}$$

Consequently, the variance of $\bar{I}(\omega)$ is inversely proportional to the number of periodograms averaged, and as K increases, the variance approaches zero.