# Discrete Mathematics

## Algebraic Structures

H. Turgut Uyar Ayşegül Gençata Yayımlı Emre Harmancı

2001-2016



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## Topics

Algebraic Structures
- Introduction
- Algebraic Families
- Groups

Lattices
- Partially Ordered Sets
- Lattices
- Boolean Algebra



## Algebraic Structures

- algebraic structure: <set, operations, constants>
- carrier set
- operations: binary, unary
- constants: identity, zero

## Operations

- every operation is a function
- binary operation:
  $\circ : S \times S \to T$
- unary operation:
  $\Delta : S \to T$
- closed: $T \subseteq S$

example

- subtraction is closed on $\mathbb{Z}$
- subtraction is not closed on $\mathbb{Z}^+$



## Constants

Definition
identity: 1
$x \circ 1 = 1 \circ x = x$

- left identity: $1_l \circ x = x$
- right identity: $x \circ 1_r = x$

Definition
zero: 0
$x \circ 0 = 0 \circ x = 0$

- left zero: $0_l \circ x = 0$
- right zero: $x \circ 0_r = 0$



## Examples of Constants

- identity for $< \mathbb{N}, max >$ is 0
- zero for $< \mathbb{N}, min >$ is 0
- zero for $< \mathbb{Z}^+, min >$ is 1



## Examples of Constants

| $\circ$ | a | b | c |
|---|---|---|---|
| a | a | b | b |
| b | a | b | c |
| c | a | b | a |

- $b$ is a left identity
- $a$ and $b$ are right zeros

## Constants

**Theorem**

$$\exists 1_l \wedge \exists 1_r \Rightarrow 1_l = 1_r$$

**Proof.**
$1_l \circ 1_r = 1_l = 1_r$ □

**Theorem**

$$\exists 0_l \wedge \exists 0_r \Rightarrow 0_l = 0_r$$

**Proof.**
$0_l \circ 0_r = 0_l = 0_r$ □



## Inverse

- $x \circ y = 1$:
  $x$ is a *left inverse* of $y$
  $y$ is a *right inverse* of $x$
- $x \circ y = y \circ x = 1$:
  $x$ is an inverse of $y$
  $y$ is an inverse of $x$



## Inverse

**Theorem**
$\circ$ *associative*

$$w \circ x = x \circ y = 1 \Rightarrow w = y$$

**Proof.**

$$\begin{aligned} w &= w \circ 1 \\ &= w \circ (x \circ y) \\ &= (w \circ x) \circ y \\ &= 1 \circ y \\ &= y \end{aligned}$$

□



## Algebraic Families

- algebraic family: structure and axioms
- axioms: associativity, commutativity, inverses, ...

## Algebraic Family Examples

- axioms:
- $x \circ y = y \circ x$
- $(x \circ y) \circ z = x \circ (y \circ z)$
- $x \circ 1 = x$
- structures for which these axioms hold:
- $< \mathbb{Z}, +, 0 >$
- $< \mathbb{Z}, \cdot, 1 >$
- $< \mathcal{P}(S), \cup, \emptyset >$



## Subalgebra

- $A = < S, \circ, \Delta, k >$
  $A' = < S', \circ', \Delta', k' >$
- $A'$ is a subalgebra of $A$:
- $S' \subseteq S$
- $k' = k$
- $\forall a, b \in S' \; a \circ' b = a \circ b \in S'$
- $\forall a \in S' \; \Delta' a = \Delta a \in S'$



## Subalgebra Examples

- $< \mathbb{Z}^+, +, 0 >$ is a subalgebra of $< \mathbb{Z}, +, 0 >$
- $< \mathbb{N}, -, 0 >$ is not a subalgebra of $< \mathbb{Z}, -, 0 >$



## Semigroups

Definition
semigroup: $< S, \circ >$

- $\forall a, b, c \in S \; (a \circ b) \circ c = a \circ (b \circ c)$

## Semigroup Example

- $< \Sigma^+, \& >$
- $\Sigma$: alphabet, $\Sigma^+$: strings of length at least 1
- $\&$: string concatenation



## Monoids

Definition

monoid: $< S, \circ, 1 >$

- $\forall a, b, c \in S \ (a \circ b) \circ c = a \circ (b \circ c)$
- $\forall a \in S \ a \circ 1 = 1 \circ a = a$



## Monoid Example

- $< \Sigma^*, \&, \epsilon >$
- $\Sigma$: alphabet, $\Sigma^*$: strings of any length
- $\&$: string concatenation
- $\epsilon$: empty string



## Groups

Definition

group: $< S, \circ, 1 >$

- $\forall a, b, c \in S \ (a \circ b) \circ c = a \circ (b \circ c)$
- $\forall a \in S \ a \circ 1 = 1 \circ a = a$
- $\forall a \in S \ \exists a^{-1} \in S \quad a \circ a^{-1} = a^{-1} \circ a = 1$
- *Abelian group*: $\forall a, b \in S \ a \circ b = b \circ a$

## Group Examples

- $< \mathbb{Z}, +, 0 >$ is a group
- $< \mathbb{Q}, \cdot, 1 >$ is not a group
- $< \mathbb{Q} - \{0\}, \cdot, 1 >$ is a group



## Group Example

- $a \circ b = a + b + ab$
- is $< \mathbb{Z}, \circ >$ a group?
- is $\circ$ associative?

$$\begin{aligned} (a \circ b) \circ c &= (a + b + ab) + c + (a + b + ab) \cdot c \\ &= a + b + ab + c + ac + bc + abc \\ &= a + b + c + bc + ab + ac + abc \\ &= a + (b + c + bc) + a \cdot (b + c + bc) \\ &= a \circ (b \circ c) \end{aligned}$$



## Group Example

- is there an identity element?

$$a \circ 0 = a + 0 + a \cdot 0 = a$$

- does every element have an inverse?

$$\begin{aligned} && a \circ a^{-1} &= 0 \\ \Rightarrow && a + a^{-1} + a \cdot a^{-1} &= 0 \\ \Rightarrow && a + a^{-1} \cdot (1 + a) &= 0 \\ \Rightarrow && a^{-1} &= -\frac{a}{1+a} \end{aligned}$$

-1 doesn't have an inverse, not a group



## Group Example: Permutations

- permutation: a bijective function on a set
- $A = \{a_1, a_2, \ldots, a_n\}$

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ p(a_1) & p(a_2) & \ldots & p(a_n) \end{pmatrix}$$

- permutation composition: $\diamond$

## Permutation Example

- $A = \{1, 2, 3\}$

$$p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad p_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$



## Permutation Composition Example

- $A = \{1, 2, 3\}$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$p_3 \diamond p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$



## Group Example: Permutations

- permutation composition is associative
- identity permutation: $1_A$

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

- $Perm(A)$: set of all permutations of the elements of $A$
- $< Perm(A), \diamond, 1_A >$ is a group



## Group Example: Permutation

- $A = \{1, 2, 3, 4\}$

| $A$ | $1_A$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ | $p_9$ | $p_{10}$ | $p_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| 2 | 2 | 2 | 3 | 3 | 4 | 4 | 1 | 1 | 3 | 3 | 4 | 4 |
| 3 | 3 | 4 | 2 | 4 | 2 | 3 | 3 | 4 | 1 | 4 | 1 | 3 |
| 4 | 4 | 3 | 4 | 2 | 3 | 2 | 4 | 3 | 4 | 1 | 3 | 1 |

| | $p_{12}$ | $p_{13}$ | $p_{14}$ | $p_{15}$ | $p_{16}$ | $p_{17}$ | $p_{18}$ | $p_{19}$ | $p_{20}$ | $p_{21}$ | $p_{22}$ | $p_{23}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 4 |
| 2 | 1 | 1 | 2 | 2 | 4 | 4 | 1 | 1 | 2 | 2 | 3 | 3 |
| 3 | 2 | 4 | 1 | 4 | 1 | 2 | 2 | 3 | 1 | 3 | 1 | 2 |
| 4 | 4 | 2 | 4 | 1 | 2 | 1 | 3 | 2 | 3 | 1 | 2 | 1 |

## Group Example: Permutation

- $p_8 \diamond p_{12} = p_{12} \diamond p_8 = 1_A$:
  $p_{12} = p_8^{-1}$, $p_8 = p_{12}^{-1}$
- $p_{14} \diamond p_{14} = 1_A$:
  $p_{14} = p_{14}^{-1}$
- $G =< \{1_A, p_1, \ldots, p_{23}\}, \diamond, 1_A >$ is a group



## Group Example: Permutation

- $G' =< \{1_A, p_2, p_6, p_8, p_{12}, p_{14}\}, \diamond, 1_A >$

| $\diamond$ | $1_A$ | $p_2$ | $p_6$ | $p_8$ | $p_{12}$ | $p_{14}$ |
|---|---|---|---|---|---|---|
| $1_A$ | $1_A$ | $p_2$ | $p_6$ | $p_8$ | $p_{12}$ | $p_{14}$ |
| $p_2$ | $p_2$ | $1_A$ | $p_8$ | $p_6$ | $p_{14}$ | $p_{12}$ |
| $p_6$ | $p_6$ | $p_{12}$ | $1_A$ | $p_{14}$ | $p_2$ | $p_8$ |
| $p_8$ | $p_8$ | $p_{14}$ | $p_2$ | $p_{12}$ | $1_A$ | $p_6$ |
| $p_{12}$ | $p_{12}$ | $p_6$ | $p_{14}$ | $1_A$ | $p_8$ | $p_2$ |
| $p_{14}$ | $p_{14}$ | $p_8$ | $p_{12}$ | $p_2$ | $p_6$ | $1_A$ |

- $G'$ is a subgroup of $G$



## Cancellation in Groups

**Theorem**
$a \circ c = b \circ c \Rightarrow a = b$
$c \circ a = c \circ b \Rightarrow a = b$

**Proof.**

$$\begin{array}{rrcl} & a \circ c & = & b \circ c \\ \Rightarrow & (a \circ c) \circ c^{-1} & = & (b \circ c) \circ c^{-1} \\ \Rightarrow & a \circ (c \circ c^{-1}) & = & b \circ (c \circ c^{-1}) \\ \Rightarrow & a \circ 1 & = & b \circ 1 \\ \Rightarrow & a & = & b \end{array}$$

□



## Basic Theorem of Groups

**Theorem**
*The unique solution of the equation $a \circ x = b$ is:*
$x = a^{-1} \circ b$

**Proof.**

$$\begin{array}{rrcl} & a \circ x & = & b \\ \Rightarrow & a^{-1} \circ (a \circ x) & = & a^{-1} \circ b \\ \Rightarrow & 1 \circ x & = & a^{-1} \circ b \\ \Rightarrow & x & = & a^{-1} \circ b \end{array}$$

□

## Ring

Definition
ring: $< S, +, \cdot, 0 >$

- $\forall a, b, c \in S \;\; (a + b) + c = a + (b + c)$
- $\forall a \in S \;\; a + 0 = 0 + a = a$
- $\forall a \in S \;\; \exists(-a) \in S \;\; a + (-a) = (-a) + a = 0$
- $\forall a, b \in S \;\; a + b = b + a$
- $\forall a, b, c \in S \;\; (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- $\forall a, b, c \in S$
  - $a \cdot (b + c) = a \cdot b + a \cdot c$
  - $(b + c) \cdot a = b \cdot a + c \cdot a$



## Field

Definition
field: $< S, +, \cdot, 0, 1 >$

- all properties of a ring
- $\forall a, b \in S \;\; a \cdot b = b \cdot a$
- $\forall a \in S \;\; a \cdot 1 = 1 \cdot a = a$
- $\forall a \in S \;\; \exists a^{-1} \in S \;\; a \cdot a^{-1} = a^{-1} \cdot a = 1$



## References

Grimaldi

- Chapter 5: Relations and Functions
  - 5.4. Special Functions
- Chapter 16: Groups, Coding Theory, and Polya's Method of Enumeration
  - 16.1. Definitions, Examples, and Elementary Properties
- Chapter 14: Rings and Modular Arithmetic
  - 14.1. The Ring Structure: Definition and Examples



## Partially Ordered Set

Definition
partial order relation:

- reflexive
- anti-symmetric
- transitive

- *partially ordered set* (poset):
  a set with a partial order relation defined on its elements

## Partial Order Examples

Example (set of sets, $\subseteq$)

- $A \subseteq A$
- $A \subseteq B \wedge B \subseteq A \Rightarrow A = B$
- $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$



## Partial Order Examples

Example ($\mathbb{Z}$, $\leq$)

- $x \leq x$
- $x \leq y \wedge y \leq x \Rightarrow x = y$
- $x \leq y \wedge y \leq z \Rightarrow x \leq z$



## Partial Order Examples

Example ($\mathbb{Z}^+$, $|$)

- $x|x$
- $x|y \wedge y|x \Rightarrow x = y$
- $x|y \wedge y|z \Rightarrow x|z$



## Comparability

- $a \preceq b$: *a precedes b*
- $a \preceq b \vee b \preceq a$: *a and b are comparable*
- total order (linear order):
  all elements are comparable with each other

## Comparability Examples

**Example**

- $\mathbb{Z}^+, |$: 3 and 5 are not comparable
- $\mathbb{Z}, \leq$: total order



## Hasse Diagrams

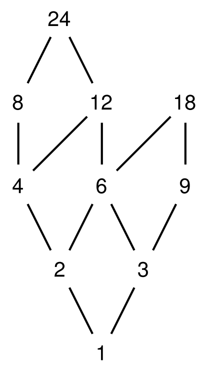
- $a \ll b$: *a immediately precedes b*
  $\neg \exists x \; a \preceq x \preceq b$
- Hasse diagram:
  - draw a line between $a$ and $b$ if $a \ll b$
  - preceding element is below



## Hasse Diagram Examples

**Example**

$\{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$
the relation $|$



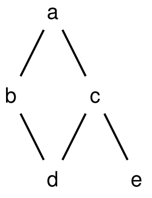



## Consistent Enumeration

- consistent enumeration:
  $f : S \to \mathbb{N}$
  $a \preceq b \Rightarrow f(a) \leq f(b)$
- there can be more than one consistent enumeration

## Consistent Enumeration Examples

### Example




- $\{a \longmapsto 5, b \longmapsto 3, c \longmapsto 4, d \longmapsto 1, e \longmapsto 2\}$
- $\{a \longmapsto 5, b \longmapsto 4, c \longmapsto 3, d \longmapsto 2, e \longmapsto 1\}$



## Maximal - Minimal Elements

### Definition

maximal element: $max$

$\forall x \in S \;\; max \preceq x \Rightarrow x = max$

### Definition

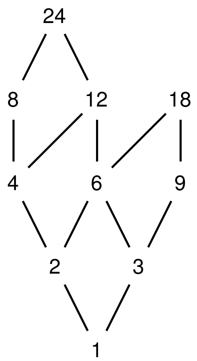
minimal element: $min$

$\forall x \in S \;\; x \preceq min \Rightarrow x = min$



## Maximal - Minimal Element Examples

### Example




$max: \;\; 18, 24$

$min: \;\; 1$



## Bounds

### Definition

$A \subseteq S$

$M$ is an upper bound of $A$:

$\forall x \in A \;\; x \preceq M$

$M(A)$: set of upper bounds of $A$

$sup(A)$ is the supremum of $A$:

$\forall M \in M(A) \;\; sup(A) \preceq M$

### Definition

$A \subseteq S$

$m$ is a lower bound of $A$:

$\forall x \in A \;\; m \preceq x$

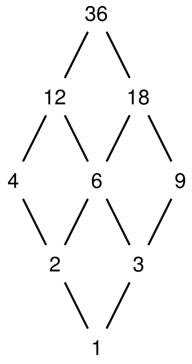
$m(A)$: set of lower bound of $A$

$inf(A)$ is the infimum of $A$:

$\forall m \in m(A) \;\; m \preceq inf(A)$

## Bound Example

**Example (factors of 36)**




inf = greatest common divisor
sup = least common multiple



## Lattice

**Definition**

lattice: $< L, \wedge, \vee >$
$\wedge$: meet, $\vee$: join

- $a \wedge b = b \wedge a$
  $a \vee b = b \vee a$
- $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
  $(a \vee b) \vee c = a \vee (b \vee c)$
- $a \wedge (a \vee b) = a$
  $a \vee (a \wedge b) = a$



## Poset - Lattice Relationship

- If $P$ is a poset, then $< P, inf, sup >$ is a lattice.
  - $a \wedge b = inf(a, b)$
  - $a \vee b = sup(a, b)$
- Every lattice is a poset where these definitions hold.



## Duality

**Definition**

dual:
$\wedge$ instead of $\vee$, $\vee$ instead of $\wedge$

**Theorem (Duality Theorem)**

*Every theorem has a dual theorem in lattices.*

## Lattice Theorems

**Theorem**

$a \wedge a = a$

**Proof.**

$a \wedge a = a \wedge (a \vee (a \wedge b))$ $\square$



## Lattice Theorems

**Theorem**

$a \preceq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$

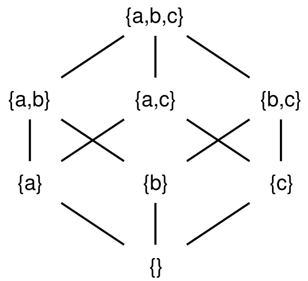


## Lattice Examples

**Example**

$< \mathcal{P}\{a, b, c\}, \cap, \cup >$

$\subseteq$ relation






## Bounded Lattice

**Definition**

lower bound of lattice $L$: 0

$\forall x \in L \; 0 \preceq x$

**Definition**

upper bound of lattice $L$: $I$

$\forall x \in L \; x \preceq I$

**Theorem**

*Every finite lattice is bounded.*

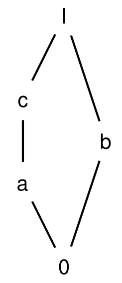
## Distributive Lattice

- *distributive lattice*:
  - $\forall a, b, c \in L \; a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
  - $\forall a, b, c \in L \; a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$



## Counterexamples

**Example**

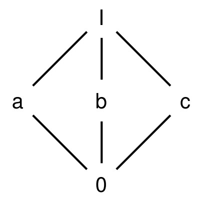



$a \vee (b \wedge c) = a \vee 0 = a$
$(a \vee b) \wedge (a \vee c) = I \wedge c = c$



## Counterexamples

**Example**




$a \vee (b \wedge c) = a \vee 0 = a$
$(a \vee b) \wedge (a \vee c) = I \wedge I = I$



## Distributive Lattice

**Theorem**

*A lattice is nondistributive if and only if it has a sublattice isomorphic to any of these two structures.*

## Join Irreducible

Definition

join irreducible element:

$a = x \vee y \Rightarrow a = x \vee a = y$

- *atom*: a join irreducible element which immediately succeeds the minimum



## Join Irreducible Example

Example (divisibility relation)

- prime numbers and 1 are join irreducible
- 1 is the minimum, the prime numbers are the atoms



## Join Irreducible

Theorem

*Every element in a lattice can be written as the join of join irreducible elements.*



## Complement

Definition

$a$ and $x$ are complements:

$a \wedge x = 0$ and $a \vee x = I$

## Complemented Lattice

**Theorem**

*In a bounded, distributive lattice the complement is unique, if it exists.*

**Proof.**

$a \wedge x = 0, a \vee x = I,\ a \wedge y = 0, a \vee y = I$

$$\begin{aligned} x &= x \vee 0 = x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y) = I \wedge (x \vee y) \\ &= x \vee y = y \vee x = I \wedge (y \vee x) \\ &= (y \vee a) \wedge (y \vee x) = y \vee (a \wedge x) = y \vee 0 = y \end{aligned}$$

□



## Boolean Algebra

**Definition**

Boolean algebra:

$< B, +, \cdot, \overline{x}, 1, 0 >$

$$\begin{array}{ll} a+b=b+a & a \cdot b = b \cdot a \\ (a+b)+c = a+(b+c) & (a \cdot b) \cdot c = a \cdot (b \cdot c) \\ a+0=a & a \cdot 1 = a \\ a+\overline{a}=1 & a \cdot \overline{a} = 0 \end{array}$$



## Boolean Algebra - Lattice Relationship

**Definition**

A Boolean algebra is a finite, distributive, complemented lattice.



## Duality

**Definition**

dual:

$+$ instead of $\cdot$, $\cdot$ instead of $+$

0 instead of 1, 1 instead of 0

**Example**

$(1+a) \cdot (b+0) = b$

dual of the theorem:

$(0 \cdot a) + (b \cdot 1) = b$

## Boolean Algebra Examples

Example

$B = \{0, 1\}, + = \vee, \cdot = \wedge$

Example

$B = \{$ factors of 70 $\}, + = lcm, \cdot = gcd$



## Boolean Algebra Theorems

$$\begin{array}{ll} a + a = a & a \cdot a = a \\ a + 1 = 1 & a \cdot 0 = 0 \\ a + (a \cdot b) = a & a \cdot (a + b) = a \\ (a + b) + c = a + (b + c) & (a \cdot b) \cdot c = a \cdot (b \cdot c) \end{array}$$

$$\begin{array}{ll} \overline{\overline{a}} = a & \\ \overline{a + b} = \overline{a} \cdot \overline{b} & \overline{a \cdot b} = \overline{a} + \overline{b} \end{array}$$



## References

Required Reading: Grimaldi

- Chapter 7: Relations: The Second Time Around
  - 7.3. Partial Orders: Hasse Diagrams
- Chapter 15: Boolean Algebra and Switching Functions
  - 15.4. The Structure of a Boolean Algebra