Discrete Mathematics

Sets

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Topics

Set

Definition

set: a collection of elements that are

- distinct
- unordered
- non-repeating

Set Representation

- *explicit representation*
  elements are listed within braces: $\{a_1, a_2, \ldots, a_n\}$
- *implicit representation*
  elements that validate a predicate: $\{x \mid x \in G,\ p(x)\}$
- $\emptyset$: empty set
- $a \in S$: a is an element of S
  $a \notin S$: a is not an element of S
- $|S|$: number of elements in S (cardinality)

Notation Examples

$\{3, 8, 2, 11, 5\}$	$\{x \mid x \in \mathbb{Z}^+,\ 20 < x^3 < 100\} \equiv \{3, 4\}$
$11 \in \{3, 8, 2, 11, 5\}$	$\{2x - 1 \mid x \in \mathbb{Z}^+,\ 20 < x^3 < 100\} \equiv \{5, 7\}$
$\lvert\{3, 8, 2, 11, 5\}\rvert = 5$	$\{n \mid n \in \mathbb{N},\ \exists k \in \mathbb{N}\ [n = 2k]\}$

Set Dilemma

- There is a barber who lives in a small town.
  He shaves all those men who don't shave themselves.
  He doesn't shave those men who shave themselves.
  Does the barber shave himself?
- yes $\rightarrow$ but he doesn't shave men who shave themselves
  $\rightarrow$ no
- no $\rightarrow$ but he shaves all men who don't shave themselves
  $\rightarrow$ yes

Set Dilemma

- S: set of sets that are not an element of themselves
  $S = \{A \mid A \notin A\}$
  $S \stackrel{?}{\in} S$
- $S \in S \rightarrow$ but the predicate is not valid $\rightarrow$ no
- $S \notin S \rightarrow$ but the predicate is valid $\rightarrow$ yes

Subset

Definition

$A \subseteq B \Leftrightarrow \forall x\ [x \in A \rightarrow x \in B]$

- set equality:
  $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$
- proper subset:
  $A \subset B \Leftrightarrow (A \subseteq B) \wedge (A \neq B)$
- $\forall S\ [\emptyset \subseteq S]$

Subset

$A \not\subseteq B$

$$\begin{aligned} A \not\subseteq B &\Leftrightarrow \neg\forall x\ [x \in A \rightarrow x \in B] \\ &\Leftrightarrow \exists x\ \neg[x \in A \rightarrow x \in B] \\ &\Leftrightarrow \exists x\ \neg[\neg(x \in A) \vee (x \in B)] \\ &\Leftrightarrow \exists x\ [(x \in A) \wedge \neg(x \in B)] \\ &\Leftrightarrow \exists x\ [(x \in A) \wedge (x \notin B)] \end{aligned}$$

Power Set

Definition

power set $\mathcal{P}(S)$: set of all subsets of S, including $\emptyset$ and S

example

$\mathcal{P}(\{1,2,3\}) = \{\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}\}$

- $|S| = n \Rightarrow |\mathcal{P}(S)| = 2^n$

Set Operations

complement

$\overline{A} = \{x \mid x \notin A\}$

intersection

$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$

- if $A \cap B = \emptyset$ then A and B are disjoint

union

$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$

Set Operations

difference
$A - B = \{x \mid (x \in A) \wedge (x \notin B)\}$

- $A - B = A \cap \overline{B}$
- *symmetric difference*:
  $A \triangle B = \{x \mid (x \in A \cup B) \wedge (x \notin A \cap B)\}$

Cartesian Product

Cartesian product
$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$

$A \times B \times C \times \cdots \times K = \{(a, b, \ldots, k) \mid a \in A, b \in B, \ldots, k \in K\}$

- $|A \times B \times C \times \cdots \times K| = |A| \cdot |B| \cdot |C| \cdots |K|$

Cartesian Product Example

$A = \{a_1, a_2, a_3, a_4\}$
$B = \{b_1, b_2, b_3\}$

$$\begin{aligned} A \times B = \{ \quad & (a_1, b_1),\ (a_1, b_2),\ (a_1, b_3), \\ & (a_2, b_1),\ (a_2, b_2),\ (a_2, b_3), \\ & (a_3, b_1),\ (a_3, b_2),\ (a_3, b_3), \\ & (a_4, b_1),\ (a_4, b_2),\ (a_4, b_3) \quad \} \end{aligned}$$

Equivalences

Double Complement
$\overline{\overline{A}} = A$

Commutativity
$A \cap B = B \cap A$ $\qquad$ $A \cup B = B \cup A$

Associativity
$(A \cap B) \cap C = A \cap (B \cap C)$ $\qquad$ $(A \cup B) \cup C = A \cup (B \cup C)$

Idempotence
$A \cap A = A$ $\qquad$ $A \cup A = A$

Inverse
$A \cap \overline{A} = \emptyset$ $\qquad$ $A \cup \overline{A} = \mathcal{U}$

Equivalences

Identity

$A \cap \mathcal{U} = A$ $\quad$ $A \cup \emptyset = A$

Domination

$A \cap \emptyset = \emptyset$ $\quad$ $A \cup \mathcal{U} = \mathcal{U}$

Distributivity

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $\quad$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Absorption

$A \cap (A \cup B) = A$ $\quad$ $A \cup (A \cap B) = A$

DeMorgan's Laws

$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\quad$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$

DeMorgan's Law

Theorem

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof.

$$\begin{aligned}
\overline{A \cap B} &= \{x | x \notin (A \cap B)\} \\
&= \{x | \neg(x \in (A \cap B))\} \\
&= \{x | \neg((x \in A) \wedge (x \in B))\} \\
&= \{x | \neg(x \in A) \vee \neg(x \in B)\} \\
&= \{x | (x \notin A) \vee (x \notin B)\} \\
&= \{x | (x \in \overline{A}) \vee (x \in \overline{B})\} \\
&= \{x | x \in \overline{A} \cup \overline{B}\} \\
&= \overline{A} \cup \overline{B}
\end{aligned}$$

$\square$

Example

Theorem

$A \cap (B - C) = (A \cap B) - (A \cap C)$

Proof.

$$\begin{aligned}
(A \cap B) - (A \cap C) &= (A \cap B) \cap \overline{(A \cap C)} \\
&= (A \cap B) \cap (\overline{A} \cup \overline{C}) \\
&= ((A \cap B) \cap \overline{A}) \cup ((A \cap B) \cap \overline{C}) \\
&= \emptyset \cup ((A \cap B) \cap \overline{C}) \\
&= (A \cap B) \cap \overline{C} \\
&= A \cap (B \cap \overline{C}) \\
&= A \cap (B - C)
\end{aligned}$$

$\square$

Example

Theorem

$$\begin{aligned}
& A \subseteq B \\
\Leftrightarrow\ & A \cup B = B \\
\Leftrightarrow\ & A \cap B = A \\
\Leftrightarrow\ & \overline{B} \subseteq \overline{A}
\end{aligned}$$

Example

$A \subseteq B \Rightarrow A \cup B = B$.

$A \cup B = B \Leftrightarrow A \cup B \subseteq B \wedge B \subseteq A \cup B$

$B \subseteq A \cup B$

$$\begin{aligned} x \in A \cup B &\Rightarrow x \in A \vee x \in B \\ A \subseteq B &\Rightarrow x \in B \\ &\Rightarrow A \cup B \subseteq B \end{aligned}$$

$\square$

Example

$A \cup B = B \Rightarrow A \cap B = A$.

$A \cap B = A \Leftrightarrow A \cap B \subseteq A \wedge A \subseteq A \cap B$

$A \cap B \subseteq A$

$$\begin{aligned} y \in A &\Rightarrow y \in A \cup B \\ A \cup B = B &\Rightarrow y \in B \\ &\Rightarrow y \in A \cap B \\ &\Rightarrow A \subseteq A \cap B \end{aligned}$$

$\square$

Example

$A \cap B = A \Rightarrow \overline{B} \subseteq \overline{A}$.

$$\begin{aligned} z \in \overline{B} &\Rightarrow z \notin B \\ &\Rightarrow z \notin A \cap B \\ A \cap B = A &\Rightarrow z \notin A \\ &\Rightarrow z \in \overline{A} \\ &\Rightarrow \overline{B} \subseteq \overline{A} \end{aligned}$$

$\square$

Example

$\overline{B} \subseteq \overline{A} \Rightarrow A \subseteq B$.

$$\begin{aligned} \neg(A \subseteq B) &\Rightarrow \exists w\ [w \in A \wedge w \notin B] \\ &\Rightarrow \exists w\ [w \notin \overline{A} \wedge w \in \overline{B}] \\ &\Rightarrow \neg(\overline{B} \subseteq \overline{A}) \end{aligned}$$

$\square$

Principle of Inclusion-Exclusion

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$

Theorem

$$
\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| \quad = \quad & \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| \\
& + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \\
& \cdots + -1^{n-1} |A_i \cap A_j \cap \cdots \cap A_n|
\end{aligned}
$$

Inclusion-Exclusion Example

- sieve of Eratosthenes
- a method to identify prime numbers

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26	27	28	29	30			
2	3		5		7		9		11		13		15		17
	19		21		23		25		27		29				
2	3		5		7				11		13				17
	19				23		25				29				
2	3		5		7				11		13				17
	19				23						29				

Inclusion-Exclusion Example

- number of primes between 1 and 100
- numbers that are not divisible by 2, 3, 5 and 7

- A_2: set of numbers divisible by 2
- A_3: set of numbers divisible by 3
- A_5: set of numbers divisible by 5
- A_7: set of numbers divisible by 7

- $|A_2 \cup A_3 \cup A_5 \cup A_7|$

Inclusion-Exclusion Example

- $|A_2| = \lfloor 100/2 \rfloor = 50$
- $|A_3| = \lfloor 100/3 \rfloor = 33$
- $|A_5| = \lfloor 100/5 \rfloor = 20$
- $|A_7| = \lfloor 100/7 \rfloor = 14$

- $|A_2 \cap A_3| = \lfloor 100/6 \rfloor = 16$
- $|A_2 \cap A_5| = \lfloor 100/10 \rfloor = 10$
- $|A_2 \cap A_7| = \lfloor 100/14 \rfloor = 7$
- $|A_3 \cap A_5| = \lfloor 100/15 \rfloor = 6$
- $|A_3 \cap A_7| = \lfloor 100/21 \rfloor = 4$
- $|A_5 \cap A_7| = \lfloor 100/35 \rfloor = 2$

Inclusion-Exclusion Example

- $|A_2 \cap A_3 \cap A_5| = \lfloor 100/30 \rfloor = 3$
- $|A_2 \cap A_3 \cap A_7| = \lfloor 100/42 \rfloor = 2$
- $|A_2 \cap A_5 \cap A_7| = \lfloor 100/70 \rfloor = 1$
- $|A_3 \cap A_5 \cap A_7| = \lfloor 100/105 \rfloor = 0$
- $|A_2 \cap A_3 \cap A_5 \cap A_7| = \lfloor 100/210 \rfloor = 0$

Inclusion-Exclusion Example

$$
\begin{aligned}
|A_2 \cup A_3 \cup A_5 \cup A_7| &= (50 + 33 + 20 + 14) \\
&- (16 + 10 + 7 + 6 + 4 + 2) \\
&+ (3 + 2 + 1 + 0) \\
&- (0) \\
&= 78
\end{aligned}
$$

- number of primes: $(100 - 78) + 4 - 1 = 25$

Subset Cardinality

- $A \subset B \Rightarrow |A| < |B|$
- not necessarily true for infinite sets

example

$\mathbb{Z}^+ \subset \mathbb{N}$

but

$|\mathbb{Z}^+| = |\mathbb{N}|$

- how can we compare the cardinalities of infinite sets?

Counting Sets

- to compare $|S_1|$ and $|S_2|$, pair off elements of S_1 and S_2
- if all elements can be paired, then $|S_1| = |S_2|$

$|\mathbb{Z}^+| = |\mathbb{N}|$

$\mathbb{Z}^+$	1	2	3	4	5	6	7	...
$\mathbb{N}$	0	1	2	3	4	5	6	...

Counting Sets Example

$|\mathbb{Q}| = |\mathbb{N}|$

	1	2	3	4	5	...
1	1/1	2/1	3/1	4/1	5/1	...
2	1/2	2/2	3/2	4/2	5/2	...
3	1/3	2/3	3/3	4/3	5/3	...
4	1/4	2/4	3/4	4/4	5/4	...
5	1/5	2/5	3/5	4/5	5/5	...
⋮	⋮	⋮	⋮	⋮	⋮	

- pair off row-wise:
  $1/1 \to 0 \quad 2/1 \to 1 \quad 3/1 \to 2 \quad 4/1 \to 3 \quad 5/1 \to 4 \quad \ldots$
- pair off diagonally:
  $1/1 \to 0 \quad 2/1 \to 1 \quad 1/2 \to 2 \quad 3/1 \to 3 \quad 2/2 \to 4$
  $1/3 \to 5 \quad 4/1 \to 6 \quad 3/2 \to 7 \quad 2/3 \to 8 \quad 1/4 \to 9 \quad \ldots$

Uncountable Sets

$|\mathbb{R}| \stackrel{?}{=} |\mathbb{N}|$

- $\{x \mid x \in \mathbb{R},\ 0 < x \leq 1\}$
- elements represented by non-terminating expansions:
  $0.4\overline{9}$ instead of 0.5

$$\begin{array}{lcl} 0.a_{11}a_{12}a_{13}a_{14}\ldots & \to & 0 \\ 0.a_{21}a_{22}a_{23}a_{24}\ldots & \to & 1 \\ 0.a_{31}a_{32}a_{33}a_{34}\ldots & \to & 2 \\ \vdots & & \\ 0.a_{n1}a_{n2}a_{n3}a_{n4}\ldots & \to & n-1 \\ \vdots & & \end{array}$$

- consider $0.b_1b_2b_3\ldots$ where

$$b_k = \begin{cases} 3 & \textit{if } a_{kk} \neq 3 \\ 7 & \textit{if } a_{kk} = 3 \end{cases}$$

- $\forall k \in \mathbb{N}\ r \neq r_k$
- *Cantor's Diagonal Construction*

Infinity

- $|\mathbb{R}|$ is uncountable
- $|\mathbb{R}| > |\mathbb{N}|$
- C: set of all possible computer programs
- P: set of all possible problems
- $|C| = |\mathbb{N}|$
- $|P| = |\mathbb{R}|$
- there are problems which cannot be solved using computers

References

Required reading: Grimaldi

- Chapter 3: Set Theory
  - 3.1. Sets and Subsets
  - 3.2. Set Operations and the Laws of Set Theory
- Chapter 8: The Principle of Inclusion and Exclusion
  - 8.1. The Principle of Inclusion and Exclusion
- Appendix 3: Countable and Uncountable Sets