2.5 Complex Eigenvalues

Real Canonical Form

A semisimple matrix $A \in L(\mathbb{R}^n)$ with complex conjugate eigenvalues can be diagonalized using the procedure previously described. However, the eigenvectors corresponding to the conjugate eigenvalues are themselves complex conjugate and the calculations involve working in complex n-dimensional space. There is nothing wrong with this in principle, however the manipulations may be a bit messy.

Example: Diagonalize the matrix $A = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$.

Eigenvalues are roots of the characteristic polynomial. $\tau = 4$. $\delta = 5$. $\Delta = -4$. The eigenvalues are $\lambda_+ = 2 + i$ and $\lambda_- = 2 - i$.

Eigenvectors are solutions of $Av = \lambda v$. Obtain $v_+ = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$ and $v_- = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$. Then from $AP = P\Lambda$ we need to compute $\Lambda = P^{-1}AP$. The transformation matrix $P = (v_+|v_-)$. Computing Λ requires care since we have to do matrix multiplication and complex arithmetic at the same time.

If we now want to solve an initial value problem for a linear system involving the matrix A, we have to compute $e^{\Lambda t} = \begin{pmatrix} e^{(2+i)t} & 0 \\ 0 & e^{(2-i)t} \end{pmatrix}$ and $e^{tA} = Pe^{\Lambda t}P^{-1}$. This matrix product is pretty messy to compute by hand. Even using a symbolic algebra system, we may have to do some work to convert our answer for e^{tA} into real form.

⟨Carry out the matrix product in Mathematica instead using ComplexDiagonalization1.nb. Discuss the commands Eigenvalues, Eigenvectors, notation for parts of expressions, Transpose, MatrixForm, Inverse and the notation for matrix multiplication. Obtain Λ and $e^{t\Lambda}$.⟩ ■

Alternatively, there is the Real Canonical Form that allows us to stay in the real number system. Suppose A has eigenvalue $\lambda = a + ib$, eigenvector $v = u + iw$ and their complex conjugates. Then writing $Av = \lambda v$ in real and imaginary parts:

$$A(u + iw) = (a + ib)(u + iw)$$

Taking real and imaginary parts

$Au = au - bw$

$Aw = bu + aw$

Consider the transformation matrix $P = (u|w)$. These equation can be written

$$A(u|w) = (u|w)\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (u|w)B.$$

The exponential of the 2×2 block on the right was computed at the end of section 2.3 (Meiss, Eq. 2.31).

Example. Let $A = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$. Find its real canonical form and compute e^{tA}. We have already found the eigenvalues and eigenvectors. Setting $v_+ = u + iw$ we have

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The transformation matrix and its inverse are

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Find

$$AP = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, \quad \mathrm{B} = P^{-1}AP = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}.$$

Using Meiss 2.31

$$e^{\mathrm{B}t} = e^{2t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Compute $e^{tA} = Pe^{Bt}P^{-1}$. Find

$$e^{t\mathrm{B}}P^{-1} = e^{2t}\begin{pmatrix} \sin t & \cos t - \sin t \\ \cos t & -\sin t - \cos t \end{pmatrix},$$

$$e^{tA} = e^{2t}\begin{pmatrix} \sin t + \cos t & -2\sin t \\ \sin t & \cos t - \sin t \end{pmatrix}. \quad \blacksquare$$

Diagonalizing an arbitrary semisimple matrix

Suppose A has k real eigenvalues and $n-k$ pairs of complex conjugate ones. Let $v_i,\ 1 \le i \le k$ be the corresponding real eigenvectors and $\{u_i | w_i\},\ k+1 \le i \le n$, be the real and imaginary parts of the complex conjugate eigenvectors. The transformation matrix

$$P = (v_1 | \ldots | v_k | u_{k+1} | w_{k+1} | \ldots | u_n | w_n)$$

is nonsingular and

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_n \end{pmatrix}$$

where

$$B_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix} \text{ for } k+1 \le j \le n.$$

The solution of the initial value problem $\dot{x} = Ax$ will involve the matrix exponential

$$e^{tA} = \begin{pmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{tB_n} \end{pmatrix}.$$

In this way we compute the matrix exponential of any matrix that is diagonalizable.

2.6 Multiple Eigenvalues

The commutator of A and B is $[A, B] = AB - BA$. If the commutator is zero then A and B commute.

Fact. If $[A, B] = 0$ and $[A, C] = 0$, then $[A, B + C] = 0$.

Proof. $[A, B + C] = A(B + C) - (B + C)A = (AB - BA) + (AC - CA) = 0$. □

Generalized Eigenspaces

Let $T \in L(E)$ where $E = \mathbb{R}^n$ or $\mathbb{C}^n$. Recall that eigenvalue λ_k and eigenvector v_k satisfy $Tv_k = \lambda_k v_k$. This can be rewritten as

$$(T - \lambda_k I)v_k = 0.$$

Suppose λ_k has algebraic multiplicity 1. Then the associated eigenspace is

$E_k = \{cv_k : c \in \mathbb{R}\} = \ker(T - \lambda_k I)$.

A space E is invariant under the action of T if $v \in E$ implies $Tv \in E$. For example, E_k is invariant under T by the fact above.

Suppose λ_k is an eigenvalue of T with algebraic multiplicity $n_k \geq 1$. Define the generalized eigenspace of λ_k as

$E_k = \ker(T - \lambda_k I)^{n_k}$.

The symbol E_k refers to generalized eigenspace but coincides with eigenspace if $n_k = 1$.

A nonzero solution v to $(T - \lambda_k I)^{n_k} v = 0$ is a generalized eigenvector of T.

Lemma 2.5 (Invariance). Each of the generalized eigenspaces of a linear operator T is invariant under T.

Proof. Suppose $v \in E_k$ so that $(T - \lambda_k I)^{n_k}\, v = 0$. Since T and $T - \lambda_k I$ commute

$$\begin{aligned}(T - \lambda_k I)^{n_k} T v &= (T - \lambda_k I)^{n_k - 1} T (T - \lambda_k I) v \\ &= \cdots \\ &= T(T - \lambda_k I)^{n_k} v \\ &= 0. \quad \square\end{aligned}$$

Let F_1 and F_2 be vector spaces. The direct sum $F_1 \oplus F_2$ is the vector space with elements (v_1, v_2), where $v_1 \in F_1$ and $v_2 \in F_2$, and operations of vector addition and scalar multiplication defined by $(v_1, v_2) + (w_1, w_2) = (v_1 + w_1, v_2 + w_2)$ and $k(v_1, v_2) = (kv_1, kv_2)$, where also $w_1 \in F_1$ and $w_2 \in F_2$ and $k \in$ scalars. For example, $\mathbb{R} \oplus \mathbb{R} = \mathbb{R}^2$.

Theorem 2.6 (Primary Decomposition). Let T be a linear operator on a complex vector space E with distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ and let E_j be the generalized eigenspace of T with eigenvalue λ_j. Then $\dim(E_j)$ is the algebraic multiplicity of λ_j and E is the direct sum of the generalized eigenspaces, i.e. $E = E_1 \oplus E_2 \oplus \ldots \oplus E_n$.

Proof. This is proved in Hirsch and Smale.

Remark. We can choose a basis $\{v_1, \dots, v_{n_j}\}$ for each eigenspace. By theorem 2.6, these can be combined to obtain a basis $\{v_1, \dots, v_n\}$ for E.

Warning. The labeling for generalized eigenvectors given above is Meiss' notation. Note that the eigenvectors are *relabeled* to give the basis for E. This keeps the notation simple but the labels must be interpreted correctly depending on context.

Semisimple-Nilpotent Decomposition

Shift notation from T as linear operator and refer to matrix A instead. Let $E = \mathbb{R}^n$ or $\mathbb{C}^n$ and $A \in L(E)$. Let $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ be the diagonal matrix with the eigenvalues of A repeated according to multiplicity. Let $\{v_1, \dots, v_n\}$ be a basis for E of generalized eigenvectors of A. Consider the transformation matrix $P = (v_1| \dots |v_n)$ and define

$S = P\Lambda P^{-1}$.

S is a semisimple matrix. Multiply by P on the right to obtain $SP = P\Lambda$. The i^th component of this result is $Sv_i = \lambda_j v_i$, where $\lambda_1, \dots \lambda_r$ are the distinct eigenvalues of A and $v_i \in E_j$. Think of S as the diagonalizable part of A.

Consider an arbitrary $v \in E_j$ Then v can be expressed as a linear combination of the basis vectors for E_j: $v = \sum_{k=1}^{n_j} c_k v_k$. We then have

$Sv = \sum_{k=1}^{n_j} c_k S v_k = \sum_{k=1}^{n_j} c_k \ \lambda_j \ v_k = \lambda_j v.$

Within E_j, S acts as a multiple of the identity operator. In particular, E_j is invariant under the action of S.

Lemma 2.7 Let $N = A - S$, where $S = P\Lambda P^{-1}$. Then N commutes with S and is nilpotent with order at most m, the maximum of the algebraic multiplicities of A.

Remark 1. N is nilpotent of order k means the same thing as N has nilpotency k.

Remark 2. Since the generalized eigenspace E_j of A is invariant under the action of both A and S, it is also invariant under the action of N.

Proof. See the text; plan to give it in class. Note that there are two parts: (1) show $[S, N] = 0$ and (2) show N nilpotent.

Theorem 2.8. A matrix A on a complex vector space E has a unique decomposition $A = S + N$, where S is semisimple, N is nilpotent and $[S, N] = 0$.

Proof. Not in lecture. See text.

The Exponential

Let $A \in L(E)$, then by lemma 2.7 $A = S + N$ where $[S, N] = 0$. Further, we have $S = P\Lambda P^{-1}$, $e^{tS} = Pe^{t\Lambda}P^{-1}$ and $N^m = 0$, where m is the maximum algebraic multiplicity of the eigenvalues. Then, using the law of exponents for commuting matrices and the series definition of the exponential

[1] $e^{tA} = e^{tS}e^{tN} = Pe^{\Lambda t}P^{-1}\sum_{k=0}^{m-1}\frac{(tN)^k}{k!}$

This formula allows us to compute the exponential of an arbitrary matrix. Combine this result with the fundamental theorem to find an analytical solution for any linear system.

Example. Solve the initial value problem $\dot{x} = Ax$ with $x(0)$ given and $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$.

By the fundamental theorem, $x(t) = e^{tA}x(0)$. We need to compute e^{tA}. $\tau = \text{tr}(A) = 6$ and $\delta = \det(A) = 9$. The characteristic equation is $\lambda^2 - 6\lambda + 9 = 0$. The root $\lambda = 3$ has multiplicity 2. Then

$$\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = 3I.$$

Every matrix commutes with the identity matrix, so that $S = P\Lambda P^{-1} = \Lambda = 3I$. Then

$$N = A - S = A - 3I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Notice that $N^n = N^2 = 0$. N has nilpotency 2. Then using [1]

$$e^{tA} = e^{tS}(I + tN) = e^{3t}\begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix},$$

$$x(t) = e^{tA}x(0) = e^{3t}\begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = e^{3t}\begin{pmatrix} (1-t)x_1(0) + tx_2(0) \\ -tx_1(0) + (1+t)x_2(0) \end{pmatrix}.$$

Notice that if $x_1(0) = x_2(0) = c_0$ the straight line solution $e^{3t}c_0\begin{pmatrix}1\\1\end{pmatrix}$ is obtained, where $\begin{pmatrix}1\\1\end{pmatrix}$ is the eigenvector associated with λ. However the full phase portrait is most easily visualized using a computer.

⟨phase portrait drawn by a computer⟩

Example. Solve the initial value problem $\dot{x} = Ax$, $x(0) = x_0$ where $A = \begin{pmatrix}-1 & 1 & -2\\ 0 & -1 & 4\\ 0 & 0 & 1\end{pmatrix}$.

Since A is upper triangular, the eigenvalues can be read off the main diagonal. $\lambda_1 = -1$ has multiplicity $n_1 = 2$ and $\lambda_2 = 1$ has multiplicity $n_2 = 1$. The generalized eigenspace associated with λ_1 is $E_1 = \ker(A - \lambda_1 I)^2 = \ker(A + I)^2$. Find

$$(A+I)^2 = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 8\\ 0 & 0 & 4\end{pmatrix}.$$

A choice for generalized eigenvectors spanning E_1 is $v_1 = (1,0,0)^T$ and $v_2 = (0,1,0)^T$. The generalized eigenspace associated with λ_2 is $E_2 = \ker\,(A - I)$. Find

$$A - I = \begin{pmatrix}-2 & 1 & -2\\ 0 & -2 & 4\\ 0 & 0 & 0\end{pmatrix}.$$

Let $v_3 = (0,2,1)^T$. The transformation matrix is

$$P = (v_1|v_2|v_3) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}.$$

Notice that P is block diagonal. Its inverse P^{-1} is also block diagonal, with each block the inverse of the corresponding block in P. Then

$$P^{-1} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix}.$$

We are now ready to find S and N. $S = P\Lambda P^{-1}$ where $\Lambda = \operatorname{diag}(-1,-1,1)$. Obtain

$$S = \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 4\\ 0 & 0 & 1\end{pmatrix} \text{ and } N = A - S = \begin{pmatrix}0 & 1 & -2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$ It's easy to check $N^2 = 0$.

Then e^{tA} is given by

$e^{tA} = e^{tS}e^{tN} = Pe^{t\Lambda}P^{-1}(I + Nt)$, where $e^{t\Lambda} = \mathrm{diag}(e^{-t}, e^{-t}, e^{t})$.

The solution of the initial value problem is $x(t) = e^{tA}x_0$.

Jordan Form

Let $A \in L(E)$ where $E = \mathbb{R}^n$ or $\mathbb{C}^n$. A cannot always be diagonalized by a similarity transformation, but it can always be transformed into Jordan canonical form, which gives a simple form for the nilpotent part of A. Finding a basis of generalized eigenvectors that reduces A to this form is generally difficult by hand, but computer algebra systems like Mathematica have built in commands that perform the computation. Finding the Jordan form is not necessary for the solution of linear systems and is not described by Meiss in chapter 2. However, it is the starting point of some treatments of center manifolds and normal forms, which systematically simplify and classify systems of nonlinear ODEs. This subsection follows the first part of section 1.8 in Perko closely. The following theorem is described by Perko and proved in Hirsch and Smale:

Theorem (The Jordan Canonical Form). Let A be a real matrix with real eigenvalues λ_j, $j = 1, \dots, k$ and complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j, j = k+1, \dots, n$. Then there exists a basis $\{v_1, \dots, v_k, u_{k+1}, w_{k+1}, \dots u_n, w_n\}$ for $\mathbb{R}^{2n-k}$, where $v_j, j = 1, \dots, n$ are generalized eigenvectors of A, the first k of these are real and $u_j = \mathrm{Re}(v_j)$, $w_j = \mathrm{Im}(v_j)$ for $j = k + 1, \dots, n$. The matrix $P = (v_1|\dots|v_k|u_{k+1}|w_{k+1}|\dots|u_n|w_n)$ is invertible and

$$P^{-1}AP = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_r \end{pmatrix},$$

where the elementary Jordan blocks $B = B_j, j = 1, \dots, r$ are either of the form

$$B = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \lambda & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix}, \qquad [2]$$

for λ one of the real eigenvalues of A or of the form

$$B = \begin{pmatrix} D & I_2 & 0 & \dots & 0 \\ 0 & D & I_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & D & I_2 \\ 0 & \dots & \dots & 0 & D \end{pmatrix}, \qquad [3]$$

with

$$D = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

for $\lambda = a + ib$ one of the complex eigenvalues of A.

The Jordan form yields some explicit information about the form of the solution on the initial value problem

$$\dot{x} = Ax, \quad x(0) = x_0 \qquad [4]$$

which, according to the Fundamental Solution Theorem, is given by

$$x(t) = e^{At}x_0 = P\,\mathrm{diag}(e^{B_j t})P^{-1}x_0.$$

If $B_j = B$ is an $m \times m$ matrix of form [2] and λ is a real eigenvalue of A, then $B = \lambda I + N$ where

$$N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

is nilpotent of order m and

$$N^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ & & \dots & & & \\ 0 & \dots & \dots & 0 & & 0 \end{pmatrix}, \qquad \dots. \qquad N^{m-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ & & \dots & & & \\ 0 & \dots & \dots & 0 & & 0 \end{pmatrix}$$

Then

$$e^{Bt} = e^{\lambda t}e^{Nt} = e^{\lambda t}\begin{pmatrix} 1 & t & t^2/2! & \dots & t^{m-1}/(m-1)! \\ 0 & 1 & t & \dots & t^{m-2}/(m-2)! \\ \dots & & & & \\ 0 & \dots & \dots & 1 & t \end{pmatrix}.$$

Similarly, if $B_j = B$ is an $2m \times 2m$ matrix of form [3] and $\lambda = a + ib$ is a complex eigenvalue of A, then $B = \mathrm{diag}(D) + N$ where

$$N = \begin{pmatrix} 0 & I_2 & 0 & \dots & 0 \\ 0 & 0 & I_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & I_2 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

is nilpotent of order m and

$$e^{Bt} = \mathrm{diag}(e^{Dt})e^{Nt} = e^{at}\mathrm{diag}(R)\begin{pmatrix} I_2 & I_2 t & t^2/2! & \dots & I_2 t^{m-1}/(m-1)! \\ 0 & I_2 & I_2 t & \dots & I_2 t^{m-2}/(m-2)! \\ \dots \\ 0 & \dots & \dots & I_2 & I_2 t \end{pmatrix}$$

$$= e^{at}\begin{pmatrix} R & Rt & Rt^2/2! & \dots & Rt^{m-1}/(m-1)! \\ 0 & R & Rt & \dots & Rt^{m-2}/(m-2)! \\ \dots \\ 0 & 0 & 0 & \dots & R \end{pmatrix},$$

where R is the rotation matrix

$$R = e^{Dt} = \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}.$$

This form of the solution to [3] leads to the following result.

Corollary. Each coordinate in the solution $x(t)$ of the initial value problem [4] is a linear combination of functions of form

$t^k e^{at}\cos(bt)$ or $t^k e^{at}\sin(bt)$,

where $\lambda = a + ib$ is an eigenvalue of the matrix A and $0 \le k \le n-1$.

More precisely, we have $0 \le k \le m-1$, where m is the largest order of the elementary Jordan blocks.

2.7 Linear Stability

Let $A \in L(E)$. The solution of $\dot{x} = Ax$, $x(0) = x_0$ is $x(t) = e^{At}x_0$, and each component is a sum of terms proportional to an exponential $e^{\lambda_j t}$, for an eigenvalue λ_j of A. The real parts of these eigenvalues determine whether the terms are exponentially growing or decaying. Denote the generalized eigenvectors $v_j = u_j + i\, w_j$ and define

$E^u = \mathrm{span}\{u_j, w_j : \mathrm{Re}(\lambda_j) > 0\}$ is the unstable eigenspace,

$E^c = \mathrm{span}\{u_j, w_j : \mathrm{Re}(\lambda_j) = 0\}$ is the center eigenspace and

$E^s = \mathrm{span}\{u_j, w_j : \mathrm{Re}(\lambda_j) < 0\}$ is the stable eigenspace.

According to Lemma 2.5 each of the generalized eigenspaces is invariant under the action of A. E^u is the direct sum of the generalized eigenspaces corresponding to eigenvalues with positive real part and it is also invariant under the action of A. Similarly, E^c and E^s are invariant. E is the direct sum $E = E^u \oplus E^c \oplus E^s$.

We can consider the action of A in each subspace by considering restricted operators. $A|_{E^u}$ denotes the restriction of A to E^u, etc. This corresponds to the fact that $\Lambda = P^{-1}AP$ is block diagonal. For example, Λ can always be brought to Jordan canonical form.

A system is <u>linearly</u> <u>stable</u> if all its solutions are bounded as $t \to \infty$. If $x_0 \in E^s$ then $e^{tA}x_0$ is always bounded.

Lemma 2.9. If A is an $n \times n$ matrix and $x_0 \in E^s$, the stable space of A, then there are constants $K \geq 1$ and $\alpha > 0$ such that

$|e^{tA}x_0| \leq Ke^{-\alpha t}|x_0|, t \geq 0.$

Consequently, $e^{tA}x_0 \to 0$ as $t \to \infty$.

Remark. This result is very reasonable. From [1], each component of the solution will be proportional to $e^{\lambda t}$for some eigenvalue λ, and by hypothesis $Re(\lambda) = a < 0$. α is chosen so that for each such eigenvalue $a < -\alpha < 0$. The maximum power of t that appears in any component of $e^{tA|_{E^s}}$ is $m - 1$, where m is the maximum multiplicity of any eigenvalue in $A|_{E^s}$. For t sufficiently large, the exponentially decaying terms must dominate the powers of t. For details of the proof see Meiss.

A linear system is <u>asymptotically</u> <u>linearly</u> <u>stable</u> if all of its solutions approach 0 as $t \to \infty$.

Theorem 2.10 (Asymptotic Linear Stability). $\lim_{t\to\infty} e^{tA}x_0 = 0$ for all x_0 if and only if all eigenvalues of A have negative real parts.

Proof. If all eigenvalues have negative real part, lemma 2.9 implies $\lim_{t\to\infty} e^{tA}x_0 = 0$. If an eigenvalue λ has positive real part, then there is a straight line solution, $x(t) = ce^{\lambda t}v$ where v is an eigenvector of λ, that grows without bound. If there is an eigenvalue $\lambda = i\beta$ with zero real part, the solutions in this subspace have terms of the form $t^j e^{i\beta t}$ that do not go to zero.□

A system with no center subspace is hyperbolic. Lemma 2.9 and Theorem 2.10 describe properties of these systems that depend only on the signs of their eigenvalues.

In contrast, the stability of systems with a center subspace can be affected by the nilpotent part of A. The proof of theorem 2.10 suggests why these systems cannot be asymptotically stable. Solutions of the 2D center described in section 2.2 are bounded. However, center systems with nonzero nilpotent parts have solutions that are unbounded (Perko, section 1.9, problem 5(d)).

Routh-Hurwitz Stability Criteria

These criteria determine whether the roots of a polynomial have all negative real parts. When applied to the characteristic polynomial associated with a linear system of equations, they test for asymptotic stability of the equilibrium point. In the 2D case, the characteristic polynomial is

$$p(\lambda) = \lambda^2 - \tau\lambda + \delta.$$

It is easy to see that all of the eigenvalues have negative real parts if $\tau < 0$ and $\delta > 0$. All of the coefficients of the characteristic polynomial must be positive. In the 3D case, the characteristic polynomial is

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - \tau\lambda^2 + \sigma\lambda - \delta.$$

All of the eigenvalues have negative real parts if and only if $\tau < 0$ and $\tau\sigma < \delta < 0$. See Meiss, problem 2.11. The positivity of the coefficients of the characteristic polynomial is necessary but not sufficient. Analogous stability criteria are available for higher order polynomials.

In some cases, it may be much easier to study the stability of a linear system using these criteria than by finding the eigenvalues.

2.8 Nonautonomous Linear Systems and Floquet Theory

Let $A \in L(\mathbb{R}^n)$. The initial value problem for an autonomous linear system

$$\dot{x} = Ax, \ x(0) = x_0, \qquad [1]$$

can arise from linearization about an equilibrium point. As we shall see in chapter 5, when A is hyperbolic this system gives a good approximation to the behavior of nearby trajectories. ⟨sketch⟩ The solution of [1] is given by the Fundamental Solution Theorem

$$x(t) = e^{tA}x_0.$$

The initial value problem for the non-autonomous linear system

$$\dot{x} = A(t)x,\ \ x(t_0) = x_0, \qquad [2]$$

can arise from linearization about a periodic orbit. In this case $x(t) = y(t) - \gamma(t)$, where $\gamma(t)$ is the periodic orbit and $y(t)$ is another trajectory close to the orbit. ⟨sketch⟩ Higher order terms are dropped to arrive at [2], but the solution of [2] may give a good approximation to the behavior of the nearby trajectory. See chapter 4 . Floquet theory discusses the solution of [2] when A is periodic. Let the period be T.

The fundamental matrix solution corresponding to [2] is the solution of the initial value problem

$$\frac{d}{dt}\Phi = A(t)\Phi,\ \ \Phi(t_0, t_0) = I. \qquad [3]$$

$\Phi(t, t_0) \in L(\mathbb{R}^n)$ is the solution at time t of the initial value problem that begins at time t_0. Note that, if $\Phi(t, t_0)$ solves [3] then $x(t) = \Phi(t, t_0)x_0$ solves [2].

A trajectory that starts at an initial time r and ends at time t may be decomposed into two parts. The first from time r to time s and the second from time s to time t. Mathematically, Φ satisfies

$$\Phi(t, r) = \Phi(t, s)\Phi(s, r).$$

Then $x(s) = \Phi(s, r)x(r)$ and $x(t) = \Phi(t, s)x(s) = \Phi(t, s)\Phi(s, r)x(r) = \Phi(t, r)x(r)$.

Suppose that A has period T and consider the solution of [3] with $t_0 = 0$. The monodromy matrix is the solution of the initial value problem after one period

$$M = \Phi(T, 0).$$

Then the solution of [2] with $t_0 = 0$ after one period is $x(T) = \Phi(T, 0)x_0 = Mx_0$. Consider the trajectory during the second period. It is the solution of the initial value problem

$$\dot{x} = A(t)x,\ \ x(T) = Mx_0.$$

Define a new time variable $\tau = t - T$, and use $A(\tau + T) = A(\tau)$ to see that this is the same as [2] with $t_0 = 0$ and x_0 replaced by Mx_0. Therefore the solution of [2], with $t_0 = 0$, after two periods is $x(2T) = M^2x_0$. After n periods $x(nT) = M^n x_0$.

The eigenvalues of M are the Floquet multipliers. Suppose the initial condition x_0 of [2], with t_0=0, is also an eigenvector of M and μ is the corresponding eigenvalue. Then

$$x(T) = Mx_0 = \mu x_0 = e^{\ln \mu} x_0,$$

where $ln\ \mu$ is the corresponding Floquet exponent. Meiss discusses the fact that the monodromy matrix is nonsingular; see Theorem 2.11. Therefore the Floquet multipliers are all nonzero and the Floquet exponents are well defined. However, it is perfectly possible for a Floquet multiplier to be negative, in this case the corresponding exponent would be pure imaginary.

Floquet theory will be concerned with the logarithm of the monodromy matrix. We next define the matrix logarithm. Begin with a preliminary remark.

The matrix exponential e^A was defined by a series expansion patterned after the Maclaurin series for e^x where $x \in \mathbb{R}$. A similar procedure will be used to define the logarithm of a nilpotent matrix. Recall $(1-x)^{-1} = 1 + x + x^2 + \cdots$, converging for $|x| < 1$. If we now integrate both sides and evaluate the constant of integration, we find $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^2}{3} - \cdots$, also converging for $|x| < 1$. Now replace $1 \to I$ and $x \to -\widetilde{N}$, where $\widetilde{N}$ is a nilpotent matrix, to obtain

$$\ln\left(I + \widetilde{N}\right) = -\sum_{j=1}^{\infty} \frac{1}{j} (-\widetilde{N})^j. \qquad [4]$$

There are no convergence issues because $\widetilde{N}$ is nilpotent!

Lemma 2.12 Any nonsingular matrix A has a (possibly complex) logarithm

$$\ln A = P\ln(\Lambda)\mathrm{P}^{-1} - \sum_{\mathrm{j}=1}^{\mathrm{m}-1} \frac{1}{\mathrm{j}} (-\mathrm{S}^{-1}\mathrm{N})^{\mathrm{j}}.$$

Here, $A = S + N$ is the semisimple-nilpotent decomposition, $\ln(\Lambda) = \mathrm{diag}(\ln \lambda_1, \ldots, \ln \lambda_n)$ with eigenvalues λ_i of A repeated according to multiplicity, m is the maximum algebraic multiplicity of any eigenvalue and P is the matrix of generalized eigenvectors of A.

Proof. As usual let $\Lambda = P^{-1}AP = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and define $\ln(\Lambda)$ as given above. Since A is nonsingular, none of the eigenvalues are zero. Then

$$S = P\Lambda P^{-1} = Pe^{\ln\Lambda}P^{-1} = e^{P\ln(\Lambda)P^{-1}},$$

thus $\ln S = P\ln(\Lambda)P^{-1}$. This is the formula for the logarithm of a semisimple matrix. In general,

$$A = S + N = S(I + S^{-1}N).$$

Since $[S, N] = 0$, $(S^{-1}N)^m = (S^{-1})^m N^m = 0$, so $S^{-1}N$ is nilpotent. Let $S^{-1}N = \widetilde{N}$ in the remark above. Then the logarithm of $I + S^{-1}N$ is given by [4]. By analogy with $\ln(ab) = \ln a + \ln b$, we claim $B = \ln S + \ln(I + S^{-1}N)$ is the logarithm of A. If $\ln S$ and $\ln(I + S^{-1}N)$ commute, then

$e^B = e^{\ln S + \ln(I+S^{-1}N)} = e^{\ln S} e^{\ln\,(I+S^{-1}N)} = S(I+S^{-1})N = A,$

and B is the logarithm of A.

To see that $\ln S$ and $\ln\,(I+S^{-1}N)$ commute, note that in each generalized eigenspace E_j the action of $\ln S$ is multiplication by $\ln \lambda_j$ and therefore is proportional to the identity matrix. E_j is also invariant under the action of $\ln\,(I+S^{-1}N)$, which given by [4] with $\widetilde{N} = S^{-1}N$. Every matrix commutes with the identity matrix, and therefore $\ln S$ and $\ln\,(I+S^{-1}N)$ commute. □

Remark. The logarithm of a complex number is many-valued. Consider $z \in \mathbb{C}$. If $z = re^{i\theta}$ then $\ln z = r + i(\theta + 2\pi n)$ with n an integer. To obtain the logarithm function, a consistent choice must be made for the imaginary part. In the same way, when A has an eigenvalue λ that is not positive, a consistent choice must be made for the imaginary part of $\ln \lambda$ when $\ln \Lambda$ is formed.

Example. Find the logarithm of $A = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$.

The roots of the characteristic equation give $\lambda_\pm = \cos t \pm i \sin t$. Then $\Lambda = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$ and, recalling Euler's formula, $\ln \Lambda = \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix}$. Write $Av_+ = \lambda_+ v_+$ and solve for the components of v_+ to find $v_+ = (1, i)^T$ and $v_- = \bar{v}_+$. The transformation matrices are $P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ and $P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. Finally $\ln A = P(\ln \Lambda)P^{-1} = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} = t\sigma$. ■

Example. Find the logarithm of $A = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$.

Since A is upper triangular, the eigenvalues are on the main diagonal. A has 1 eigenvalue, $\lambda = 1/2$, with multiplicity $m = 2$. The associated generalized eigenspace is all of $\mathbb{R}^2$. The simplest choice for a basis is $v_1 = (1,0)^T$ and $v_2 = (0,1)^T$. Then the transformation matrices are $P = P^{-1} = I$. We thus have

$$\ln S = \ln \Lambda = \begin{pmatrix} -\ln 2 & 0 \\ 0 & -\ln 2 \end{pmatrix}.$$

A has a nilpotent part so we have another term in the logarithm to compute. Note $S = \Lambda = \frac{1}{2} I$.

Let $N = A - S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\tilde{N} = S^{-1}N = 2I\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. According to [4] $\ln(I + \tilde{N}) = \tilde{N}$

$\ln A = \ln S + \ln(I + \tilde{N}) = \begin{pmatrix} -\ln 2 & 2 \\ 0 & -\ln 2 \end{pmatrix}$. ■

Recall that we are trying to characterize the solutions $x(t)$ of the initial value problem for a T-periodic linear system [2]. The fundamental matrix solution $\Phi(t, t_0)$ is the solution of the matrix initial value problem [3]. Any solution of [2] can be written $x(t) = \Phi(t, t_0)x_0$. The monodromy matrix is the solution of [3], with $t_0 = 0$, after one period: $M = \Phi(T, 0)$.

Theorem 2.13 (Floquet). Let M be the monodromy matrix for a T-periodic linear system $\dot{x} = A(t)x$ and $TB = \ln M$ its logarithm. Then there exists a T-periodic matrix $\mathcal{P}$ such that the fundamental solution is

$\Phi(t, 0) = \mathcal{P}(t)e^{tB}$.

Proof. Give as in the text.

Remark. Note that $\mathcal{P}(0) = \Phi(0,0) = I$. This implies $\mathcal{P}(nT) = I$ for n an integer. Then $x(nT) = \Phi(nT, 0)x_0 = e^{nTB}x_0 = M^n x_0$. However, when t is not an integer multiple of a period the matrices $\mathcal{P}$ and e^{tB} may be complex (consider $t = T/2$ when e^{tB} is the square root of M).

Alternatively there is a real form of Floquet's theorem. It is based upon the fact that the square of any real matrix A has a real logarithm (Exercise #21).

Theorem 2.14. Let Φ be the fundamental matrix solution for the time T-periodic linear system [2]. Then there exists a real $2T$-periodic matrix Q and a real matrix R such that

$\Phi(t, 0) = Q(t)e^{tR}$.

Proof. From exercise 21, for any nonsingular matrix M there is a real matrix $2R$ such that $M^2 = e^{2TR}$. Define $Q(t) = \Phi(t, 0)e^{-tR}$, and then

$Q(t + 2T) = \Phi(t + 2T, 0)e^{-(t+2T)R} = \Phi(t, 0)M^2M^{-2}e^{-tR} = Q(t)$.

Therefore, Q is $2T$-periodic. □