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THE MATHEMATICS USED IN MATHEMATICAL PSYCHOLOGY

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Introduction. The main issue in applying mathematics to psychological problems today, and most likely for some time to come, is the formulation of these problems in mathematical terms. The solution of difficult but well-formulated mathematical problems and the analysis of complex applied problems in terms of precise and well-confirmed theories are more secondary efforts. We do not yet have the basic concepts and variables staked out in a way that makes the introduction of mathematics the relatively straightforward business that it has become in much of physical science. We are in a situation somewhat analogous to sixteenth, or hopefully seventeenth, century physics, but the analogy is far from complete. We resemble the early physicists in our effort, often fumbling and always slow, to isolate and purify the fundamental variables from the myriad, vague, commonsense psychological ideas and concepts. We differ in the range of techniques available to us. The modern electronics technology, including high speed computers, provides us with a control over experimental conditions and a computational capacity for data analysis incomparably more extensive and subtle than those with which the early physicists had to contend. In addition, most of the mathematics and statistics we now use was quite unknown three centuries ago.

Applications to what psychology? The current applications of mathematics to parts of psychology are precisely that—applications to portions of the total field. To the great satisfaction of many who do not necessarily view with favor the increasing mathematical complication of the psychological literature, huge portions of both academic and applied psychology are essentially free from mathematical inroads. The main areas that have been affected are those usually grouped together as "experimental" psychology, which is a misnomer because experiments are also performed in the "nonexperimental" areas. By convention, however, experimental psychology equals basic research into such fundamental psychological processes as learning, sensation, perception, and motivation. What is popularly viewed as psychology—abnormal, child, personality, and much of social psychology—has not been seriously influenced by any mathematical developments other than classical statistics, especially testing of hypotheses which is used extensively, if not always well, throughout psychology. (I shall not comment here about the role of either statistics or computers in psychological research. Both are extremely influential, and the mathematical community, whether it likes it or not, is bound to be involved to some extent in the problems that they create, but it is enough for this paper to deal just with mathematical applications.)

With our attention confined to mathematical attempts to understand the data that arise from laboratory experiments that are designed to elucidate fundamental psychological processes, certain features of these experiments should be made explicit because they exert a considerable impact upon the mathematics that we use. Laboratory experiments elicit behavior which differs in various ways from most ordinary, on-going behavior. Although a number of exceptions can be cited, by and large time is rendered discrete in the laboratory, i.e., the temporal pattern of events is structured in some fashion into trials. It is obvious that ordinary experience is not so neatly packaged. The stimulation to the subject—not his total environment, but those aspects of it to which we want him to attend—is usually delivered as discrete, repeatable bursts of some sort or other. The subject's possible responses, either those he is instructed to make or those we choose to observe, are frequently restricted to a discrete set (more often than not, a small finite one). Again, this is sometimes the case in nature, but certainly not the rule. Many ordinary situations possess a certain open-endedness or freedom of response which makes them difficult to study; life's richness is sufficiently impoverished in the laboratory that we can bring to bear our conceptual and calculating tools. Finally, the information feedback and the rewarding or punishing outcomes of an experiment are ordinarily discrete and delivered to the subject immediately after he responds. This is certainly not typical of modern life, in which people continually face long delayed and, more often than not, diffuse feedback and rewards. For a more exact characterization of this class of experiments, (see [4]).

Thus, there is little doubt that significant differences exist between the laboratory experiments we are trying to model in mathematical terms and everyday behavior. Most psychologists hope that one day we may gain some insight into socially significant behavior through the phenomena and laws discovered by means of our restricted and often artificial experimental designs, but the considerable difference between the two types of behavior is a raw fact that cannot be denied, and the bridges linking them do not seem to be easy to construct.

The reasons for these experimental abstractions are not primarily mathematical, although to a minor extent the general dispersion among scientists of elementary mathematics may have had its impact upon experimental designs. By and large, however, these restrictions to discrete time, stimuli, responses, and outcomes have been imposed by experimentalists so that 1) experiments can be completed in reasonable periods of time, and 2) the resulting data records will be reliable and amenable to certain types of analysis. As disturbing as it may seem, much of what we do in a psychological laboratory is forced upon us by data recording and analysis problems.

From a mathematician's point of view, the effect of these procedures is to limit somewhat the kinds of mathematics he can effectively employ: the continuous mathematics of classical analysis is not particularly well suited to the discreteness of most psychological experiments. For example, psychological problems are rarely formulated as differential equations, although such equations do arise sometimes in an incidental way in the solution of a problem cast in other terms. But as far as actual formulation is concerned, the standard methods of classical physics are little used and when they are the results are not usually very interesting.

Enter numbers. In a way, I have gotten ahead of my story, for to talk about formulating a problem in terms of classical analysis presupposes that the variables can be represented by numbers or vectors. This we take for granted in much of physics, but for psychology and the other social and behavioral sciences one of the most perplexing problems is how to introduce numbers in a meaningful way. Perhaps we are too wedded to the familiar and should not try to bring in numbers at all, and ultimately we may be forced by our subject matter to other mathematical representations, but at present we seem to be relatively helpless until we have a scalar or vector representation. There are reasons to believe that the attempt to represent some psychological notions by numbers is not a totally foolish goal. To be sure, the vector or scalar nature of our variables is far less certain than it is for variables such as mass, velocity, etc. Nevertheless, concepts such as the utility of a commodity, the loudness of a tone, or the brightness of a light—subjective notions that to some extent parallel objective attributes, in these cases, monetary worth, acoustic energy, and light energy—all seem to have an intensive nature reminiscent of numerical scales in physics. Mathematical psychologists and others in related areas are devoting some effort to discovering whether or not this is a poor analogy or, to put it positively, to finding out when we can justify the assignment of numbers to stimuli and outcomes so as to create useful scales of utility, sensation, and the like. I use the phrase "justify the assignment of numbers" intentionally. It is worse than useless for psychologists to parallel superficially textbook problems of physics by saying: let H_i denote the amount of hostility possessed by person i, let A_{ij} denote the amount of aggression expressed by i towards j, etc. There are no reasons that I know of to suppose that either of these notions, hostility or aggression, and many others like them, are scalar quantities. To begin a problem in this way is tantamount to throwing most of it away. At our present state of knowledge, to "suppose" a measurement problem out of existence is little more than a feeble joke.

So one of our main tasks is to see whether we can sensibly represent some psychological concepts numerically. There are two major techniques: one involves probability concepts and leads to the main body of mathematical psychology; the other, which has been much more incidental, attempts a more fundamental approach. This I discuss first.

Fundamental measurement. If we examine theories of physical measurement, as represented for example by the work of N. R. Campbell [6], we find an important distinction between fundamental and derived measurement. The essential feature of fundamental measurement is that only assumptions about qualitative observations are made: no numbers enter into the axioms of the theory. From these assumptions a numerical representation of the observations is shown to exist. The axioms characterize observables such as the deflection of a beam balance when objects are placed upon the two pans. The best known system, that called extensive measurement, was devised to account for the measurement of mass, length, etc. It involves a set Ω of objects that are to be measured, a binary operation $\circ$ of "combination" or "concatenation" of any two objects in Ω to form a third, and a qualitative ordering $\geqq$ of "not less than" over Ω such as that determined by the deflection of a balance. Typical axioms are: if a and b are in Ω, then $a \circ b$ is also in Ω; the relation $\geqq$ is a weak ordering of Ω; if $a \geqq b$, then $a \circ c \geqq b \circ c$ for all c in Ω; and so on. Ultimately, one states a set of axioms that is sufficient to prove a representation and uniqueness theorem of the following form. There exists a function $\phi: \Omega \rightarrow$ real numbers such that

i. *$a \geqq b$ (qualitatively) if and only if $\phi(a) \geqq \phi(b)$ (numerically)*;

ii. *for all a, b in Ω, $\phi(a \circ b) = \phi(a) + \phi(b)$*;

iii. *if ϕ and ϕ^* are two functions that both satisfy* (i) *and* (ii), *then there exists a positive constant α such that $\phi = \alpha\phi^*$. That is, the representation is unique up to a similarity transformation or, in the language of modern measurement theory, the variable in question can be represented as a ratio scale.*

It would be most convenient for psychology to have an analogous theory in which the axioms turned out to be empirically verifiable statements about the behavior of human or animal subjects. Unfortunately, one apparently cannot reinterpret simply the axioms of extensive measurement in psychological terms. It is easy to give sensible psychological interpretations of Ω and $\geqq$, but it is less easy to assign natural meanings to the concatenation operation. Our failure to devise empirically acceptable interpretations of extensive measurement when, nonetheless, we feel strongly that fundamental measurement ought to be possible has forced us to reconsider some of Campbell's ideas [28]. He seemed to feel that fundamental measurement is synonymous with extensive measurement, that is, with a system leading to the three assertions above. In this, it is generally agreed today, he was incorrect. As I have suggested, fundamental measurement involves a system of axioms that is stated without any reference to the real numbers, that has an empirical interpretation which permits the axioms to be checked directly, and from which it is possible to establish a numerical representation of the undefined objects and relations of this system so that condition (i) above is satisfied, (ii) is replaced by some appropriate condition, and (iii) may be weakened somewhat to another fairly restrictive group of transformations such as the positive linear transformations. Extensive measurement is one example of fundamental measurement, but there are others.

Most of these examples of fundamental, nonextensive measurement have arisen from the study of preference, a topic of concern in economics and statistics as well as psychology. One development was triggered by von Neumann and Morgenstern's [**33**] work on the expected utility hypothesis. Although their theory is not an example of fundamental measurement (because probabilities occur in the statement of the axioms), Savage's generalization [**26**] is—in fact, it provides for the simultaneous fundamental measurement of both utility and subjective probability. Suppes and his collaborators [**9**], following up an idea of the philosopher Ramsey [**24**], have devised a system for the fundamental measurement of utility that differs from Savage's in having only one chance event. Pfanzagl [**23**] has presented a rather different axiomatization that involves, essentially, the notion of bisection of two stimuli; it is closely similar to an axiomatization of means given by Aczél [**1**]. Recently, Tukey and I [**20**] have developed a system of fundamental measurement in which the basic ingredients are a weak ordering of objects having at least two independent components. A physical realization that would satisfy our axioms, according to classical physics, is the ordering of objects by momentum as measured by, say, a ballistic pendulum; the two components of the objects are, in ordinary terms, mass and velocity. Potential psychological examples—none of which has yet been explored empirically—involve subject-determined orderings of stimuli that have at least two independent coordinates. Preference of rats between pairs of outcomes consisting of different amounts of food paired with different levels of shock might be an example. We have shown that if the ordering satisfies certain "reasonable" axioms, then a numerical representation exists that is additive over the components and it is unique up to positive linear transformations; the theory of extensive measurement can be extracted as a special case when the two components are the same and one further axiom (involving the existence of a null object) is added.

Work of this type, which I believe promises ultimately to help isolate fundamental psychological variables, constitutes only a tiny portion of current research in mathematical psychology. For the most part, numbers enter in a different way, somewhat analogous to what Campbell called derived measurement. Typical physical examples of derived measurement are the usual definitions of density and momentum in which the measure in question is expressed in terms of two or more quantities that have been fundamentally measured already. His concept of derived measurement has to be stretched somewhat to encompass most of our work, and so I shall not press this point in what follows.

Probabilistic models. The main sources of numbers in most of mathematical psychology are the relative frequencies of responses and the times that it takes for responses to occur. Relative frequencies, which are much the more common measure, are interpreted as estimates of conditional response probabilities. This immediately leads one to consider probabilistic theories of behavior. Such theories are conceptually distinct, and in general quite different, from those postulated in statistics as models for hypothesis testing, analysis of variance, and the like.

Some debate, much of it unpublished, concerns the appropriateness of probability models in psychology. The current trend is viewed with alarm by some, who point out how ready such theories are to incorporate our confusion, ignorance, and experimental errors into what purports to be a description of the subject's behavior. But, as it is very difficult to characterize clearly the conditions under which it is appropriate to view an organism as a probability mechanism, many of us are uncertain just what to do about these criticisms. In one way or another we must cope with the fact that even under carefully controlled laboratory conditions individual subjects do not respond consistently to repeated presentations of exactly the same stimuli and outcomes. In general, the technique in classical physics of adding a dash of normal probability theory to an underlying deterministic theory has not proven very effective. Human and animal behavior appears to exhibit a more complex probabilistic structure than can be encompassed by deterministic theories flavored with a moderately uniform overlay of randomness, and so many of us have been driven to study probability models. I use the word "driven" advisedly because I, at least, was originally unsympathetic to such models. A few examples of these models will serve to illustrate the kinds of mathematics employed.

Learning models. Consider, first, "simple learning." This term should be taken with a goodly dose of salt, for most learning experiments have little immediate bearing upon what we ordinarily call learning—acquisition of concepts, insights, and understanding. Typically, we place the subject in a repetitive choice situation in which his responses are differentially rewarded, and we study the slowly evolving change of performance as the reward schedule unfolds itself to him. If the change evolves too rapidly, then we revamp the experiment to slow it down to the point where we can observe the transient behavior develop. Our problem, then, is to explain the laws whereby the response probabilities change over trials as a function of the subject's past experience *in the experiment.*

A variety of theories have been proposed and are under active study. I shall mention three of them. The first, known as stimulus sampling theory, is due largely to W. K. Estes ([**10**]; for a survey see [**3**]). These models postulate a mechanism whereby the subject "samples" environmental cues which are each "conditioned" or "attached" to particular responses. A decision rule describes how a response is selected on the basis of the cues sampled. Then, depending upon which outcome occurs, the conditioning of cues to responses is changed in a systematic fashion. Such mechanisms lead to Markov chains to describe the transitions of the response probabilities, and so all of that well-developed theory can be brought to bear upon it. Much attention has been paid to the detailed sequential properties of the responses, which are very complex indeed; it is amazing how well some of the models account for these subtleties in the data.

A second class of learning models begins directly with assumptions about the trial-to-trial changes in the response probabilities. For example, Bush and Mosteller [5] studied stochastic processes generated by path-independent linear operators, and numerous other authors in later publications have extended our knowledge of these processes considerably (see [27]). Let $p_n(r)$ denote the probability of response r on trial n; then they supposed that

$$p_{n+1}(r) = (1 - \theta)p_n(r) + \theta\lambda,$$

where θ and λ are constants that depend upon the particular response and outcome that occurred on trial n. The linearity of the model is obvious; by path-independence is meant the assumption that the relevant past history of the process is completely summarized by the response probability $p_n(r)$ and the events that occurred on trial n. Such nonstationary stochastic processes are relatively complicated, and only a few simple cases are well understood. For example, although the asymptotic mean is known to exist [14], we do not have an explicit expression for it in the most general two response situation.

Another class of path-independent response models are the commutative ones that I have looked into [16], [19]. These learning operators are most simply stated in the form

$$f(p_{n+1}) = \beta f(p_n),$$

where f is a real-valued, strictly monotonic transformation of the unit interval and β is a positive constant that depends upon the response and outcome on trial n. By the form of the operator, this class of models is clearly both path-independent and commutative. Some fairly general results about these operators have been deduced using functional equation techniques, some of which are treated in Aczél's recent book on functional equations [2].

Once a stochastic learning model is formulated, the main task is to determine explicit (computable) expressions for various of its properties, which can then be used to estimate parameters from data and to evaluate the adequacy of the model. Typical properties are asymptotic means and variances, expected number of choices of a particular response, expected number of runs of responses of a given length, etc. In the linear model, it is common to set up the obvious expression for the desired random variable and then to calculate its expectation over the branching process described by the model. When this works, it does so mainly because the operators are linear; it rarely, if ever, works with the nonlinear models. For the commutative models it is possible to formulate and solve functional equations for some properties of interest (see

[**13**] and [**29**]). Certain asymptotic results have been obtained by using a technique based upon chains of infinite order [**15**].

Although parameter estimation and the testing of goodness-of-fit are problems whenever a theory is evaluated by data, these issues have grown especially significant and troublesome for the stochastic processes used in learning. Complicated mathematical problems often develop when we try to work out optimal estimators, and yet simpler procedures have been shown to lead to confusion and misinterpretation of data [**27**]. This last point blends into the question of measuring the goodness-of-fit of a model to a set of data, which question is not at all well formulated at present. Mostly ad hoc and intuitive evaluations are made and, while everyone agrees that they are not really satisfactory, little in the statistical literature seems helpful. Much interesting work could be done by someone with a strong statistical bent and a taste for unformulated, complex problems.

Preference models. The experimental study of preference is curious for this reason: the outcomes and stimuli of the experiment are either the same or very closely related. On each trial the subject is presented with a set of potential outcomes among which he is to choose. He receives either the one chosen or, in the case of what are known as "risky alternatives," a chance mechanism intervenes to determine which outcome is finally delivered.

Statisticians and economists have studied a variety of algebraic models for preference behavior, among them von Neumann and Morgenstern's expected utility model and Savage's generalization of it which were mentioned earlier. Psychologists who have become interested in preference problems have turned mostly to probabilistic models, primarily because the experimental data exhibit patterns of inconsistencies which are difficult to cope with in a deterministic framework. Broadly speaking, two approaches have been taken. One is to state, in terms of observable choice probabilities, properties that might reasonably be expected to be observed. These do not constitute a complete theory of behavior; rather, they are testable propositions that might become theorems in a more complete theory of behavior. A simple example is strong stochastic transitivity, which is one of several possible generalizations of ordinary transitivity. It asserts that if the probability of choosing a over b, $p(a, b)$, $\geqq 1/2$ and if $p(b, c) \geqq 1/2$, then $p(a, c) \geqq \max [p(a, b), p(b, c)]$. The other approach is to formulate more complete theories of behavior. One familiar example is the class of random utility models in which it is supposed that a person's preferences are determined at any instant of time by a numerical utility function which varies from instant to instant in some random fashion. The outcome having the highest momentary utility is the one that is momentarily preferred. Some of us are trying to deduce testable properties from the different models, to establish the network of relations among the models and properties and to uncover equivalent restatements of models. At the same time, experimentalists are attempting to discover which of the conditions seem to be satisfied by human beings; the results to date, however, have tended to be ambiguous. I suspect that our experimental studies of preference are not yet adequate to answer the questions we want to ask.

Psychophysical models. The oldest branch of psychology and the one that, over the long haul, has used mathematics more consistently than any other is psychophysics, the study of the way in which responses depend upon continuous physical attributes of the stimulation. These experiments always involve the presentation of one of several possible stimuli on each trial, to which the subject responds from a set of responses that is systematically coordinated with the physical properties of the stimuli. Perhaps the simplest example is a detection experiment in which a faint stimulus or no stimulus is presented on a trial and the subject reports, in effect, "yes" or "no" depending upon whether or not he believes the stimulus to have been presented.

As an example of the problems facing the theorist, consider the plot of the conditional probability of responding "yes" when the stimulus is presented versus the conditional probability of responding "yes" when it is absent. If we vary either the outcomes to the subject or the presentation probability of the stimulus, we find that the resulting data points appear to arise from a continuous underlying curve that goes from (0, 0) to (1, 1) in the unit square and lies above the chance line $y=x$. The various theories proposed for the detection process do indeed predict the existence of such a curve, and different theories predict different ones. For example, signal detectability theory (for summaries see [**11**] and [**18**]) says that the curve becomes a straight line when we plot the probabilities on normal-normal paper, whereas a "low" threshold theory [**17**], [**18**], which postulates a discrete underlying process, says that it consists of two straight line segments in the ordinary plot. Although these predictions are conceptually quite different, considerable data are needed to select between them.

Quite a variety of models now exist for different psychophysical procedures and attempts are being made to systematize and unify them. For a period, information theory seemed to provide a unifying approach, but recent studies suggest that the unity was more apparent than real.

One nice feature of much of mathematical psychophysics, in contrast to other areas, is that it uses comparatively elementary mathematics and yet provides significant psychological insights that probably cannot be achieved otherwise. Therefore, it seems particularly well suited for inclusion in undergraduate courses on mathematical psychology and as a source of simple to intermediate level problems for undergraduate mathematics courses aimed at behavioral scientists.

Latency models. As I pointed out earlier, the elapsed time for a response to occur—the response latency—is a second obvious supply of numbers for the psychologist. In the simplest case, one presents a stimulus and requires the subject to respond to it as rapidly as possible. If the experiment is very carefully done and if the subject is in good physical and emotional condition, then highly regular and reproducible distributions of response latencies are found. The provisos are important: latency is a much more skitterish measure than response frequency, and correspondingly greater care is needed in the collection of latency data.

The theoretical problem is to devise hypothetical mechanisms to account for the forms of the observed distributions, thereby permitting us to infer something about the nature of the underlying processing of sensory information. The main idea so far is that the resulting latency is built up from a chain of elementary random events—often exponential ones (see [21]). Theories of this type tend to parallel and sometimes use results from the theory of queues. All in all, the main mathematical device is the Laplace transform in one guise or another. I suspect, however, that new ideas will soon enter, because it has recently been observed that when subjects are paid differentially depending upon the latency of the response, the distributions have highly peaked modes and high tails—of the form $t^{-\delta}$ rather than $e^{-\lambda t}$. It is not clear that such distributions can arise from additive chains of independent random variables. Any ideas that arise to account for these data can be expected to have an impact upon our conceptualizations of the choice processes involved in learning, preference, and psychophysics.

Psychometrics. During the 1930's and 40's, the main domain of the mathematically inclined psychologist was psychometrics, which is somewhat like both psychophysics and statistics. Many of the psychometric models are formally the same as those used in psychophysics [32]; others, especially those used in the theory of tests, are basically multivariate statistical models [30], [31]. Throughout psychometrics, one collects relatively little data from each of a relatively large number of subjects. Advantage is taken of the internal consistencies and correlations exhibited within the population of subjects, and the analysis proceeds upon the assumption that a common structure underlies all of the subjects, who differ from one another only in the values of certain parameters. This approach is to be contrasted, for example, with that of much of psychophysics in which relatively large amounts of data are collected from relatively few subjects. There, each subject is studied individually in detail. Most of the psychometric techniques postulate that the subjects can be represented as points in a Euclidean vector space and, of course, one of the main mathematical techniques is matrix algebra. Factor analysis and other multi-dimensional scaling procedures are most readily exposited in the language of matrices, and some moderately deep results of matrix theory have been used.

Nonnumerical models. In the final class of models that I shall mention, numbers play no role. This is not an especially coherent or extensive class of models, and I shall mention only two examples.

In psycholinguistics—which studies the interplay between the speaker, the hearer, and their language—Chomsky and Miller ([8]; see also [7] and [22]) among others, have employed techniques of abstract algebra to describe aspects of the basic grammatical structure of languages. Specifically, they have treated grammars as concatenation algebras and have used results from the theory of recursive functions. The mathematics gets quite involved and a number of interesting new results have been proven. A strong interplay exists between theoretical psycholinguists and those working on the theory of automata. So far, few predictions that are amenable to experimentation have been derived from these pretty mathematical structures, and so relatively little experimentation has yet resulted. But work in this area is active and rapid developments should take place in the next few years.

A different sort of structural model has received attention in the area known as sociometrics, which is, loosely, the study of the structure of qualitative relationships among people. The main tool that Harary [12] and others have used in constructing this theory is the topological theory of graphs (for a summary see [25]). The results tend to be mathematically sophisticated, but the theory is relatively ineffectual in establishing connections between the mathematics and the empirical world. The main problem, I believe, is the failure to introduce any behavioral assumptions into the model. The structural assumptions do not characterize in any way the entities, the human beings, represented by the points of the graph, and so little in the way of prediction is possible.

Concluding remarks. Up to this point, I have treated the applications of mathematics to psychology from the psychologist's vantage point. Now let me turn about and consider briefly the extent to which different categories of mathematics have been used.

1. FUNDAMENTALS, including set theory, functions, relations, orderings, axiomatics, Boolean algebra, and the like. Fundamental mathematical notions and results are used throughout mathematical psychology; without a moderate grounding in them one cannot read much of the literature. Many problems are formulated initially in terms of sets, relations, and nonnumerical functions. Axiomatic systems are not uncommon and precise, if sometimes relatively elementary, reasoning is usual. Such knowledge is mandatory, and it should be absorbed as early as possible.

2. ANALYSIS. Analysis presents a curious problem because psychological theories are rarely formulated in its terms. A differential or integral equation is hardly ever a starting point, although difference equations sometimes are. Nevertheless, a student must know at least the calculus or suffer an impossible disadvantage, because no one gives a second thought to using derivatives or integrals in the solution of problems cast in other terms. Moreover, one cannot penetrate deeply into probability theory, stochastic processes, or advanced statistics without a firm grip on classical analysis, especially the theory of real variables. As I mentioned earlier, transform theory has proved important in the study of latencies and, along with a little complex variable theory, it is im-

portant in the analysis of tracking behavior. But this topic is sufficiently special that few students need study the theory of functions of a complex variable at present.

It is difficult to know what analysis a student should be advised to take. Although it is handy to have quite a bit, there are as yet few penetrating uses of it except indirectly via probability theory. Nonetheless, a student really should know enough to be able to read some quite sophisticated papers on functional equations, which are beginning to be used in many places, and on stochastic processes. Moreover, to the extent that we begin to formulate and study continuous time processes, classical analysis will prove essential. I mention this because some research of this sort has recently begun to appear.

3. PROBABILITY THEORY. Elementary probability concepts occur extensively in psychology; they are employed with neither comment nor apology. Some of the major theorems—e.g., the central limit theorem—are used frequently. Any well trained mathematical psychologist should have under his belt a solid course on probability and mathematical statistics. For many students, if not most, some additional work on stochastic processes is mandatory. In particular, they should be well grounded in the theory of Markov chains, especially for a finite number of states. These chains arise naturally in some theories of the learning process. In addition, they should be exposed to a number of the nonstationary stochastic processes that are also used. In general, these processes are not yet part of the standard textbook fare and, in many cases, they are not yet completely understood. Psychology offers numerous difficult, well-formulated, but unsolved problems in the theory of stochastic processes. Some interesting functional equations have arisen in the study of stochastic processes for learning; perhaps they are destined to serve in psychology the role of differential equations in physics.

4. ALGEBRA. Elementary algebra is used everywhere and complete facility is presupposed. Matrices are applied in psychometrics and multivariate analysis. Concatenation algebras are used in psycholinguistics. But the main body of abstract algebra—including groups, rings, fields, lattice theory, and more exotic notions, but excluding Boolean algebra—has been little used. Here and there a group or a lattice crops up, but by and large abstract algebra has not proved particularly important except for the sophistication engendered by its study.

5. TOPOLOGY AND GEOMETRY. Neither is much used. The only really serious attempt that I know of to employ topological notions is the application of graph theory to sociometrics. A bit of point set topology occurs now and then and occasionally fixed point theorems have been employed to establish the existence of something or other. Aside from trivial uses of high school geometry and a little work with non-Euclidean geometry in vision, geometry has not played much of a role in mathematical psychology.

In closing, it should be pointed out that no fundamentally new mathematical ideas have yet arisen from work on psychological problems. To date, we are parasites on mathematics. I doubt that this will always be so, but the time span is likely to be such that our contributions to mathematics will have little or no impact on anyone doing research in mathematics today. The reason that I believe an influence is bound to occur ultimately is that psychological problems exist that, on the one hand, seem to be meaningful, if not yet precisely stated, and about which we can experiment, if not yet incisively, and on the other hand, that do not seem to fit comfortably into the existing mathematical language. In some cases this may simply reflect a lack of ingenuity, but in others I suspect that the appropriate mathematics does not yet exist. The history of physics favors such a belief, which of course has the dubious virtue of being capable of neither proof nor disproof. Perhaps a few words about where I see trouble will make it a little more meaningful.

The language of sets does not always seem adequate to formulate psychological problems. Put so baldly, the statement is almost heretical since, in practice, set theory is the accepted way to formulate mathematical problems . . . and, hence, applied mathematical problems. Still, we should not forget that set theory is really quite new—less than a century old. It could be an interim theory. Certainly when I think about certain psychological problems, I wish it weren't the way it is. The boundaries of many of my "sets," and of ones that my subjects ordinarily deal with, are a good deal fuzzier than those of mathematics. Consider an experiment in which the subject is presented with a set of possible responses. This is a nice, unambiguous set of the sort that we are all conditioned to expect. But in a theoretical analysis of the subject's behavior, it often seems far more reasonable to consider not this set, but the one he considered before making his choice. It is quite difficult to pin down just what elements are and are not members of that set, and I am not sure that it is possible in principle. Do we merely lack techniques adequate to answer that question today, or is it basically impossible to answer it? Even supposing that it is impossible, I do not believe that this means that attempts to understand behavior are ridiculous, although ultimately it may be deemed inappropriate to try to cast theories of behavior in current mathematical language.

To take another example, we all deal effectively with the uncertainties of everyday life in terms of extremely imprecise concepts such as "likely," "fairly likely," and so on. As theorists, we often try to cope with this sort of behavior by phrasing it in the language of probability, but I suspect that most of us do not really feel that the mathematics meshes especially well with the problem. The categories of uncertainty are not really well-defined sets and their fuzziness is not particularly well summarized by probability notions. Perhaps we can make the existing concepts work, but I doubt that we should count on it.

Even assuming that there are profound troubles, mathematical psychologists are not about to call a moratorium until the troubles are resolved. Some, indeed most of us, will skirt around those aspects of behavior for which the difficulties are especially pronounced. Others will try to tackle them more directly and to the extent of their success, they will enrich mathematics as well as psychology.

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THE SOLUTION OF A SECOND ORDER LINEAR DIFFERENTIAL EQUATION NEAR A REGULAR SINGULAR POINT

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The standard undergraduate course in ordinary differential equations suffers from many ills. In many cases it seems as an anticlimax after an upgraded calculus course. There is usually too much emphasis on specific integration techniques and not enough on existence and uniqueness theorems. It has not kept up with the computer age and therefore does not sufficiently emphasize numerical integration techniques. There is not enough attention paid to qualitative analysis, i.e., the determination of properties of the solution such as bounds, asymptotic expansions, stability, etc., without explicitly solving the equation. It is not necessary to postpone such a course in differential equations until after an advanced calculus course to cure these ills. One possible key is the early introduction of the Green's function and the subsequent reduction of the differential equation to an integral equation. Once this is done the existence and uniqueness theorems usually follow by a simple iteration procedure which uses nothing more sophisticated than the term-by-term integration of a uniformly convergent series. The iteration procedure is very basic, conceptually simple, and sets the tone for a large class of numerical integration techniques. Also, when the problem is cast in this way, many of the properties of the solution can be investigated without explicitly solving the equation.

The purpose of this paper is to illustrate these remarks in connection with the solution of a second order ordinary differential equation near a regular singular point, a case which is very important in applications but one which is commonly mishandled. The present approach is not new, but it does not appear