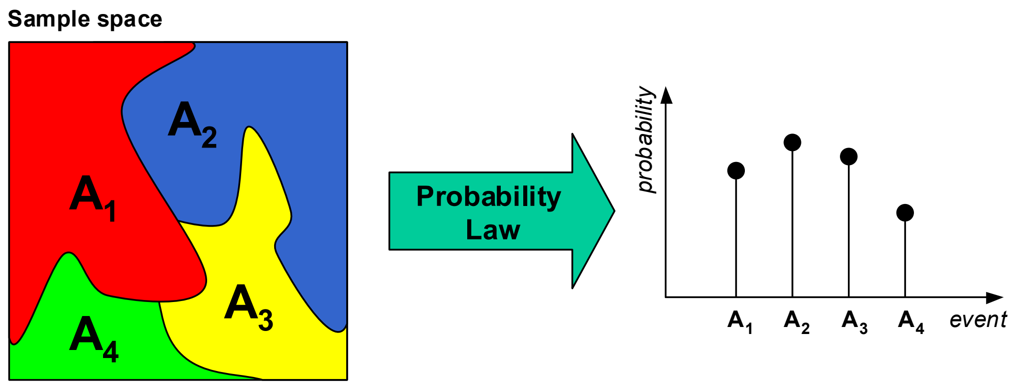
LECTURE 2: Review of Probability and Statistics

- **Probability**
  - Definition of probability
  - Axioms and properties
  - Conditional probability
  - Bayes Theorem
- **Random Variables**
  - Definition of a Random Variable
  - Cumulative Distribution Function
  - Probability Density Function
  - Statistical characterization of Random Variables
- **Random Vectors**
  - Mean vector
  - Covariance matrix
- **The Gaussian random variable**

Basic probability concepts

Definitions (informal)

- Probabilities are numbers assigned to events that indicate "***how likely***" it is that the event will occur when a random experiment is performed
- A **probability law** for a random experiment is a rule that assigns probabilities to the events in the experiment
- The **sample space** S of a random experiment is the set of all possible outcomes

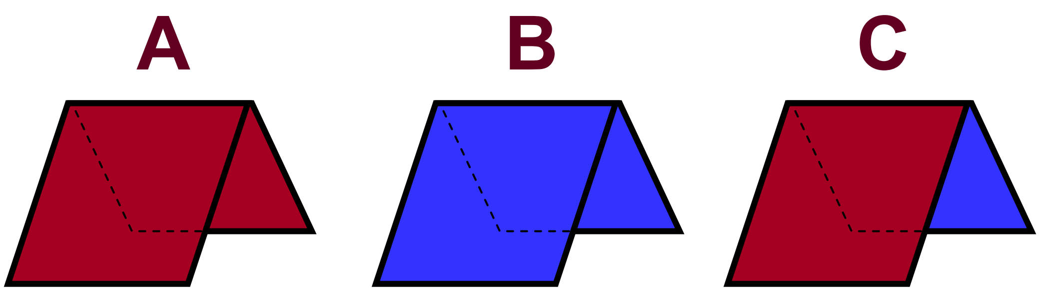


Axioms of probability

- Axiom I: $0 \le P[A_i]$
- Axiom II: $P[S] = 1$
- Axiom III: if $A_i \cap A_j = \varnothing$, then $P[A_i \cup A_j] = P[A_i] + P[A_j]$

Warming-up exercise

- **I come to class with three colored cards**
  - One BLUE on both sides
  - One RED on both sides
  - One BLUE on one side, RED on the other



- **I shuffle the three cards, then pick one and show you one side only. The side visible to you is RED**
  - Obviously, the card has to be either A or C, *right*?
- **I am willing to bet $1 that the other side of the card has the same color, and need someone in the audience to bet another $1 that it is the other color**
  - Obviously, on the average we will end up even, *right*?
  - Let's try it!

More properties of probability

PROPERTY 1: $P[A^C] = 1 - P[A]$

PROPERTY 2: $P[A] \leq 1$

PROPERTY 3: $P[\varnothing] = 0$

PROPERTY 4: given $\{A_1, A_2, \ldots A_N\}$, if $\{A_i \cap A_j = \varnothing \;\; \forall i,j\}$ then $P[\bigcup_{k=1}^{N} A_k] = \sum_{k=1}^{N} P[A_k]$

PROPERTY 5: $P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]$

PROPERTY 6: $P[\bigcup_{k=1}^{N} A_k] = \sum_{k=1}^{N} P[A_k] - \sum_{j<k}^{N} P[A_j \cap A_k] + \ldots + (-1)^{N+1} P[A_1 \cap A_2 \cap \ldots \cap A_N]$

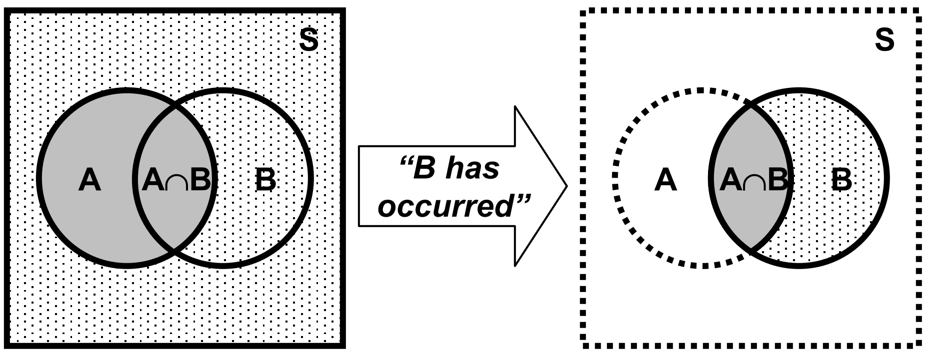
PROPERTY 7: if $A_1 \subset A_2$, then $P[A_1] \leq P[A_2]$

Conditional probability

- **If A and B are two events, the probability of event A when we already know that event B has occurred is defined by the relation**

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \quad \text{for} \quad P[B] > 0$$

- **This conditional probability P[A|B] is read:**
  - the "conditional probability of A conditioned on B", or simply
  - the "probability of A given B"

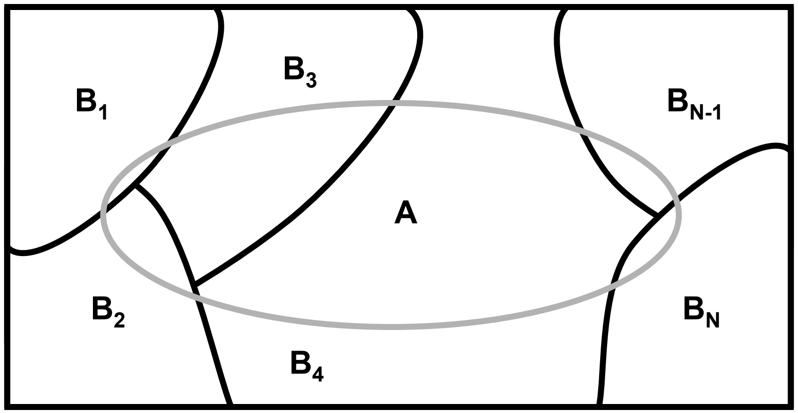


- **Interpretation**
  - The new evidence "*B has occurred*" has the following effects
    - The original sample space S (the whole square) becomes B (the rightmost circle)
    - The event A becomes A∩B
  - P[B] simply re-normalizes the probability of events that occur jointly with B

Theorem of total probability

- Let B_1, B_2, …, B_N be mutually exclusive events whose union equals the sample space S. We refer to these sets as a partition of S.
- An event A can be represented as:

$$A = A \cap S = A \cap (B_1 \cup B_2 \cup ... \cup B_N) = (A \cap B_1) \cup (A \cap B_2) \cup ... (A \cap B_N)$$



- Since B_1, B_2, …, B_N are mutually exclusive, then

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + ... + P[A \cap B_N]$$

- and, therefore

$$P[A] = P[A \mid B_1]P[B_1] + ... P[A \mid B_N]P[B_N] = \sum_{k=1}^{N} P[A \mid B_k]P[B_k]$$

Bayes Theorem

- **Given B_1, B_2, …, B_N, a partition of the sample space S. Suppose that event A occurs; what is the probability of event B_j?**
  - Using the definition of conditional probability and the Theorem of total probability we obtain

$$P[B_j \mid A] = \frac{P[A \cap B_j]}{P[A]} = \frac{P[A \mid B_j] \cdot P[B_j]}{\sum_{k=1}^{N} P[A \mid B_k] \cdot P[B_k]}$$

Rev. Thomas Bayes (1702-1761)

- **This is known as Bayes Theorem or Bayes Rule, and is (one of) the most useful relations in probability and statistics**
  - Bayes Theorem is definitely the fundamental relation in Statistical Pattern Recognition

Bayes Theorem and Statistical Pattern Recognition

- **For the purpose of pattern <u>classification</u>, Bayes Theorem can be expressed as**

$$P[\omega_j \mid x] = \frac{P[x \mid \omega_j] \cdot P[\omega_j]}{\sum_{k=1}^{N} P[x \mid \omega_k] \cdot P[\omega_k]} = \frac{P[x \mid \omega_j] \cdot P[\omega_j]}{P[x]}$$

  - where ω_j is the i[th] class and ***x*** is the feature vector
- **A typical decision rule (class assignment) is to choose the class ω_i with the highest $P[\omega_i|x]$**
  - Intuitively, we will choose the class that is more "likely" given feature vector ***x***
- **Each term in the Bayes Theorem has a special name, which you should be familiar with**
  - $P[\omega_j]$ **Prior probability** (of class ω_i)
  - $P[\omega_j \mid x]$ **Posterior Probability** (of class ω_i given the observation ***x***)
  - $P[x \mid \omega_j]$ **Likelihood** (conditional probability of observation ***x*** given class ω_i)
  - $P[x]$ A normalization constant that does not affect the decision

Stretching exercise

- **Consider a clinical problem where we need to decide if a patient has a particular medical condition on the basis of an *imperfect* test:**
  - Someone with the condition may go undetected (*false-negative*)
  - Someone free of the condition may yield a positive result (*false-positive*)
- **Nomenclature**
  - The true-negative rate *P(NEG|¬COND)* of a test is called its SPECIFICITY
  - The true-positive rate *P(POS|COND)* of a test is called its SENSITIVITY

	TEST IS POSITIVE	TEST IS NEGATIVE	ROW TOTAL
HAS CONDITION	*True-positive* *P(POS|COND)*	*False-negative* *P(NEG|COND)*	
FREE OF CONDITION	*False-positive* *P(POS|¬COND)*	*True-negative* *P(NEG|¬COND)*	
COLUMN TOTAL			

- **PROBLEM**
  - Assume a population of **10,000** where **1** out of every 100 people has the condition
  - Assume that we design a test with **98%** specificity and **90%** sensitivity
  - Assume you are required to take the test, which then yields a POSITIVE result
  - **What is the probability that you have the condition**?
    - SOLUTION A: Fill in the joint frequency table above
    - SOLUTION B: Apply Bayes rule

Stretching exercise

- **Consider a clinical problem where we need to decide if a patient has a particular medical condition on the basis of an *imperfect* test:**
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	TEST IS POSITIVE	TEST IS NEGATIVE	ROW TOTAL
HAS CONDITION	*True-positive* *P(POS|COND)* **100×0.90**	*False-negative* *P(NEG|COND)* **100×(1-0.90)**	**100**
FREE OF CONDITION	*False-positive* *P(POS|¬COND)* **9,900×(1-0.98)**	*True-negative* *P(NEG|¬COND)* **9,900×0.98**	**9,900**
COLUMN TOTAL	**288**	**9,712**	**10,000**

- **PROBLEM**
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  - Assume that we design a test with **98%** specificity and **90%** sensitivity
  - Assume you are required to take the test, which then yields a POSITIVE result
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    - SOLUTION A: Fill in the joint frequency table above
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Stretching exercise

- **SOLUTION B: Apply Bayes theorem**

$$P[\mathrm{COND}\,|\,\mathrm{POS}] =$$

$$= \frac{P[\mathrm{POS}\,|\,\mathrm{COND}] \cdot P[\mathrm{COND}]}{P[\mathrm{POS}]} =$$

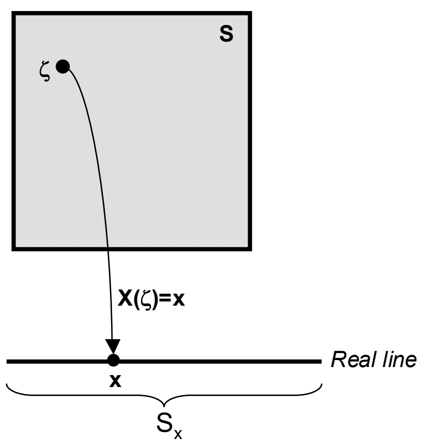
$$= \frac{P[\mathrm{POS}\,|\,\mathrm{COND}] \cdot P[\mathrm{COND}]}{P[\mathrm{POS}\,|\,\mathrm{COND}] \cdot P[\mathrm{COND}] + P[\mathrm{POS}\,|\,\neg\mathrm{COND}] \cdot P[\neg\mathrm{COND}]} =$$

$$= \frac{0.90 \cdot 0.01}{0.90 \cdot 0.01 + (1 - 0.98) \cdot 0.99} =$$

$$= 0.3125$$

Random variables

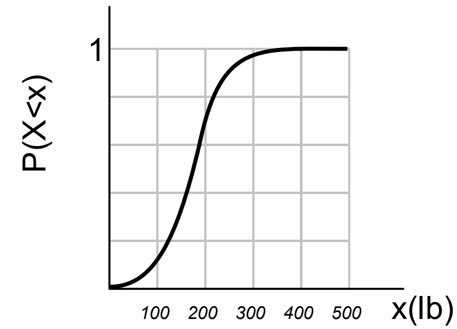
- **When we perform a random experiment we are usually interested in some measurement or numerical attribute of the outcome**
  - When we sample a population we may be interested in their weights
  - When rating the performance of two computers we may be interested in the execution time of a benchmark
  - When trying to recognize an intruder aircraft, we may want to measure parameters that characterize its shape
- **These examples lead to the concept of *random variable***
  - **A random variable X is a function that assigns a real number X(ζ) to each outcome ζ in the sample space of a random experiment**
    - This function X(ζ) is performing a mapping from all the possible elements in the sample space onto the real line (real numbers)
  - The function that assigns values to each outcome is fixed and deterministic
    - as in the rule "*count the number of heads in three coin tosses*"
    - the randomness the observed values is due to the underlying randomness of the argument of the function X, namely the outcome ζ of the experiment
  - Random variables can be
    - Discrete: the resulting number after rolling a dice
    - Continuous: the weight of a sampled individual



Cumulative distribution function (cdf)

- **The cumulative distribution function $F_X(x)$ of a random variable X is defined as the probability of the event $\{X \le x\}$**

$$F_X(x) = P[X \le x] \text{ for } -\infty < x < +\infty$$



cdf for a person's weight

  - Intuitively, $F_X(b)$ is the long-term proportion of times in which $X(\zeta) \le b$

- **Properties of the cdf**

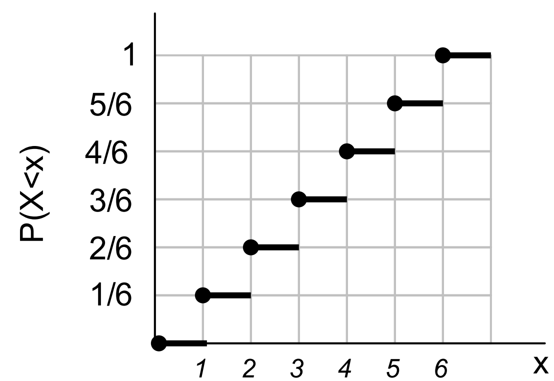
$$0 \le F_X(x) \le 1$$

$$\lim_{x\to\infty} F_X(x) = 1$$

$$\lim_{x\to-\infty} F_X(x) = 0$$

$$F_X(a) \le F_X(b) \text{ if } a \le b$$

$$F_X(b) = \lim_{h\to 0} F_X(b+h) = F_X(b^+)$$

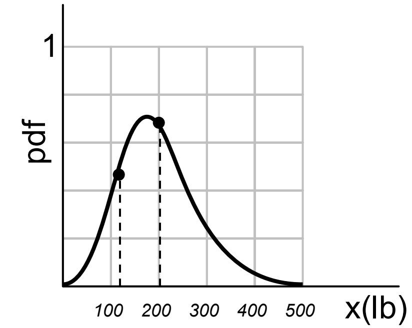


cdf for rolling a dice

Probability density function (pdf)

- The <u>probability density function</u> of a continuous random variable X, if it exists, is defined as the derivative of $F_X(x)$

$$f_X(x) = \frac{dF_X(x)}{dx}$$



pdf for a person's weight

- For discrete random variables, the equivalent to the pdf is the <u>probability mass function</u>:

$$f_X(x) = \frac{\Delta F_X(x)}{\Delta x}$$

- Properties

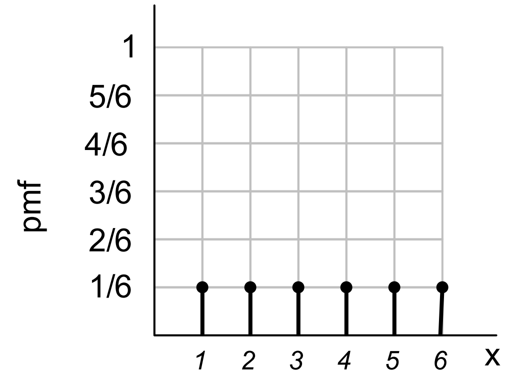
$$f_X(x) > 0$$

$$P[a < x < b] = \int_a^b f_X(x)dx$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

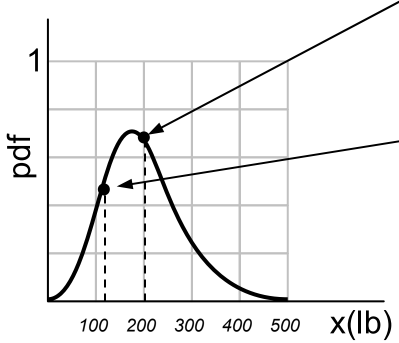
$$1 = \int_{-\infty}^{+\infty} f_X(x)dx$$

$$f_X(x|A) = \frac{d}{dx}F_X(x|A) \ \text{ where } \ F_X(x|A) = \frac{P[\{X<x\} \cap A]}{P[A]} \ \text{ if } P[A] > 0$$

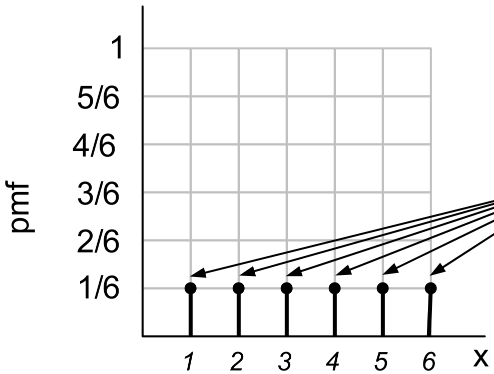


pmf for rolling a (fair) dice

Probability density function Vs. Probability



pdf for a person's weight



pmf for rolling a (fair) dice

- ***What is the probability of somebody weighting 200 lb****?*
  - *According to the pdf, this is about 0.62*
  - *This number seems reasonable, right?*
- ***Now, what is the probability of somebody weighting 124.876 lb?***
  - *According to the pdf, this is about 0.43*
  - *But, intuitively, we know that the probability should be zero (or very, very small)*
- ***How do we explain this paradox?***
  - *The pdf DOES NOT define a probability, but a probability DENSITY!*
  - *To obtain the actual probability we must integrate the pdf in an interval*
  - *So we should have asked the question: what is the probability of somebody weighting 124.876 lb plus or minus 2 lb?*
- ***The probability mass function is a 'true' probability (reason why we call it a 'mass' as opposed to a 'density')***
  - *The pmf is indicating that the probability of any number when rolling a fair dice is the same for all numbers, and equal to 1/6, a very legitimate answer*
  - *The pmf DOES NOT need to be integrated to obtain the probability (it cannot be integrated in the first place)*

Statistical characterization of random variables

- **The cdf or the pdf are SUFFICIENT to fully characterize a random variable, However, a random variable can be PARTIALLY characterized with other measures**
  - **Expectation** $$E[X] = \mu = \int_{-\infty}^{+\infty} x f_X(x)dx$$
    - The expectation represents the center of mass of a density
  - **Variance** $$VAR[X] = E[(X - E[X])^2] = \int_{-\infty}^{+\infty} (x - \mu)^2 f_X(x)dx$$
    - The variance represents the spread about the mean
  - **Standard deviation** $$STD[X] = VAR[X]^{1/2}$$
    - The square root of the variance. It has the same units as the random variable.
  - **Nth moment** $$E[X^N] = \int_{-\infty}^{+\infty} x^N f_X(x)dx$$

Random vectors

- **The notion of a random vector is an extension to that of a random variable**
  - A vector random variable X is a function that assigns a vector of real numbers to each outcome ζ in the sample space S
  - We will always denote a random vector by a **column vector**
- **The notions of cdf and pdf are replaced by 'joint cdf' and 'joint pdf'**
  - Given random vector, $\underline{X} = [x_1\ x_2 \ldots x_N]^T$ we define
    - **Joint Cumulative Distribution Function** as:

$$F_{\underline{X}}(\underline{x}) = P_{\underline{X}}[\{X_1 \le x_1\} \cap \{X_2 \le x_2\} \cap \ldots \cap \{X_N \le x_N\}]$$

    - **Joint Probability Density Function** as:

$$f_{\underline{X}}(\underline{x}) = \frac{\partial^N F_{\underline{X}}(\underline{x})}{\partial x_1 \partial x_2 \ldots \partial x_N}$$

- **The term marginal pdf is used to represent the pdf of a subset of all the random vector dimensions**
  - A marginal pdf is obtained by integrating out the variables that are not of interest
  - As an example, for a two-dimensional problem with random vector $\underline{X}=[x_1\ x_2]^T$, the marginal pdf for x_1, given the joint pdf $f_{X1X2}(x_1x_2)$, is

$$f_{X1}(x_1) = \int_{x_2=-\infty}^{x_2=+\infty} f_{X_1X_2}(x_1x_2)dx_2$$

Statistical characterization of random vectors

- **A random vector is also fully characterized by its joint cdf or joint pdf**
- **Alternatively, we can (partially) describe a random vector with measures similar to those defined for scalar random variables**
  - ***Mean vector***

$$E[X] = \left[E[X_1]E[X_2]...E[X_N]\right]^T = \left[\mu_1\,\mu_2\,...\mu_N\right] = \mu$$

  - ***Covariance matrix***

$$COV[X] = \Sigma = E[(X-\mu)(X-\mu)^T]$$

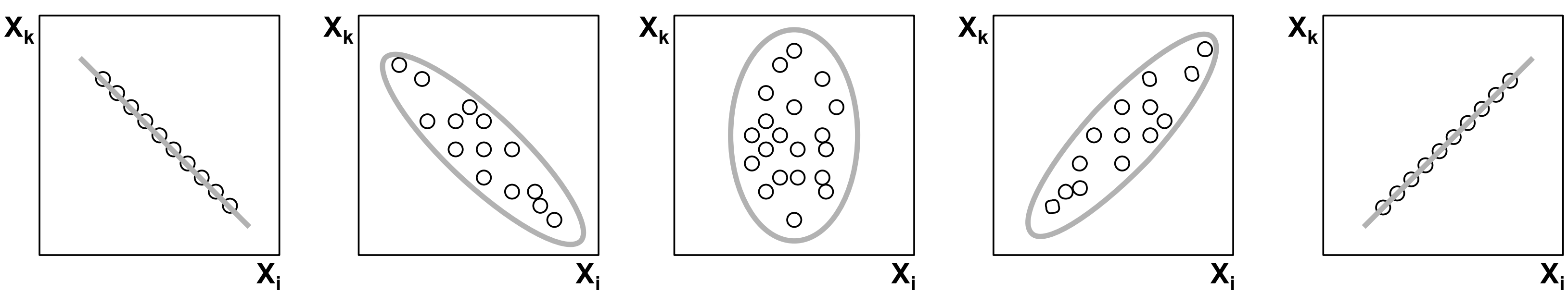
$$= \begin{bmatrix} E[(x_1-\mu_1)(x_1-\mu_1)] & ... & E[(x_1-\mu_1)(x_N-\mu_N)] \\ ... & ... & \\ E[(x_N-\mu_N)(x_1-\mu_1)] & ... & E[(x_N-\mu_N)(x_N-\mu_N)] \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & ... & c_{1N} \\ ... & ... & \\ c_{1N} & ... & \sigma_N^2 \end{bmatrix}$$

Covariance matrix (1)

- **The covariance matrix indicates the tendency of each pair of features (dimensions in a random vector) to vary together, i.e., to co-vary***
- **The covariance has several important properties**
  - If $\mathbf{x_i}$ and $\mathbf{x_k}$ tend to increase together, then $\mathbf{c_{ik}}>0$
  - If $\mathbf{x_i}$ tends to decrease when $\mathbf{x_k}$ increases, then $\mathbf{c_{ik}}<0$
  - If $\mathbf{x_i}$ and $\mathbf{x_k}$ are **uncorrelated**, then $\mathbf{c_{ik}}=0$
  - $|\mathbf{c_{ik}}|\le\sigma_i\sigma_k$, where σ_i is the standard deviation of $\mathbf{x_i}$
  - $\mathbf{c_{ii}} = \sigma_i^2 = \mathrm{VAR}(\mathbf{x_i})$
- **The covariance terms can be expressed as**

$$c_{ii} = \sigma_i^2 \quad \text{and} \quad c_{ik} = \rho_{ik}\sigma_i\sigma_k$$

  - where ρ_{ik} is called the **correlation coefficient**



$C_{ik}=-\sigma_i\sigma_k$	$C_{ik}=-\frac{1}{2}\sigma_i\sigma_k$	$C_{ik}=0$	$C_{ik}=+\frac{1}{2}\sigma_i\sigma_k$	$C_{ik}=\sigma_i\sigma_k$
$\rho_{ik}=-1$	$\rho_{ik}=-\frac{1}{2}$	$\rho_{ik}=0$	$\rho_{ik}=+\frac{1}{2}$	$\rho_{ik}=+1$

*from http://www.engr.sjsu.edu/~knapp/HCIRODPR/PR_home.htm

Covariance matrix (2)

- **The covariance matrix can be reformulated as***

$$\Sigma = E[(X-\mu)(X-\mu)^T] = E[XX^T] - \mu\mu^T = S - \mu\mu^T$$

$$\text{with } S = E[XX^T] = \begin{bmatrix} E[x_1x_1] & \dots & E[x_1x_N] \\ \dots & \dots & \dots \\ E[x_Nx_1] & \dots & E[x_Nx_N] \end{bmatrix}$$

  - S is called the autocorrelation matrix, and contains the same amount of information as the covariance matrix

- **The covariance matrix can also be expressed as**

$$\Sigma = \Gamma R \Gamma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & & \\ \dots & & \dots & \\ 0 & & & \sigma_N \end{bmatrix} \cdot \begin{bmatrix} 1 & \rho_{12} & \dots & \rho_{1N} \\ \rho_{12} & 1 & & \\ \dots & & \dots & \\ \rho_{1N} & & & 1 \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & & \\ \dots & & \dots & \\ 0 & & & \sigma_N \end{bmatrix}$$

  - A convenient formulation since Γ contains the scales of the features and R retains the essential information of the relationship between the features.
  - R is the correlation matrix

- **Correlation Vs. Independence**
  - Two random variables x_i and x_k are **uncorrelated** if $E[x_i x_k]=E[x_i]E[x_k]$
    - Uncorrelated variables are also called **linearly independent**
  - Two random variables x_i and x_k are **independent** if $P[x_i x_k]=P[x_i]P[x_k]$

*from [Fukunaga, 1991]

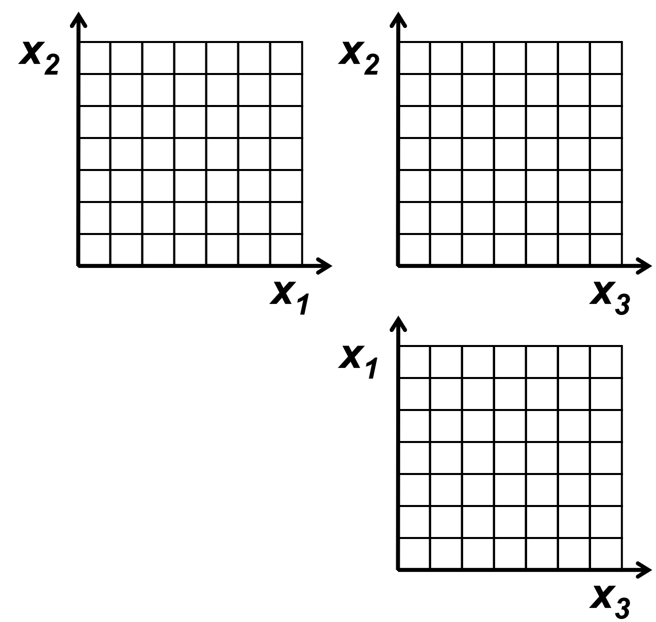
A numerical example

- **Given the following samples from a 3-dimensional distribution**
  - Compute the covariance matrix
  - Generate scatter plots for every pair of variables
    - Can you observe any relationships between the covariance and the scatter plots?

	Variables (or features)		
Examples	**x_1**	**x_2**	**x_3**
1	2	2	4
2	3	4	6
3	5	4	2
4	6	6	4

- **You may work your solution in the templates below**

Example	x_1	x_2	x_3	$x_1-\mu_1$	$x_2-\mu_2$	$x_3-\mu_3$	$(x_1-\mu_1)^2$	$(x_2-\mu_2)^2$	$(x_3-\mu_3)^2$	$(x_1-\mu_1)(x_2-\mu_2)$	$(x_1-\mu_1)(x_3-\mu_3)$	$(x_2-\mu_2)(x_3-\mu_3)$
1												
2												
3												
4												
Average												



The Normal or Gaussian distribution

- **The multivariate Normal or Gaussian distribution N(μ,Σ) is defined as**

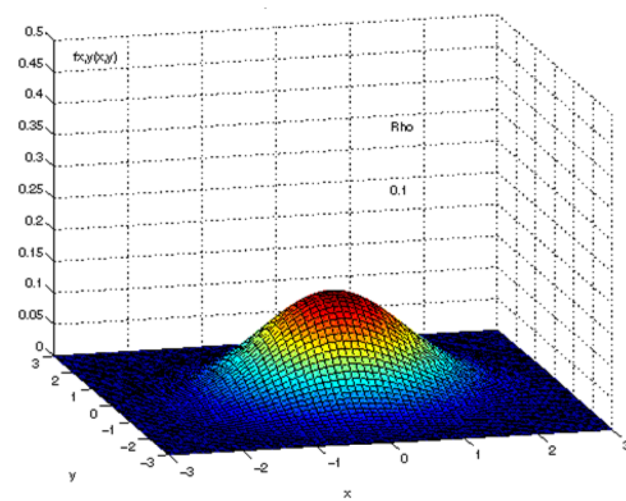
$$f_X(x) = \frac{1}{(2\pi)^{n/2}\left|\Sigma\right|^{1/2}} \exp\left[-\frac{1}{2}(X-\mu)^T \Sigma^{-1}(X-\mu)\right]$$

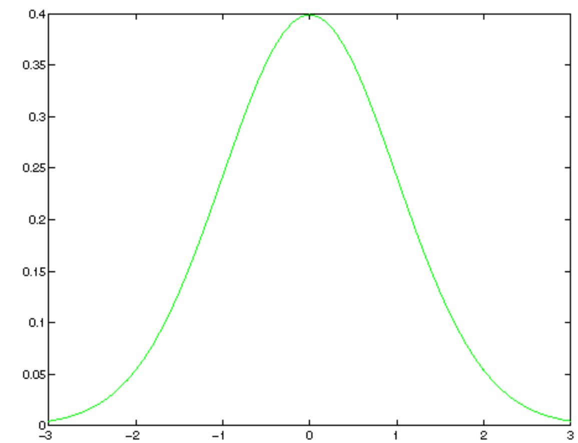
- **For a single dimension, this expression is reduced to**

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2\right]$$

- **Gaussian distributions are very popular since**
  - The parameters (μ,Σ) are **sufficient** to uniquely characterize the normal distribution
  - If the $\mathbf{x_i}$'s are mutually **uncorrelated** ($\mathbf{c_{ik}}$=0), then they are also **independent**
    - The covariance matrix becomes a diagonal matrix, with the individual variances in the main diagonal
  - **Central Limit Theorem**
  - The **marginal and conditional densities** are also Gaussian
  - Any **linear transformation** of any N jointly Gaussian rv's results in N rv's that are also Gaussian
    - For $X=[X_1\ X_2\ \ldots\ X_N]^T$ jointly Gaussian, and A an N×N invertible matrix, then Y=AX is also jointly Gaussian

$$f_Y(y) = \frac{f_X(A^{-1}y)}{\left|A\right|}$$





Central Limit Theorem

- **The central limit theorem states that given a distribution with a mean μ and variance σ^2, the sampling distribution of the mean approaches a normal distribution with a mean (μ) and a variance σ^2/N as N, the sample size, increases.**
  - **No matter** what the shape of the original distribution is, the sampling distribution of the mean approaches a normal distribution
  - Keep in mind that N is the sample size for each mean and not the number of samples

- **A uniform distribution is used to illustrate the idea behind the Central Limit Theorem**
  - Five hundred experiments were performed using am uniform distribution
    - For N=1, one sample was drawn from the distribution and its mean was recorded (for each of the 500 experiments)
      - Obviously, the histogram shown a uniform density
    - For N=4, 4 samples were drawn from the distribution and the mean of these 4 samples was recorded (for each of the 500 experiments)
      - The histogram starts to show a Gaussian shape
    - And so on for N=7 and N=10
    - As N grows, the shape of the histograms resembles a Normal distribution more closely