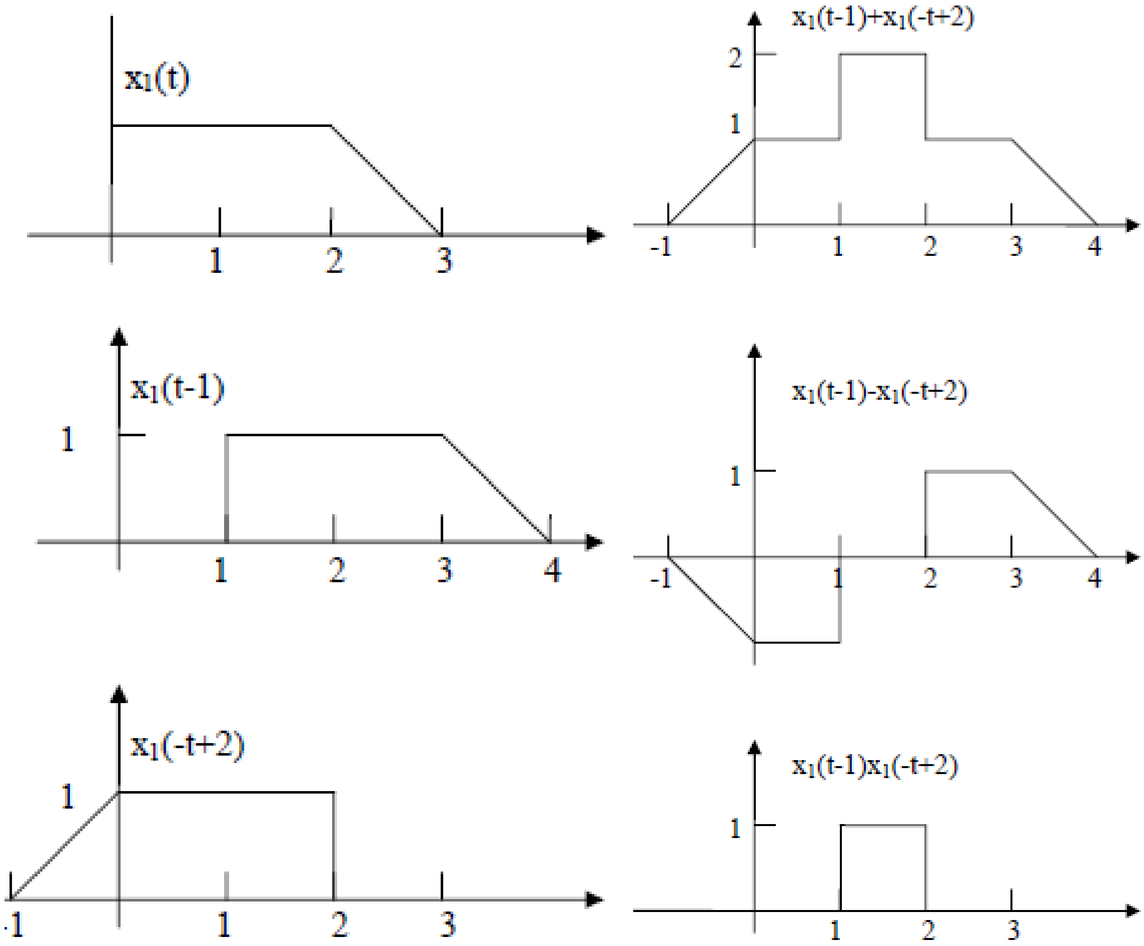
MKM 501 E – HW1

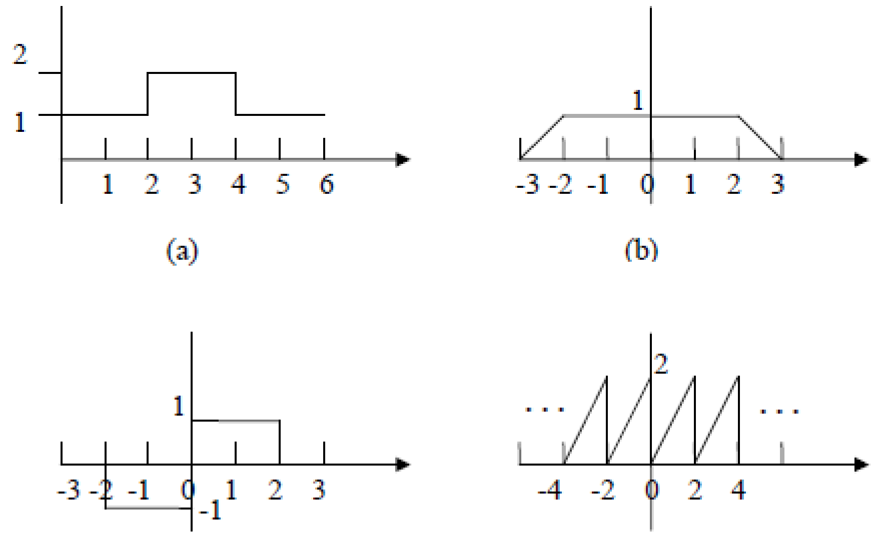
1)

Consider the signal $x_1(t)$ shown in the following figure. Plot $x_1(t-1)$, $x_1(-t+2)$, $x_1(t-1)+x_1(-t+2)$, $x_1(t-1)-x_1(-t+2)$, and $x_1(t-1)x_1(-t+2)$.



2)

Express the signals in the following figures in terms of step and ramp functions.



(a) (b)

(c) (d)

Solution:

(a) q(t)+q(t-2)-q(t-4)

(b) [r(t+3)-r(t+2)]-[r(t-2)-r(t-3)]

(c) –q(t+2)+2q(t)-q(t-2) or q(t)-q(t-2)-[q(-t)-q(-t-2)]

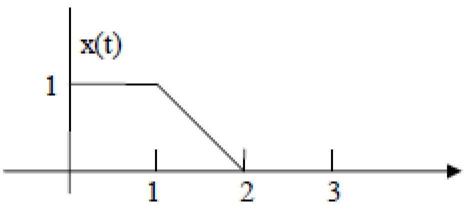
(d) The triangle in [0,2] is x(t)=r(t)-r(t-2)-2q(t-2);

$$\sum_{k=-\infty}^{\infty} x(t-2k) = \sum_{k=-\infty}^{\infty} r(t-2k) - r(t-2k-2) - 2q(t-2k-2)$$

3)

Consider a signal x(t). Define

$$x_e(t) = \frac{x(t)+x(-t)}{2} \text{ and } x_o(t) = \frac{x(t)-x(-t)}{2}$$



Show that $x_e(t)$ is even and $x_o(t)$ is odd. Note: a signal x(t) is even if x(t)=x(-t) and is odd if x(t)=-x(-t).

Solution:

$$x_e(-t) = \frac{x(-t)+x(-(-t))}{2} = \frac{x(-t)+x(t)}{2} = x_e(t)$$, so $x_e(t)$ is even;

$$x_0(-t) = \frac{x(-t)-x(-(-t))}{2} = -\frac{x(t)-x(-t)}{2} = -x_0(t)$$, so $x_o(t)$ is odd.

4)

Consider the CT signal $x(t) = 2 + \sin 1.4t - 4\cos 2.1t$

Is it periodic? If yes, find its fundamental period. (a) Express it using exclusively sine functions. What are their frequencies, corresponding magnitudes, and phases? (b) Express it using exclusively cosine functions. What are their frequencies, corresponding magnitudes, and phases?

Solution:
Yes. Its fundamental frequency is 0.7 rad/s, and its fundamental period is $2\pi/0.7$ =8.976.
Using 2= $2\sin(0\cdot t+\pi/2) = 2\cos(0\cdot t)$

$\pm\cos(\omega t) = \sin(\omega t \pm \pi/2)$

$\pm\sin(\omega t) = \cos(\omega t \mp \pi/2)$

we have

(a) $x(t) = 2\sin(0\cdot t+\pi/2) + \sin(1.4t) + 4\sin(2.1t - \pi/2)$

Frequency	0	1.4	2.1
Magnitude	2	1	4
Phase	$\pi/2$	0	- $\pi/2$

(b) $x(t) = 2\cos(0\cdot t) + \cos(1.4t - \pi/2) + 4\cos(2.1t + \pi)$

Frequency	0	1.4	2.1
Magnitude	2	1	4
Phase	0	- $\pi/2$	π

5)

Is the signal $x(t) = 2 + \sin 2 - 3\cos \pi t$ periodic? Can it be expressed using complex exponentials?

Solution:
The function is not periodic because 2 and π have no common divisor. It can be expressed using complex exponentials as
$x(t) = 2 + 0.5e^{-j\pi/2}e^{j2t} + 0.5e^{j\pi/2}e^{-j2t} + 1.5e^{j\pi}e^{j\pi t} + 1.5e^{j\pi}e^{-j\pi t}$

6)

Consider a CT system whose input and output are related by

$$y(t) = \begin{cases} u^2(t)/u(t-1) & \text{if } u(t-1) \neq 0 \\ 0 & \text{if } u(t-1) = 0 \end{cases}$$

Show that the system satisfies the homogeneity property but not the additivity property. Thus the homogeneity property does not imply the additivity property.

Solution:

If $u_1(t) = \alpha u(t)$

$$y_1(t) = \frac{\alpha^2 u^2(t)}{\alpha u(t-1)} = \alpha \frac{u^2(t)}{u(t-1)} = \alpha y(t)$$

Thus the homogeneity property holds.

If $u(t) = u_1(t) + u_2(t)$, then

$$\frac{[u_1(t)+u_2(t)]^2}{u_1(t-1)+u_2(t-1)} \neq \frac{u_1^2(t)}{u_1(t-1)} + \frac{u_2^2(t)}{u_2(t-1)}$$

Thus the additivity property does not hold.

So the homogeneity property does not imply the additivity property.

7)

Show that if $\{u_1 + u_2\} \to \{y_1 + y_2\}$, then

$$\left\{\frac{a}{b}u_1\right\} \to \left\{\frac{a}{b}y_1\right\}$$

for any integers a and b. Thus the additivity property implies $\{\alpha u_1\} \to \{\alpha y_1\}$ for any rational number α. In other words, the additivity property *almost* implies the homogeneity property.

We use $y = L[u]$ to denote $u \to y$. Let a be a positive integer, then

$$L[au] = L[u+(a-1)u] = L[u] + L[(a-1)u]$$

$$= L[u] + L[u+(a-2)u] = L[u] + L[u] + L[(a-2)u]$$

$$= \cdots = L[u] + L[u] + \cdots L[u] = aL[u]$$

$$L[u+0] = L[u] + L[0] = L[u] \Rightarrow L[0] = 0$$

$$L[au+(-au)] = L[au] + L[-au] = L[0] = 0$$

$$L[-au] = -L[au] = -aL[u]$$

Thus additivity implies $L[au] = aL[u]$ for any positive or negative integer.

Let $\alpha = a/b$ for any integers a and b,

Define $u = bv$ then

$$L[u] = L[bv] = bL[v]$$

$$\Rightarrow L[v] = \frac{1}{b}L[u]$$

or $L\left[\frac{u}{b}\right] = \frac{1}{b}L[u]$

Now

$$L[\alpha u] = L\left[\frac{a}{b}u\right] = aL\left[\frac{u}{b}\right] = \frac{a}{b}L[u]$$

Thus additivity implies

$L[\alpha u] = \alpha L[u]$ for any rational number α.

8)

Discuss whether or not each of the following equations is memoryless, linear, time-invariant, and casual:

a. $y(t) = -2 + 3u(t)$

b. $y(t) = \sqrt{u(t)}$

c. $y(t) = u(t)u(t-1)$

d. $y(t) = tu(t)$

e. $y(t) = \int_{t_0}^{t} u(\tau)\,d\tau + y(t_0)$

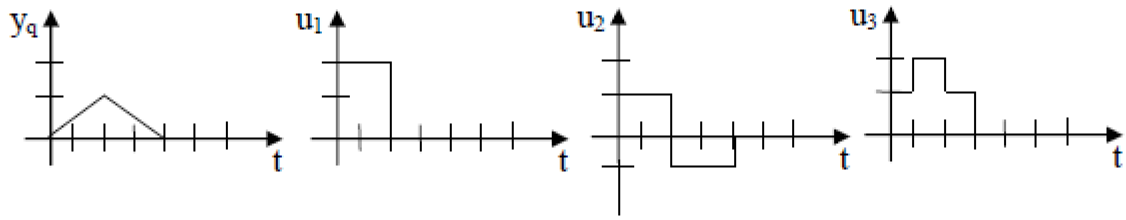
f. $y(t) = \int_{t_0}^{t} \tau u(\tau)\,d\tau + y(t_0)$

Solution:
TI: time-invariant

a. memoryless, nonlinear, TI, causal
b. memoryless, nonlinear, TI, causal
c. with memory, nonlinear, TI, causal
d. memoryless, linear, time-varying, causal
e. with memory, linear, TI, causal
f. with memory, linear, time-varying, causal

9)

Consider a CT LTI system. Suppose its step response (the zero-state response excited by a step input or $u(t) = q(t)$ is as shown in Figure (a). Find the outputs excited by the inputs shown in Figure (b) through (d).

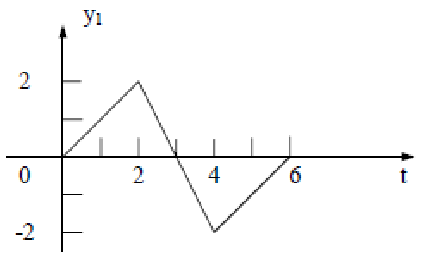


(a) (b) (c) (d)

Solution:

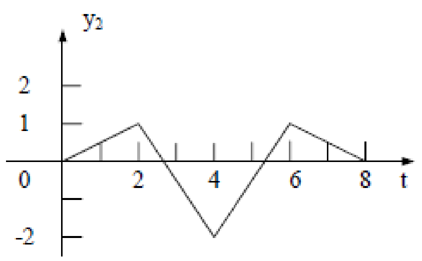
$u_1(t) = 2q(t) - 2q(t-2)$

$\Rightarrow y_1(t) = 2y_q(t) - 2y_q(t-2)$



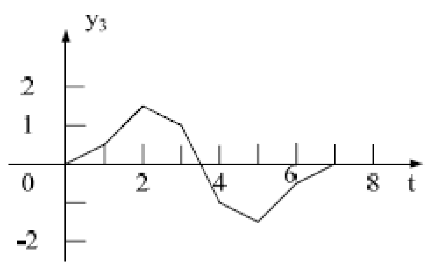
$u_2(t) = 0.5u_1(t) - 0.5u_1(t-2)$

$\Rightarrow y_2(t) = 0.5y_1(t) - 0.5y_1(t-2)$



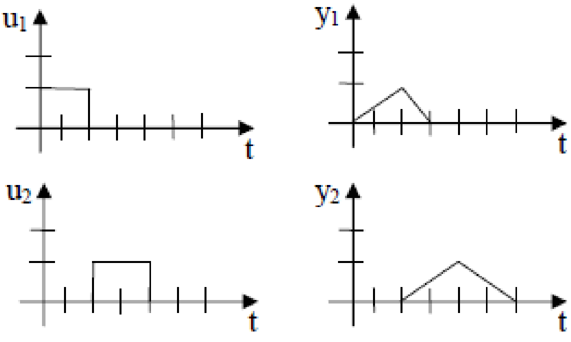
$u_3(t) = 0.5u_1(t) + 0.5u_1(t-1)$

$\Rightarrow y_3(t) = 0.5y_1(t) + 0.5y_1(t-1)$

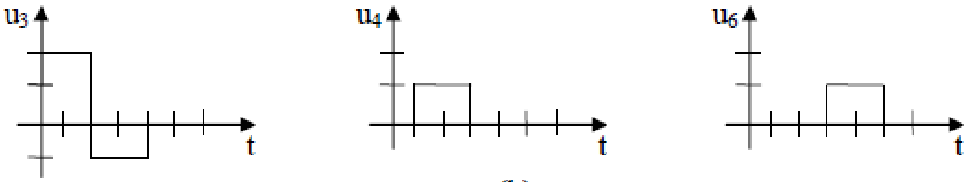


10)

Consider a CT linear system. Its zero-state responses excited by u_1 and u_2 are shown in the following figure (a). Is the system time-varying or time-invariant? Can you find the zero-state responses excited by the inputs u_3, u_4, and u_5 shown in figure (b)?

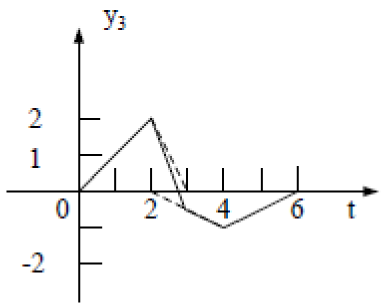


(a)



(b)

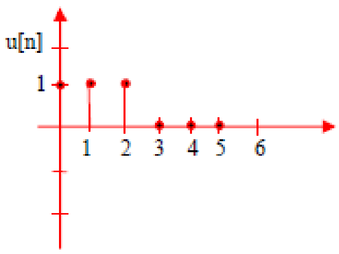
Solution:
It's time varying, thus the time shifting property cannot be used.
$u_3 = 2u_1 - u_2 \quad \Rightarrow y_3 = 2y_1 - y_2$

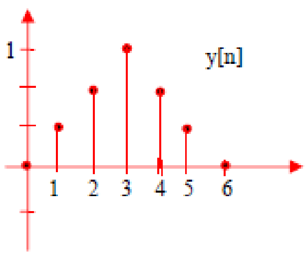


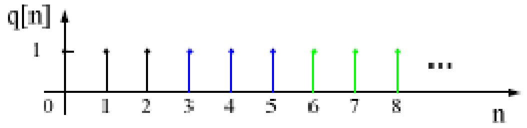
Here we have used only homogeneity and additivity properties. Although u_4 is the shifting of u_1 or u_2 by one second, y_4 is not necessarily the shifting of y_1 or y_2 by one second. Thus it is not possible to compute y_4 from the given pairs. The same conclusion applies to y_5.

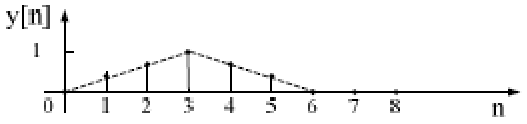
11)

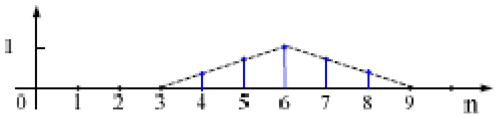
Consider a DT LTI system with the input-output pair shown below. What is its step response (the output excited by the step sequence q[n])? What is its impulse response (the output excited by the impulse sequence δ[n])?

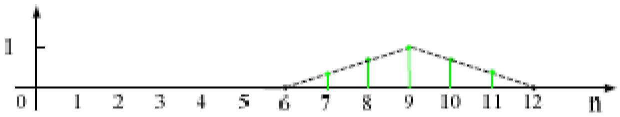


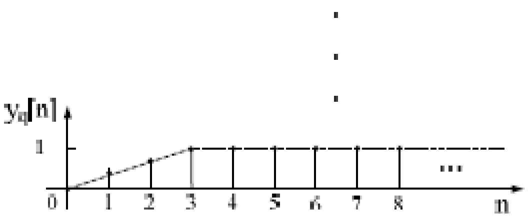




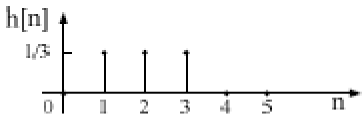








Because $\delta[n] = q[n] - q[n-1]$, the impulse response $h[n] = y_q[n] - y_q[n-1]$



12)

Consider the sequence h=[1 2 3 -2] which is located at n=0:3 and u[1 0 -2 4 4 5] which is located at n=0:5. Compute their convolution for n=-2:12.

Consider the polynomials

$$h(s) = s^3 + 2s^2 + 3s - 2$$

And

$$h(s) = s^5 - 2s^3 + 4s^2 + 4s + 5$$

Compute h(s)u(s). Verify that its coefficients are the same as those in Problem 2. Thus the multiplication of two polynomials can be computed as the convolution of their coefficients.

Solution:

$h(s)u(s) = s^8 + 2s^7 + s^6 - 2s^5 + 6s^4 + 29s^3 + 14s^2 + 7s - 10$

Its coefficients are the same as the convolution in Prob. 3.2 for n=0:8, thus the product of two polynomials can be computed using a convolution.

In MATLAB, typing

```
h=[1 2 3 -2];
u=[1 0 -2 4 4 5];
conv(h,u)
```

yields

```
1    2    1   -2    6   29   14    7  -10
```

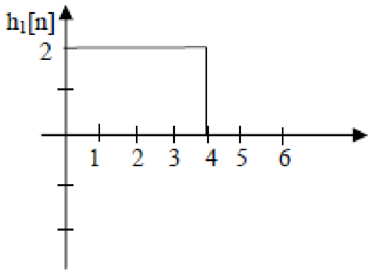
13)

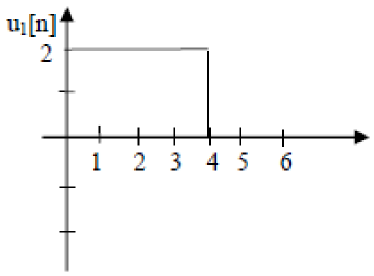
Explore the "*Joy of Convolution*" and "*Joy of Convolution (Discrete Time)*" sections at http://www.jhu.edu/~signals.

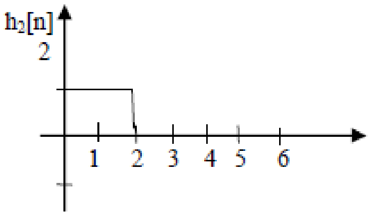
If the Java applets in the above webpages are not running properly, install the latest Java Runtime Environment (JRE) from http://java.sun.com/javase/downloads/index.jsp

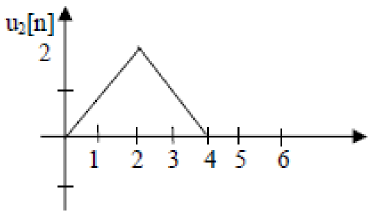
14)

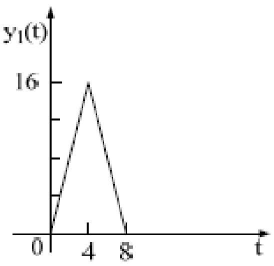
Compute the integral convolution of $h_i(t)$ and $u_i(t)$, shown below.







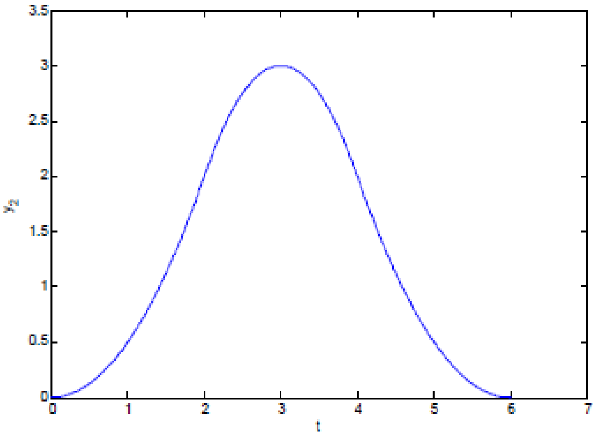




Part 2:

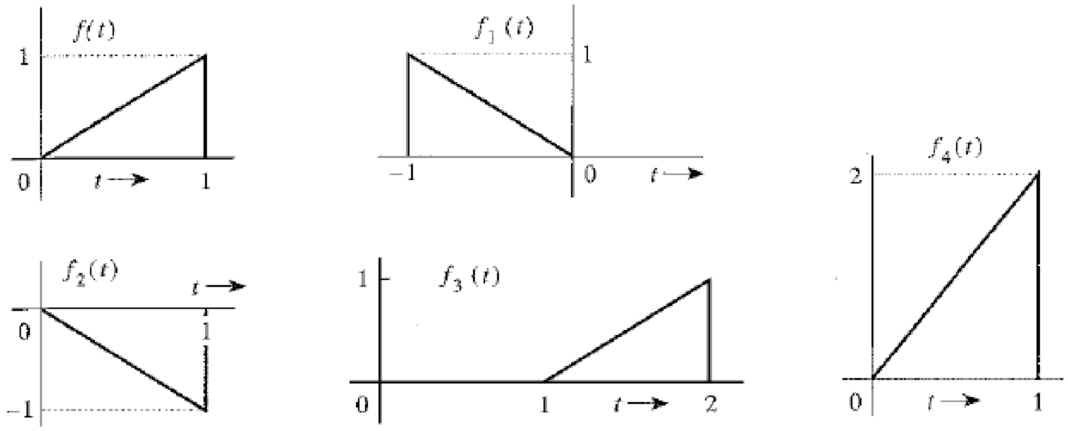
$$y_2(t) = \begin{cases} 0.5t^2, \; 0 \le t \le 2 \\ -(t-3)^2 + 3, 2 \le t \le 4 \\ 0.5(t-6)^2, 4 \le t \le 6 \\ 0, \; otherwise \end{cases}$$

$y_2(1) = 0.5, \; y_2(2) = 2, \; y_2(3) = 3$



15)

Find the energy of the signals sketched below. How does the energy change when transforming a signal by time-shifting or time-reversing it? How does the energy of a signal $f(t)$ changes when it is multiplied by a factor $a \in \mathbb{R}$.



2.1/ $E_f = \int_{-\infty}^{\infty} |f(t)|^2 dt$

In full generality, time shifting or time reversing can be represented as $f(\alpha t + \beta)$ for some scalars α, β. Likewise, this can be extended to multiplied by a scalar

$g(t) = a f(\alpha t + \beta)$ where $\alpha, \beta, a \in \mathbb{R}$ and $f(t) \in \mathbb{R} \ \forall t$

$$E_g = \int_{-\infty}^{\infty} |a f(\alpha t + \beta)|^2 dt = a^2 \int_{t=-\infty}^{\infty} f(\alpha t + \beta)^2 dt \qquad \begin{matrix} u = \alpha t + \beta \\ du = \alpha dt \end{matrix}$$

$$= \frac{a^2}{|\alpha|} \int_{u=-\infty}^{\infty} f(u)^2 du$$

Key of functions

a	b	
c	d	e

a) $f(t) = t\,(u(t) - u(t-1))$

$$E_f = \int_0^1 t^2 dt = \frac{1}{3} t^3 \Big|_0^1 = \frac{1}{3}$$

b) $f(t) = -t\,(u(t+1) - u(t))$

$$E_f = \int_{-1}^{0} t^2 dt = \frac{1}{3} t^3 \Big|_0^1 = \frac{1}{3}$$

c) $f(t) = -t\,(u(t) - u(t-1))$

$$E_f = \int_0^1 t^2 dt = \frac{1}{3} t^3 \Big|_0^1 = \frac{1}{3}$$

d) $f(t) = t - 1\,(u(t-1) - u(t-2))$

$$E_f = \int_1^2 (t-1)^2 dt$$

$$= \int_1^2 (t^2 - 2t + 1) dt$$

$$= \left(\frac{1}{3} t^3 - t^2 + t\right)\Big|_{t=1}^{2} = \left(\frac{8}{3} - 4 + 2\right) - \left(\frac{1}{3} - 1 + 1\right)$$

$$= \frac{1}{3}$$

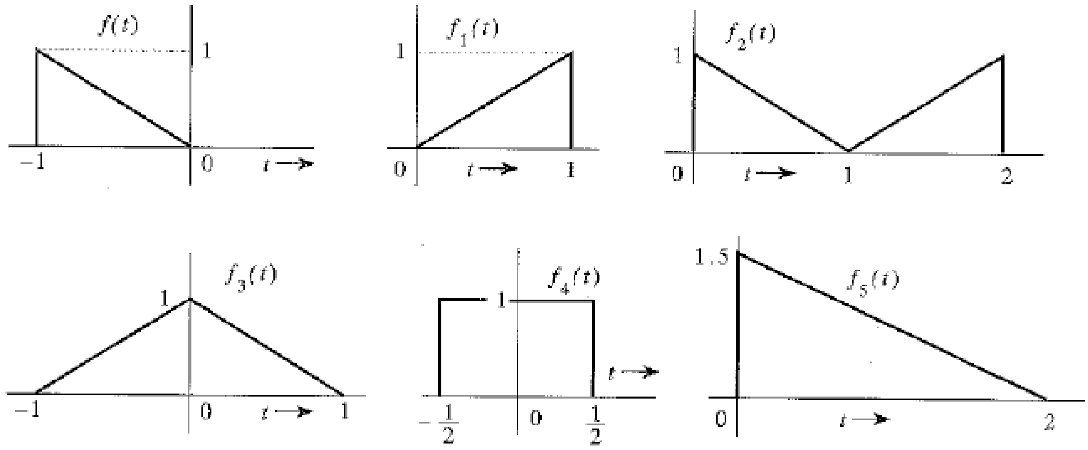
e) $f(t) = 2t\,(u(t) - u(t-1))$

$$E_f = \int_0^1 4t^2 dt = \frac{4}{3}$$

Note that this is consistent w/ the derived expression.

16)

Express the signals $f_i(t)$ $i = 1, \ldots, 5$ in the figure below using only $f(t)$, and its time-shifted, time-scaled and time-inverted versions. For example $f_2(t) = f(t-1) + f(1-t)$.



CEVABI:

2.2. $f_1(t) = f(-t)$

$f_2(t) = f(t-1) + f(1-t)$

$f_3(t) = f(t-1) + \underbrace{f(-t-1)}$

time reverse and shift left by 1

$f(-(t+1))$

$f_4(t) = f(t-1/2) + f(-(t+1/2))$

$f_5(t) = \frac{3}{2} f\left(\frac{t-2}{2}\right) = \frac{3}{2} f(t/2 - 1)$

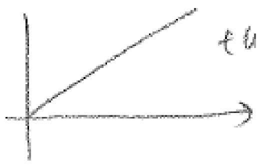
17)

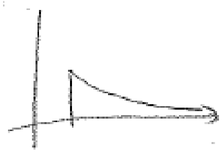
Sketch the following signals, and state if these are causal, anti-causal, or non-causal.

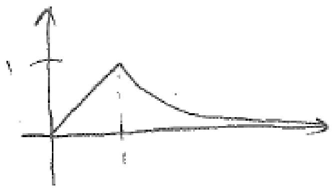
(a) $f(t) = \exp(-2t)u(t)$
(b) $f(t) = tu(t) + (\exp(1-t) - t)u(t-1)$
(c) $f(t) = \text{Re}\{\exp((1+2\pi j)t)\}u(1-t)$

2.3. a.) $f(t) = e^{-2t} u(t)$ causal $f(t)=0 \quad t<0$

b.) $f(t) = t\,u(t) + \left(e^{(1-t)} - t\right)u(t-1)$ causal $f(t)=0 \quad t<0$





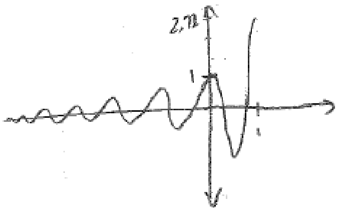


c, $\text{Re}\{e^{(1+2\pi j)t}\}\, u(1-t)$

$\text{Re}\{e^{t} e^{2\pi j t}\}\, u(1-t)$

$e^{t}\,\text{Re}\{e^{2\pi j t}\}\, u(1-t)$

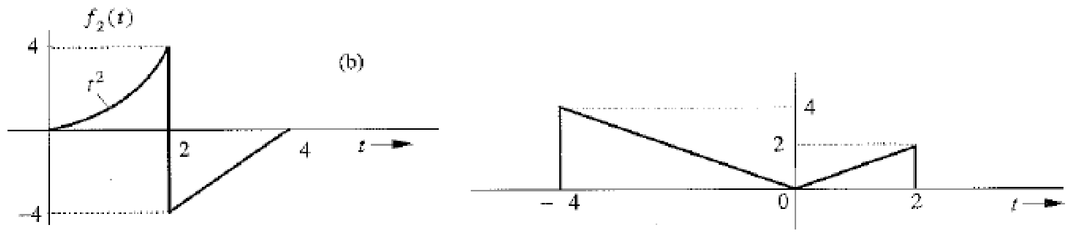
$= e^{t} \cos(2\pi t)\, u(1-t)$



non-causal $f(t) \neq 0 \quad t<0$

18)

(a) - Express the signals in the figure below by a single expression valid for all t.
(b) - Carefully sketch the signals $f_2(t-3)$, $3f_2(2t+2)$ and $f_2(2t-1)$.
(Note: the use of the step function $u(t)$ will come in handy for the first part)



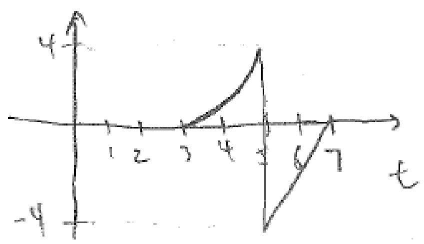
2.4 a. i) $f_2(t) = t^2\left(u(t)-u(t-2)\right) + 2(t-4)\left(u(t-2)-u(t-4)\right)$

ii) $f(t) = -t\left(u(t+4)-u(t)\right) + t\left(u(t)-u(t-2)\right)$

$= -tu(t+4) + tu(t) + tu(t) - tu(t-2)$

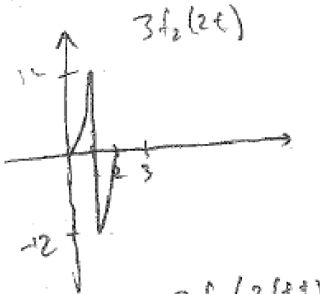
$= -t\left(u(t+4)+u(t-2)\right) + 2tu(t)$

b.) $f_2(t-3)$

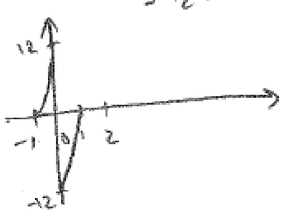


$3f_2(2t+2)$

$3f_2(2t)$

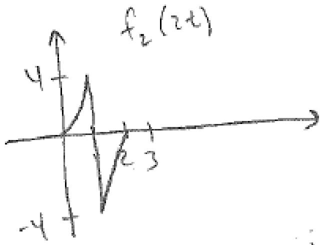


$3f_2(2(t+1)) = 3f_2(2t+2)$

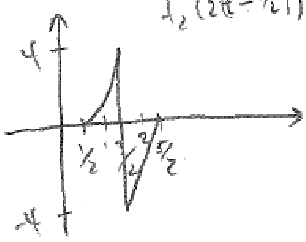


$f_2(2t-1)$

$f_2(2t)$



$f_2(2(t-\frac{1}{2})) = f_2(2t-1)$



19)

Simplify the following expressions:

(a) $\left(\frac{\sin t}{t^3+2}\right)\delta(t)$ (b) $\frac{5+jt}{2-jt}\delta(t-2)$

(c) $\int_{-\infty}^{\infty}(2t-5)\delta(t-2)dt$ (d) $\exp(3k)\cos(401\pi k)\delta(k+1)$

(e) $\int_{-\infty}^{x+1}(2t-5)\delta(t-x)dt$ (f) $\int_{-\infty}^{\infty}\exp(3x)\cos(401\pi x)\delta(x+1)dx$

2.5, a.) $\left(\frac{\sin(t)}{t^3+2}\right)\delta(t) = \frac{\sin(0)}{0+2}\delta(t)$

$= 0$

b.) $\left(\frac{5+jt}{2-jt}\right)\delta(t-2) = \left(\frac{5+2j}{2-2j}\right)\delta(t-2)$

c.) $\int_{-\infty}^{\infty}(2(2)-5)\delta(t-2)dt$

$= \int_{-\infty}^{\infty} -1\delta(t-2)dt$

$= -1$

d.) $e^{3t}\cos(401\pi t)\delta(t+1)$

$e^{-3}\cos(-401\pi)\delta(t+1)$

$= -e^{-3}\delta(t+1)$

e.) $\int_{-\infty}^{x+1}(2t-5)\delta(t-x)dt$

$= \int_{-\infty}^{x+1}(2x-5)\delta(t-x)dt$ because $x+1$ will integrate over the impulse $\delta(t-x)$

$= 2x-5$

f.) $\int_{-\infty}^{\infty} e^{3x}\cos(401\pi x)\delta(x+1)dx$

$= e^{-3}\cos(401\pi(-1))$

$= -e^{-3}$

20)

Let $f(t)$ be a signal with energy E_f. Find the energy of the signals $-f(t)$, $f(-t)$ and $f(t-T)$ where $T \in \mathbb{R}$. What is the energy of the signal $g(t) = cf(at-b)$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$? *(Hint: write the energy integral for the various signals, and use a change of variables to express it in terms of E_f)*

1) $E_f = \int_{-\infty}^{\infty} |f(t)|^2 dt$ $\qquad g(t) = cf(at-b)$

$E_g = \int_{-\infty}^{\infty} |c\, f(at-b)|^2 dt$

$= c^2 \int_{-\infty}^{\infty} |f(at-b)|^2 dt$ $\qquad u = at-b$, $du = a\,dt$

$= \frac{c^2}{|a|} \int_{-\infty}^{\infty} |f(u)|^2 du$

$= \frac{c^2}{|a|} E_f$

$-f(t)$ is $g(t)$ w/ $a=1$, $b=0$, $c=-1$ $\qquad E_g = E_f$

$f(-t)$ is $g(t)$ w/ $a=-1$, $b=0$, $c=1$ $\qquad E_g = E_f$

$f(t-T)$ is $g(t)$ w/ $a=1$, $b=T$, $c=1$ $\qquad E_g = E_f$

21)

Consider the continuous-time systems below, described in terms of an input/output relationship, where the input is $f(t)$ and the output is $y(t)$. Determine if the systems are linear or non-linear. **(Justify your answer fully, simply stating the answer will give you no credit)**

(a) $\frac{dy}{dt}(t) + 3ty(t) = t^2 f(t)$ $\qquad$ (b) $3y(t) + 2 = f(t)$

(c) $y(t) = \int_{-\infty}^{t} f(\tau)d\tau$ $\qquad$ (d) $\frac{dy}{dt}(t) + 2y(t) = f(t)\frac{df}{dt}(t)$

3] To show that a system H is linear we must show

$$k_1 x_1 + k_2 x_2 \longrightarrow \boxed{H} \longrightarrow k_1 y_1 + k_2 y_2$$

a) $\frac{dy(t)}{dt} + 3t\,y(t) = t^2 f(t)$

$\frac{dy_1(t)}{dt} + 3t\,y_1(t) = t^2 f_1(t)$ Multiply both sides by k_1 $\Rightarrow$ $k_1 \frac{dy_1(t)}{dt} + 3k_1 t\, y_1(t) = k_1 t^2 f_1(t)$

$\frac{dy_2(t)}{dt} + 3t\,y_2(t) = t^2 f_2(t)$ " " k_2 $\Rightarrow$ $k_2 \frac{dy_2(t)}{dt} + 3k_2 t\, y_2(t) = k_2 t^2 f_2(t)$

Add them together

$$\left(k_1 \frac{dy_1(t)}{dt} + k_2 \frac{dy_2(t)}{dt}\right) + 3t\underbrace{\left(k_1 y_1(t) + k_2 y_2(t)\right)}_{y(t)} = t^2 \underbrace{\left(k_1 f_1(t) + k_2 f_2(t)\right)}_{f(t)}$$

Which shows that this system is linear

---

b) $3y(t) + 2 = f(t)$

$3k_1 y_1(t) + 2k_1 = f_1(t)$

$+\ 3k_2 y_2(t) + 2k_2 = f_2(t)$

$$3\underbrace{\left(k_1 y_1(t) + k_2 y_2(t)\right)}_{y(t)} + 2\underbrace{(k_1 + k_2)}_{\neq 1} = \underbrace{\left(k_1 f_1(t) + k_2 f_2(t)\right)}_{f(t)}$$

Because of the $(k_1 + k_2)$ term, this system is nonlinear.

$f(t) = 0 \Rightarrow y(t) = \frac{-2}{3}$

c) $y(t) = \int_{-\infty}^{t} f(\tau)\,d\tau$

$k_1 y_1(t) = \int_{-\infty}^{t} f_1(\tau)\,d\tau$

$k_2 y_2(t) = \int_{-\infty}^{t} f_2(\tau)\,d\tau$ $\oplus \Rightarrow$ $\underbrace{\left(k_1 y_1(t) + k_2 y_2(t)\right)}_{y(t)} = \int_{-\infty}^{t} \underbrace{\left(k_1 f_1(\tau) + k_2 f_2(\tau)\right)}_{f(t)} d\tau$

System is linear.

d) $\frac{dy(t)}{dt} + 2y(t) = f(t)\frac{df(t)}{dt}$

If this system is denoted as H then

$$f(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

$$k_1 f_1(t) \longrightarrow \boxed{H} \longrightarrow k_1 y_1(t) \quad \text{If system is linear.}$$

Set $y(t) = k_1 y_1(t)$:

$$\frac{dy(t)}{dt} + 2y(t) = \left(\frac{d}{dt}+2\right)y(t) = \left(\frac{d}{dt}+2\right)k_1 y_1(t) \overset{(a)}{=} k_1 f_1(t)\frac{df_1(t)}{dt}$$

Set $f(t) = k_1 f_1(t)$:

$$f(t)\frac{df(t)}{dt} \overset{(b)}{=} k_1^2 f_1(t)\frac{df_1(t)}{dt}$$

Clearly, the equalities a and b are not equal so this system is nonlinear.

Ex. Let $f(t) = t$ and $g(t)$ be the corresponding output:

$$t \longrightarrow \boxed{H} \longrightarrow g(t).$$

Then to be linear, $H\{-t\} = -g(t)$ but

$$\frac{dg(t)}{dt} + 2g(t) = (t)\frac{d}{dt}(t) = t \quad w/\ f(t) = t$$

$$\frac{d(-g(t))}{dt} + 2(-g(t)) \neq (-t)\frac{d}{dt}(-t) = t \quad w/\ f(t) = -t$$

Using input $f(t) = t$ or $f(t) = -t$ produces the same non-zero output $g(t)$ so this system is nonlinear.

22)

Consider the discrete-time systems below, described in terms of an input/output relationship, where the input is $x[n]$ and the output is $y[n]$. Determine if the systems are linear or non-linear. **(Justify your answer fully, simply stating the answer will give you no credit)**

(a) $y[n] = \cos(\pi n) + x[n-4]$ (b) $y[n] = y[n-1] + x[n]$
(c) $y[n] = \sin(\pi n) + x[n-4]$ (d) $y[n] = \sum_{i=-\infty}^{n} x[n]$

4) Same thing as problem 3. Need to show that if $x[n] = k_1 x_1[n] + k_2 x_2[n]$ then $y[n] = k_1 y_1[n] + k_2 y_2[n]$.

a) $y[n] = \cos(\pi n) + x[n-4]$

$k_1 y_1[n] \neq \cos(\pi n) + x_1[n-4]$ $\quad k_2 y_2[n] \neq k_2 \cos(\pi n) + k_2 x_2[n-4]$

$$\Big(\underbrace{k_1 y_1[n] + k_2 y_2[n]}_{y[n]}\Big) = \underbrace{(k_1 + k_2)}_{\neq 1}\cos(\pi n) + \Big(\underbrace{k_1 x_1[n-4] + k_2 x_2[n-4]}_{x[n]}\Big)$$

Because of the $(k_1 + k_2)$ out front of the cosine term, this system is <u>nonlinear.</u> If $x(n) = 0 \ \forall n$, $y[n] = \cos(\pi n)$.

b) $y[n] = y(n-1] + x[n]$

$k_1 y_1[n] \neq k_1 y_1[n-1] + k_1 x_1(n]$ $\quad k_2 y_2[n] \neq k_2 y_2[n-1] + k_2 x_2[n]$

$$\Big(\underbrace{k_1 y_1[n] + k_2 y_2[n]}_{y[n]}\Big) = \Big(\underbrace{k_1 y_1[n-1] + k_2 y_2[n-1]}_{y[n-1]}\Big) + \Big(\underbrace{k_1 x_1[n] + k_2 x_2[n]}_{x[n]}\Big)$$

System is <u>linear.</u>

c) $y[n] = \sin(\pi n) + x[n-4] = x[n-4]$ since $\sin(\pi n) = 0 \ \forall n$

$k_1 y_1[n] = k_1 x_1[n-4]$ $\quad k_2 y_2[n] = k_2 x_2[n-4]$

$$\underbrace{(k_1 y_1[n] + k_2 y_2[n])}_{y[n]} = \underbrace{(k_1 x_1[n] + k_2 x_2[n])}_{x[n]}$$

System is linear

d) $y[n] = \sum_{i=-\infty}^{n} x[n]$

$k_1 y_1[n] = \sum_{i=-\infty}^{n} k_1 x_1[n]$ $\quad k_2 y_2[n] = \sum_{i=-\infty}^{n} k_2 x_2[n]$

$$\underbrace{(k_1 y_1[n] + k_2 y_2[n])}_{y[n]} = \left(\sum_{i=-\infty}^{n} k_1 x_1[n] + \sum_{i=-\infty}^{n} k_2 x_2[n]\right)$$

$$= \sum_{i=-\infty}^{n} \underbrace{(k_1 x_1[n] + k_2 x_2[n])}_{x[n]}$$

System is linear

23)

Consider the systems described below, where $f(t)$ or $f[n]$ correspond to the input (respectively for continuous and discrete-time systems) and $y(t)$ or $y[n]$ correspond to the outputs (again for the CT and DT cases respectively). Determine which systems are time-invariant, and which systems are not. **(Justify your answer fully, simply stating the answer will give you no credit)**

(a) $y(t) = f(-t)$	(b) $y(t) = \int_{-3}^{3} f(\tau)d\tau$
(c) $y(t) = f(3t)$	(d) $y[n] = 2nx[n-2]$
(e) $y[n] = (x[n] - x[n-1])^2$	(f) $y[n] = y[n-1] + x[n]$

If a system is time-invariant it means that if the system's response to input $f(t)$ is $y(t) = H\{f(t)\}$ then the system's response to input $g(t) = f(t-T)$ for any $T \in \mathbb{R}$ is $H\{g(t)\} = y(t-T)$

a) Suppose the system is time-invariant, and let $f(t)$ & $y(t)$ be a valid input/output pair. Then

$$y(t) = f(-t)$$

Now what is the systems response to $g(t) = f(t-T)$?

$$H\{g(t)\} = g(-t) = f(-t-T)$$

If the system is time-invariant then $H\{g(t)\} = y(t-T)$

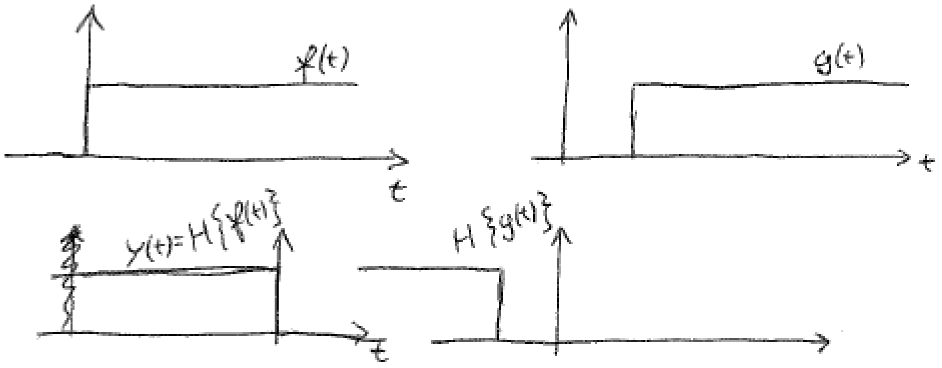
$$= f(-(t-T)) = f(-t+T)$$

since in general $f(-t+T) \neq f(-t-T)$ the system is time-varying.

Let's see in particular how the system behaves when the inputs are

$f(t) = u(t)$ and $g(t) = u(t-1)$

$H\{f(t)\} = u(-t) = y(t)$ $\qquad$ $H\{g(t)\} = u(-t-1)$



so clearly $H\{g(t)\}$ is not a shifted-to-the-right by 1 version of $y(t)$.

and so the system is TV.

b) Note that the output of the system only depends on the behavior of the input for a fixed time-interval. This gives you a clue that it might be TV.

Let $f(t) = \delta(t)$, then $H\{f(t)\} = \int_{-3}^{3} \delta(t)\,dt = 1 = y(t)$

Now let $g(t) = f(t-6)$, then $H\{g(t)\} = \int_{-3}^{3} \delta(t-6)\,dt = 0$

therefore $H\{g(t)\} \neq y(t-6)$, and so the system is TV.

c) Let $y(t) = H\{f(t)\} = f(3t)$ and let's see what is the system's output to $g(t) = f(t-T)$.

$$H\{g(t)\} = g(3t) = f(3t-T)$$

Now $y(t-T) = f(3(t-T)) = f(3t-3T)$

so in general $y(t-T) \neq H\{g(t)\}$, and so the system is TV.

For example, if $f(t) = u(t)$ then $y(t) = u(t)$ (check this)
and $H\{f(t-1)\} = u(t-1/3) \neq y(t-1) = u(t-1)$

d) The system is time-varying. As you have seen in problem 4.1, the response of a TI to a constant input must be constant, therefore if $x[n]=1$ and the system is TI $H\{x[n]\}$ should be a constant, but in this case $y[n]=2n$.

Of course you can solve the problem as before as well.

e) The system's response to an arbitrary input $x[n]$ is
$H\{x[n]\} = (x[n] - x[n-1])^2 = y[n]$

Let's see what is the output when the input is $z[n] = x[n-N]$.

$$H\{z[n]\} = (z[n] - z[n-1])^2 = (x[n-N] - \cancel{}\, x[(n-1)-N])^2$$
$$= (x[n-N] - x[n-N-1])^2$$

Now note that $$y[n-N] = (x[n-N] - x[(n-N)-1])^2$$
$$= (x[n-N] - x[n-N-1])^2$$

and so $y[n-N] = H\{z[n]\}$, since $x[\cdot]$ and N were arbitrary the system is TI.

f) In this case we don't have an explicit input-output relation, and this often causes a lot of confusion.

Let's do it carefully. Suppose $x[n]$ and $y[n]$ are a valid input output pair, meaning that $y[n] = y[n-1] + x[n]$ for all $n \in \mathbb{N}$

Now let $x'[n] = x[n-N]$, a shifted version of $x[\cdot]$. If the system is TI then $H\{x'[n]\} = \cancel{}\, y'[n] = y[n-N]$

Let's check this is indeed the case, that is $x'[n]$ & $y'[n]$ are valid input-output pairs of signals.

If so they must satisfy the input-output relation and therefore

$$y'[n] = y'[n-1] + x'[n] \qquad \forall n \in \mathbb{N}$$
$$\Leftrightarrow \quad y[n-N] = y[(n-1)-N] + x[n-N] \qquad \forall n \in \mathbb{N}$$
$$\Leftrightarrow \quad y[\underbrace{n-N}_{=m}] = y[(n-N)-1] + x[(n-N)] \qquad \forall n \in \mathbb{N}$$

But since we know that $y[n] = y[n-1] + x[n] \ \forall n \in \mathbb{N} \ \Leftrightarrow$
$y[m] = y[m-1] + x[m] \ \forall m \in \mathbb{N}$

then we see that indeed $H\{x'[n]\} = y[n-N]$

showing that the system is TI.

24)

Consider an LTI system that has impulse response $h(t) = e^{-t}u(t)$. Find the system response $y(t)$ if the input is

(a) $u(t)$ (b) $e^{-t}u(t)$ (c) $\sin(3t)u(t)$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$x(\tau) = x(\tau)$

$g(\tau) = h(-\tau)$ "flip"

$g(\tau - t) = h(t-\tau)$ "shift"

a) $x(\tau) = u(\tau)$

$g(\tau) = e^{\tau} u(-\tau)$

$g(\tau - t) = e^{\tau - t} u(t-\tau)$

$$\int_{\tau=-\infty}^{\infty} u(\tau)u(t-\tau)e^{\tau-t}\,d\tau$$

$u(\tau)$ makes the integral 0 for $\tau < 0$

$u(t-\tau)$ makes the integral 0 for $t > \tau$

so we adjust the bounds

$$\int_{\tau=0}^{t} u(\tau)u(t-\tau)e^{\tau-t}\,d\tau = u(\tau)u(t-\tau)e^{\tau-t}\Big|_{\tau=0}^{t}$$

$$= u(0)u(t)(1-e^{-t}) = u(t)(1-e^{-t})$$

b) $x(\tau) = e^{-\tau}u(\tau) \qquad h(\tau) = e^{-\tau}u(\tau)$

$h(t-\tau) = e^{\tau-t}u(t-\tau)$

$$\int_{-\infty}^{\infty} u(\tau)u(t-\tau)e^{-\tau}e^{\tau-t}\,d\tau = e^{-t}\int_0^t u(\tau)u(t-\tau)\,d\tau$$

where the bounds were set by $u(\tau)$ = and $u(t-\tau)$

$$= e^{-t}\left[u(\tau)u(t-\tau)\right]\Big|_{\tau=0}^{t} = u(t)\cdot te^{-t}$$

c) $x(\tau) = \sin(3\tau)u(\tau) \qquad h(\tau) = e^{-\tau}u(\tau)$

$h(t-\tau) = e^{\tau-t}u(t-\tau)$

$$\int_{-\infty}^{\infty} u(\tau)u(t-\tau)e^{\tau-t}\sin(3\tau)\,d\tau = \int_{\tau=0}^{t} u(\tau)u(t-\tau)e^{\tau-t}\sin(3\tau)\,d\tau$$

$u = \sin(3\tau) \qquad v = e^{\tau-t}$

$du = 3\cos(3\tau)d\tau \qquad dv = e^{\tau-t}d\tau$

$$= u(t)\left[\left(\sin(3\tau)e^{\tau-t}\right)\Big|_{\tau=0}^{t} - \int_{\tau=0}^{t} 3\cos(3\tau)e^{\tau-t}\,d\tau\right]$$

$u = 3\cos(3\tau) \qquad v = e^{\tau-t}$

$du = -9\sin(3\tau) \qquad dv = e^{\tau-t}d\tau$

$$u(t)\int_{\tau=0}^{t} \sin(3\tau)e^{t-\tau}\,d\tau = u(t)\sin(3t) - u(t)\left(3\cos(3\tau)e^{\tau-t}\right)\Big|_{\tau=0}^{t}$$

$$-u(t)\int_{\tau=0}^{t} 9\sin(3\tau)e^{\tau-t}\,d\tau$$ ← Add this term to both sides and divide by 10.

$$u(t)\int_{\tau=0}^{t} \sin(3\tau)e^{t-\tau}\,d\tau = \frac{1}{10}u(t)\left(\sin(3t) - 3\cos(3t) + 3e^{-t}\right)$$

25)

Let $p(t) = u(t+1) - u(t-1)$. Compute and sketch $(p * p)(t)$ and $(p * p * p)(t)$. Comment on your results.

3] $p(t) = u(t+1) - u(t-1)$

First consider $u(t-a) * u(t-b) = \left(\delta(t-a) * u(t)\right) * \left(\delta(t-b) * u(t)\right)$

$= \left(u(t) * u(t)\right) * \left(\delta(t-a) * \delta(t-b)\right) = \left(u(t) * u(t)\right) * \delta(t-a-b)$

$u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau) u(t-\tau)\, d\tau$ $\quad \tau > 0$, $t < \tau$

$= \int_{\tau=0}^{t} u(\tau) u(t-\tau)\, d\tau = u(t)t$

$u(t-a) * u(t-b) = \left(u(t)t\right) * \delta(t-a-b)$

$= u(t-a-b)(t-a-b)$

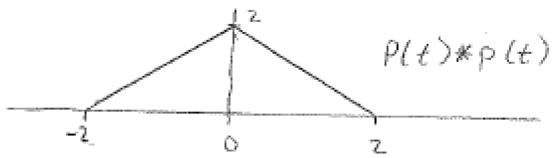
a) $p(t) * p(t) = [u(t+1) - u(t-1)] * [u(t+1) - u(t-1)]$

$= u(t+1) * u(t+1) - u(t+1) * u(t-1)$

$- u(t-1) * u(t+1) + u(t-1) * u(t-1)$

$= u(t+1) * u(t+1) - 2u(t+1) * u(t-1) + u(t-1) * u(t-1)$

$= u(t+2)(t+2) - 2u(t)t + u(t-2)(t-2)$



b) We know $u(t-a)*u(t-b) = u(t-a-b)(t-a-b)$

Now we need to know $u(t-a)*u(t-b)*u(t-c)$ since all terms in $p(t)*p(t)*p(t)$ can be expressed that way

$$\begin{aligned} u(t-a)*u(t-b)*u(t-c) &= \big(u(t)*u(t)*u(t)\big)*\big(\delta(t-a)*\delta(t-b)*\delta(t-c)\big) \\ &= \big(u(t)*u(t)*u(t)\big)*\delta(t-a-b-c) \\ &= \big(tu(t)*u(t)\big)*\delta(t-a-b-c) \end{aligned}$$

where the last line comes from our earlier result.

$x(\tau) = \tau u(\tau)$

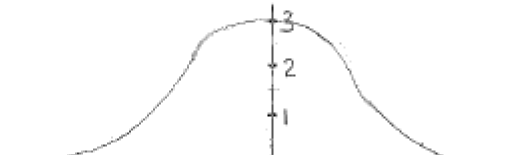
$h(t-\tau) = u(t-\tau)$

$$\int_{-\infty}^{\infty} \tau u(\tau)u(t-\tau)\,d\tau = \int_{\tau=0}^{t} \tau u(\tau)u(t-\tau)\,d\tau$$

$$= \tfrac{1}{2}t^2 u(t)$$

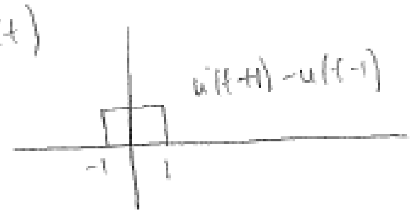
$$\begin{aligned} u(t-a)*u(t-b)*u(t-c) &= \big(\tfrac{1}{2}t^2 u(t)\big)*\delta(t-a-b-c) \\ &= \tfrac{1}{2}(t-a-b-c)^2 u(t-a-b-c) \end{aligned}$$

$$\begin{aligned} p(t)*p(t)*p(t) &= [u(t+1)-u(t-1)]*[u(t+1)-u(t-1)]*[u(t+1)-u(t-1)] \\ &= u(t+1)*u(t+1)*u(t+1) - u(t+1)*u(t+1)*u(t-1) \\ &\quad - u(t+1)*u(t-1)*u(t+1) + u(t+1)*u(t-1)*u(t-1) \\ &\quad - u(t+1)*u(t+1)*u(t+1) + u(t-1)*u(t+1)*u(t-1) \\ &\quad + u(t-1)*u(t-1)*u(t+1) - u(t-1)*u(t-1)*u(t-1) \\ &= \tfrac{1}{2}\big[(t+3)^2u(t+3) - (t+1)^2u(t+1) - (t+1)^2u(t+1) + (t-1)^2u(t-1) \\ &\quad - (t+1)^2u(t+1) + (t-1)^2u(t-1) + (t-1)^2u(t-1) - (t-3)^2u(t-3)\big] \\ &= \tfrac{1}{2}(t+3)^2u(t+3) - \tfrac{3}{2}(t+1)^2u(t+1) + \tfrac{3}{2}(t-1)^2u(t-1) - \tfrac{1}{2}(t-3)^2u(t-3) \end{aligned}$$



4.3) $p(t) = u(t+1) - u(t-1)$

$p(t)$

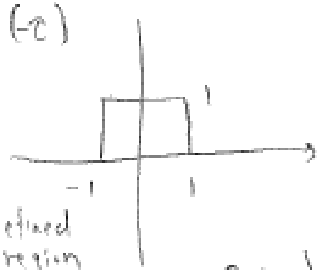


$y(t) = (p * p)\, t$

$$y(t) = \int_{-\infty}^{\infty} p(\tau)\, p(t-\tau)\, d\tau$$

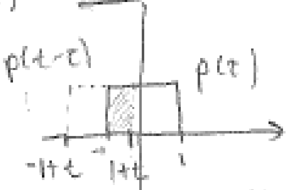
we need to find the limit of integration where this integral exists.

$p(-\tau)$



$p(\tau) = 1$, $p(t-\tau) = 1$ in defined region

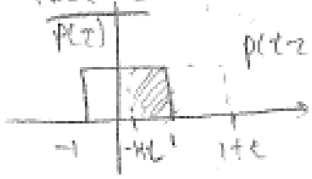
$p(t-\tau)$ case 1



$1+t \le 1$

$t \le 0$

or case 2



$-1+t \ge -1$

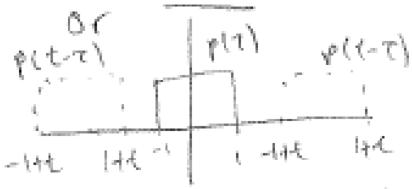
$t \ge 0$

$$y(t) = \int_{-1}^{1+t} d\tau \quad \text{case 1} \quad -2 \le t \le 0$$

$$= \int_{-1+t}^{1} d\tau \quad \text{case 2} \quad 2 \ge t \ge 0$$

case 3

or



$1+t \le -1$

$t \le -2$

$-1+t \ge 1$

$t \ge 2$

$\Rightarrow$ $1+t+1 = t+2 \quad -2 \le t \le 0$

$1-(-1+t) = -t+2 \quad 2 \ge t \ge 0$

case 3 $\quad 0 \quad t \le -2$, $t \ge 2$

$$y(t) = -\left(\frac{(-1+t)^2}{2} + 2(-1+t)\right) + \frac{-(1+t)^2}{2} + 2(1+t)$$

$$= \left(\frac{t^2}{2} - \frac{2t}{2} + \frac{1}{2} - 2 + 2t\right) + \left(\frac{-t^2}{2} + \frac{2t}{2} + \frac{1}{2}\right) + 2 + 2t$$

$$= -t^2 + 3 \qquad -1 \le t \le 1$$

Case 3

$$y(t) = \int_{-1+t}^{2} -\tau + 2 \, d\tau$$

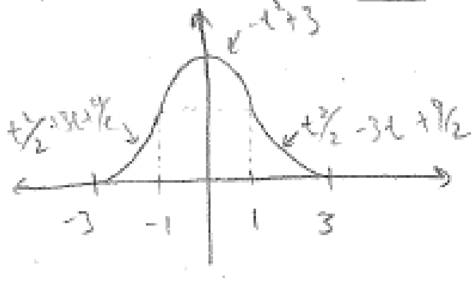
$$= -\frac{\tau^2}{2} + 2\tau \Big|_{-1+t}^{2}$$

$$= -2 + 4 - \left(-\frac{(-1+t)^2}{2} + 2(-1+t)\right)$$

$$= 2 - \left(-\frac{t^2}{2} + \frac{2t}{2} - \frac{1}{2} + 2t - 2\right)$$

$$= \frac{t^2}{2} - 3t + 9/2 \qquad 1 \le t \le 3$$

Case 4 $\quad 0 \qquad t \ \forall \text{ else}$



triangle and a box yields parabolas.

26)

Consider a system that has impulse response $h(t) = -\delta(t) + 2e^{-t}u(t)$. Compute the system's response to input $e^{t}u(-t)$ and sketch both the input and output signals.

4.4) $h(t)=-\delta(t)+2e^{-t}u(t)$ $\quad x(t)=e^{t}u(-t)$ $\quad y(t)=h(t)*x(t)$

$$y(t)=\int_{-\infty}^{\infty}[-\delta(t-\tau)+2e^{\tau-t}u(t-\tau)]e^{\tau}u(-\tau)d\tau=\underbrace{\int_{-\infty}^{\infty}-\delta(t-\tau)e^{\tau}u(-\tau)d\tau}_{y_1}+\underbrace{\int_{-\infty}^{\infty}2e^{\tau-t}e^{\tau}u(-\tau)u(t-\tau)d\tau}_{\triangleq y_2}$$

We solve this in two parts $y(t)=y_1(t)+y_2(t)$

$$y_1(t)=-\int_{-\infty}^{\infty}\delta(t-\tau)e^{\tau}u(-\tau)d\tau=-e^{t}u(-t)\int_{-\infty}^{\infty}\delta(t-\tau)d\tau=-e^{t}u(-t)$$

$$y_2(t)=\int_{-\infty}^{\infty}2e^{2\tau-t}u(-\tau)u(t-\tau)d\tau=2e^{-t}\int_{-\infty}^{\infty}e^{2\tau}u(-\tau)u(t-\tau)d\tau$$

$u(-\tau)\Rightarrow\tau<0$, $u(t-\tau)\Rightarrow\tau<t$ $\Big\}$ For $t<0$ $u(-\tau)u(t-\tau)\equiv u(t-\tau)u(-t)$; For $t\geq 0$ $u(-\tau)u(t-\tau)\equiv u(-\tau)u(t)$

$$y_2(t)=2e^{-t}u(-t)\int_{-\infty}^{\infty}\underbrace{u(t-\tau)}_{\tau<t}e^{2\tau}d\tau+2e^{-t}u(t)\int_{-\infty}^{\infty}\underbrace{u(-\tau)}_{\tau<0}e^{2\tau}d\tau$$

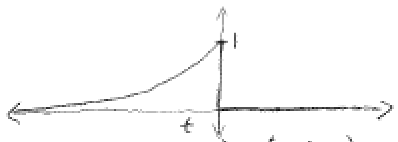
$$=2e^{-t}u(-t)\int_{-\infty}^{t}u(t-\tau)e^{2\tau}d\tau+2e^{-t}u(t)\int_{-\infty}^{0}u(-\tau)e^{2\tau}d\tau$$

$$=2e^{-t}u(-t)\left(u(t-\tau)\frac{e^{2\tau}}{2}\Big|_{-\infty}^{t}\right)+2e^{-t}u(t)\left(u(-\tau)\frac{e^{2\tau}}{2}\Big|_{-\infty}^{0}\right)$$

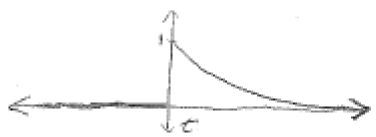
$$=e^{-t}u(-t)\left(u(0)e^{2t}-u(\infty)0\right)+e^{-t}u(t)\left(u(0)\cdot 1-u(\infty)\cdot 0\right)$$

$$=e^{t}u(-t)+e^{-t}u(t)$$

$$y(t)=y_1(t)+y_2(t)=-e^{t}\cancel{u(-t)}+e^{t}\cancel{u(-t)}+e^{-t}u(t)=e^{-t}u(t)$$



$x(t)=e^{t}u(-t)$



$y(t)=e^{-t}u(t)$

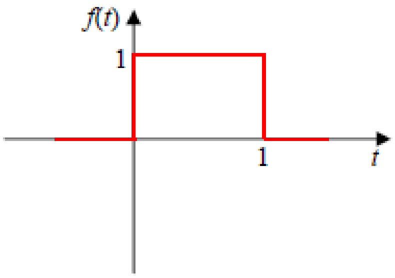
27) MATLAB assignment

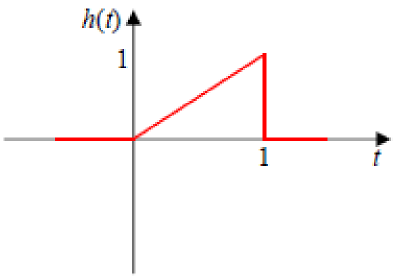
I. The goal of this part of the lab is to write a MATLAB program in which the convolution of two signals is computed.

Write a MATLAB program (m-file) in which the following two functions *f(t)* and *h(t)* are defined within the time interval of [-1 2]. Since the convolution is defined as $g(t) = f(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau) f(t-\tau) d\tau$, using your hand calculation:

- Specify the range of *t* for which the convolution result is non-zero.
- Determine appropriate ranges of τ in each case of integration and the corresponding mathematical term for $h(\tau) f(t-\tau)$ as a function of *t* and τ.

Now based on your answers to the above questions, continue your Matlab code on calculating the convolution result *g(t)*. In the same figure, plot f(t) in subplot(2,2,1), h(t) in subplot(2,2,2), and g(t) in subplot(2,2,3:4).





II. Write a MATLAB program in a new m-file to perform the convolution of the two functions given above using the *conv* command.