MKM 501 E – HW2

1) Prove each of the Fourier transform properties:

- *Linearity property*: given f(x), g(x),

$$FT(af(x)+bg(x)) = aF(\omega)+bG(\omega)$$

- Similarity property: g(x)=f(ax)

$$G(\omega) = \frac{1}{a}F(\frac{\omega}{a})$$

-*Shift formula*: given g(x) = f(x+b)

$$G(\omega) = e^{i\omega b}F(\omega)$$

- *Derivative formula*:

- If $x(t) \leftrightarrow X(\omega)$, then $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)$ for any real ω_0.

- If $x(t) \leftrightarrow X(\omega)$, then $x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$.

- **Convolution in time-domain**

If $x(t) \leftrightarrow X(\omega)$ and $v(t) \leftrightarrow V(\omega)$, then

$$x(t) \star v(t) \leftrightarrow X(\omega)V(\omega)$$

2) Compute the Fourier Transform of the Gaussian function and plot the magnitude and Power Spectrum.

Fourier Transform--Gaussian

The Fourier transform of a Gaussian function $f(x) \equiv e^{-a x^2}$ is given by

$$\begin{aligned}
\mathcal{F}_x\left[e^{-a x^2}\right](k) &= \int_{-\infty}^{\infty} e^{-a x^2} e^{-2\pi i k x}\,dx \\
&= \int_{-\infty}^{\infty} e^{-a x^2}\left[\cos(2\pi k x) - i\sin(2\pi k x)\right]dx \\
&= \int_{-\infty}^{\infty} e^{-a x^2}\cos(2\pi k x)\,dx - i\int_{-\infty}^{\infty} e^{-a x^2}\sin(2\pi k x)\,dx.
\end{aligned}$$

The second integrand is odd, so integration over a symmetrical range gives 0.
The value of the first integral is given by Abramowitz and Stegun (1972, p. 302, equation 7.4.6), so

$$\mathcal{F}_x\left[e^{-a x^2}\right](k) = \sqrt{\frac{\pi}{a}}\, e^{-\pi^2 k^2/a},$$

so a Gaussian transforms to another Gaussian.

3)

Consider the periodic signal $f(t)$, with period $T = 2$ and such that

$$f(t) = t^2, \quad \text{for } t \in [0, 2) .$$

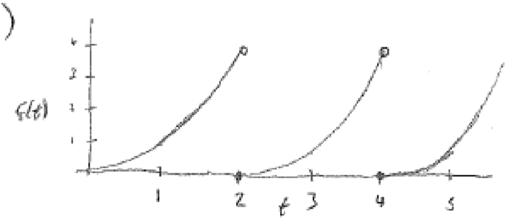
(a) - Sketch $f(t)$.

(b) - Find the trigonometric Fourier series representation $\varphi(t)$ of this signal.

(c) - Sketch the $\varphi(t)$ for all values of t.

2] $f(t) = t^2$ for $t \in [0, 2]$ $T = 2$

a)



b) $\varphi(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$ $\omega_0 = \frac{2\pi}{T} = \pi$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t)$

$a_0 = \frac{1}{T_0} \int_0^{T_0} f(t)\,dt = \frac{1}{2} \int_0^2 t^2\,dt = \frac{1}{6} t^3 \Big|_0^2 = \frac{4}{3}$

$a_n = \frac{2}{T_0} \int_0^{T_0} f(t) \cos(n\omega_0 t)\,dt = \int_0^2 t^2 \cos(n\pi t)\,dt$ $u = t^2$ $du = 2t\,dt$ $v = \frac{1}{n\pi} \sin(n\pi t)$ $dv = \cos(n\pi t)$

$= \frac{1}{n\pi} \cancelto{0}{t^2 \sin(n\pi t)} \Big|_0^2 - \frac{2}{n\pi} \int_0^2 t \sin(n\pi t)\,dt$ $u = t$ $du = dt$ $v = \frac{-1}{n\pi} \cos(n\pi t)$ $dv = \sin(n\pi t)$

$= \frac{2}{n^2\pi^2} t \cos(n\pi t) \Big|_0^2 - \frac{2}{n^2\pi^2} \int_0^2 \cos(n\pi t)\,dt$

$= \frac{4}{n^2\pi^2} - \frac{2}{n^3\pi^3} \sin(n\pi t) \Big|_0^2 = \frac{4}{n^2\pi^2}$

$b_n = \frac{2}{T_0} \int_0^{T_0} f(t) \sin(n\omega_0 t) = \int_0^2 t^2 \sin(n\pi t)\,dt$ $u = t^2$ $du = 2t\,dt$ $v = \frac{-1}{n\pi} \cos(n\pi t)$ $dv = \sin(n\pi t)$

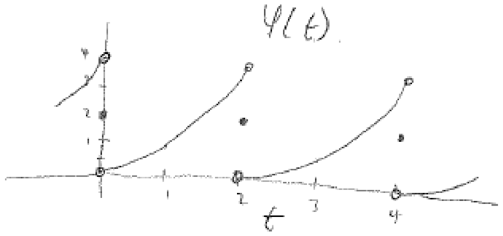
$= \frac{-1}{n\pi} t^2 \cos(n\pi t) \Big|_0^2 + \frac{2}{n\pi} \int_0^2 t \cos(n\pi t)\,dt$ $u = t$ $du = dt$ $v = \frac{1}{n\pi} \sin(n\pi t)$ $dv = \cos(n\pi t)$

$= \frac{-4}{n\pi} + \frac{2}{n^2\pi^2} \cancelto{0}{t \sin(n\pi t)} \Big|_0^2 - \frac{2}{n^2\pi^2} \int_0^2 \sin(n\pi t)\,dt$

$$= \frac{-4}{n\pi} + \frac{2}{n^2\pi^2}\cos(n\pi t)\Big|_0^2$$

$$= \frac{-4}{n\pi}$$

$$\varphi(t) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos(n\pi t) - \frac{4}{n\pi}\sin(n\pi t)$$



4) MATLAB assignment

MATLAB question: Use MATLAB to check the answers of problem 3

(a) - Plot the truncated Fourier series for $t \in [-5, 5]$ using N harmonics. Attach the plots for the cases $N = 4$,10 and 100. Comment on the results.

(b) - What is the value of the truncated Fourier series at the point $t = 2$ for the values of N in (b). What can you say about that value as N increases?

a) As N gets larger the approximation gets better. Gets very "wiggly" around end points: 0, 2.

b) As N gets larger, $\varphi(2) \to 2$.

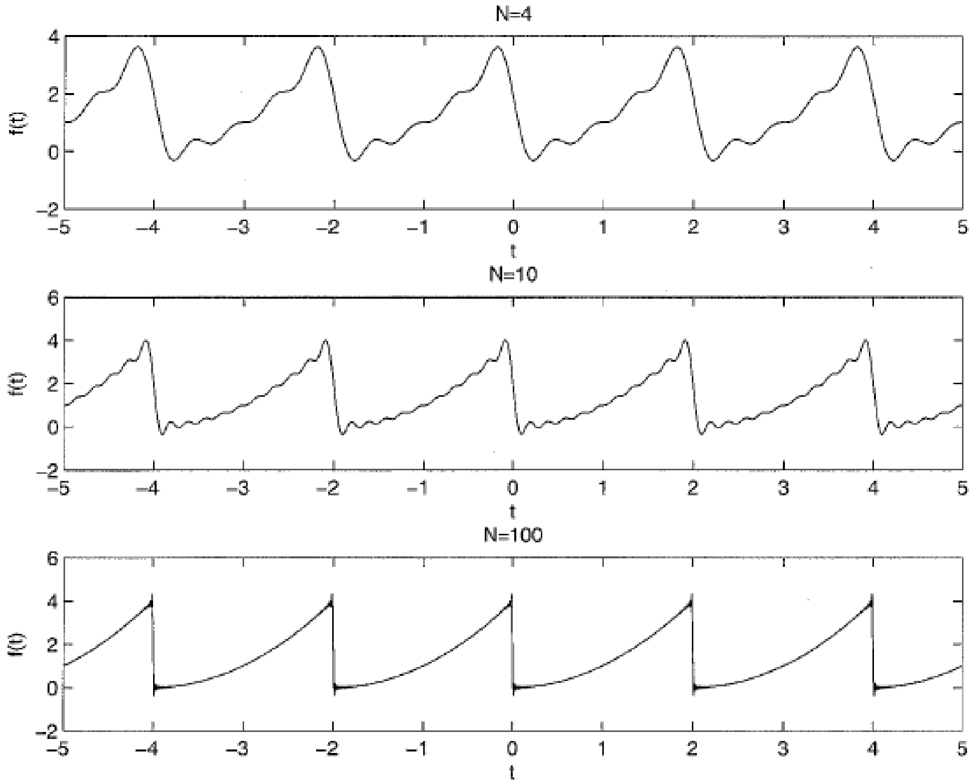
To see why consider $t = T = 2$

$$\varphi(2) = \frac{4}{3} + \sum_{n=1}^{N} \frac{4}{n^2\pi^2}\cos(2\pi n) - \frac{4}{n\pi}\sin(2\pi n)$$

$$= \frac{4}{3} + \sum_{n=1}^{N} \frac{4}{n^2\pi^2}$$

$$= \frac{4}{3} + \frac{4}{\pi^2}\underbrace{\sum_{n=1}^{N} \frac{1}{n^2}}_{=\pi^2/6 \text{ As } N\to\infty}$$

$N=4$ $\varphi(2) = 1.9103$
$N=10$ $\varphi(2) = 1.9614$
$N=100$ $\varphi(2) = 1.9960$




```
function[x1]=trigft(t,N)
x1=4/3;
for k=1:N;
   x1=x1+4./(k.*pi).^2.*cos(k.*pi.*t)-4./(k.*pi).*sin(k.*pi.*t);
end
```

5)

Consider the signal

$$x(t) = 2\cos(3\pi t - \pi/3) - \sin(2\pi t) \ .$$

(a) - Justify that $x(t)$ is periodic. What is the period T_0 and fundamental frequency ω_0?

(b) - Write the signal as a trigonometric Fourier series. Explicitly compute the trigonometric Fourier coefficients a_0 and a_k, b_k for $k = 1, 2, \ldots$.

(c) - Write $x(t)$ as an exponential Fourier series. What are the Fourier coefficients D_k?

a) Periodic functions satisfy the property $f(t) = f(t+T)$ for some T

$f(t) = 2\cos(3\pi t - \pi/3) - \sin(2\pi t)$

$f(t+T) = 2\cos(3\pi t + 3\pi T - \pi/3) - \sin(2\pi t + 2\pi T)$

$\cos(3\pi t - \pi/3) = \cos(3\pi t + 3\pi T - \pi/3)$ when $3\pi T = 2\pi k \Rightarrow T = \frac{2}{3}k$ for $k = 0, 1, \ldots$

$\sin(2\pi t) = \sin(2\pi t + 2\pi T)$ when $2\pi T = 2\pi \ell \Rightarrow T = \ell$ for $\ell = 0, 1, \ldots$

$T = \frac{2}{3}k = \ell \Rightarrow T = LCM[2/3, 1] = \underline{2}$ $\quad \omega_0 = \frac{2\pi}{T_0} = \pi$

b) $f(t) = 2\cos(3\pi t - \pi/3) - \sin(2\pi t)$

$= 2\cos(\pi/3)\cos(3\pi t) + 2\sin(\pi/3)\sin(3\pi t) - \sin(2\pi t)$

$\phi(t) = a_0 + a_1\cos(\pi t) + a_2\cos(2\pi t) + a_3\cos(3\pi t) + a_4\cos(4\pi t) + \ldots$
$+ b_1\sin(\pi t) + a_2\sin(2\pi t) + a_3\sin(3\pi t) + \ldots$

we can simply read off the coefficients:

$a_k = 0$ for all k except $k = 3: a_3 = 2\cos(\pi/3) = 1$

$b_k = 0$ for all k except $k = 2, 3: b_2 = -1$

$b_3 = 2\sin(\pi/3) = \sqrt{3}$

c) $\phi(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} D_n e^{jn\pi t}$

$= \ldots + D_{-4}e^{-4\pi j t} + D_{-3}e^{-3\pi j t} + D_{-2}e^{-2\pi j t} + D_{-1}e^{-\pi j t} + D_0 + D_1 e^{\pi j t} + D_2 e^{2\pi j t} + D_3 e^{3\pi j t} + \ldots$

$= \ldots + (D_{-3}e^{-3\pi j t} + D_3 e^{3\pi j t}) + (D_{-2}e^{-2\pi j t} + D_2 e^{2\pi j t}) + (D_{-1}e^{-\pi j t} + D_1 e^{\pi j t}) + \ldots$

$f(t) = \cos(3\pi t) + \sqrt{3}\sin(3\pi t) - \sin(2\pi t)$

$= \frac{1}{2}e^{3\pi j t} + \frac{1}{2}e^{-3\pi j t} + \frac{\sqrt{3}}{2j}e^{3\pi j t} - \frac{\sqrt{3}}{2j}e^{-3\pi j t} - \frac{1}{2j}e^{2\pi j t} + \frac{1}{2j}e^{-2\pi j t}$

$D_n = 0$ except $D_{-3} = \frac{1+\sqrt{3}j}{2}$ $\quad D_3 = \frac{1-\sqrt{3}j}{2}$ $\quad D_{-2} = \frac{-j}{2}$ $\quad D_2 = \frac{j}{2}$

6) Find the Fourier Transform of following functions and plot the amplitude and phase spectrums:

(a) $x(t) = e^{-2t}$

(b) $x(t) = \begin{cases} e^{-2t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$

(c) $x(t) = \begin{cases} e^{-2t}, & t \ge 0 \\ e^{2t}, & t < 0 \end{cases}$

(d) $x(t) = \begin{cases} e^{-3t}, & t \ge 0 \\ 2e^{4t}, & t < 0 \end{cases}$

Solution:

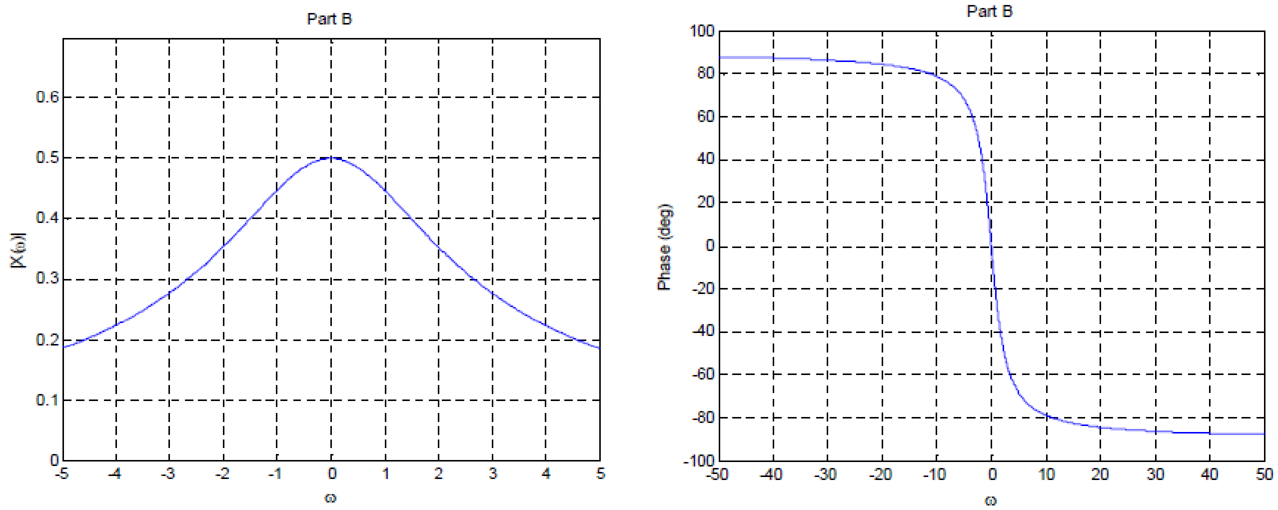
(a) $e^{-2t} \to \infty$ as $t \to -\infty$, thus its Fourier transform diverges and its spectrum is not defined.

(b) $x(t) = e^{-2t}$, for $t \geq 0$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(j\omega+2)t} dt$$

$$= \frac{1}{-(j\omega+2)} e^{-(j\omega+2)t} \Big|_{t=0}^{\infty} = \frac{1}{-(j\omega+2)}(0-1)$$

$$= \frac{1}{j\omega+2}$$



(c) $x(t) = \begin{cases} e^{-2t}, & t \geq 0 \\ e^{2t}, & t < 0 \end{cases}$

$$X(\omega) = \int_{-\infty}^{0} e^{2t} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-2t} e^{-j\omega t} dt$$

$$= \frac{1}{2-j\omega} e^{(2-j\omega)t} \Big|_{t=-\infty}^{0} + \frac{-1}{2+j\omega} e^{-(2+j\omega)t} \Big|_{t=0}^{\infty}$$

$$= \frac{1}{2-j\omega}(1-0) + \frac{-1}{2+j\omega}(0-1)$$

$$= \frac{1}{2-j\omega} + \frac{1}{2+j\omega} = \frac{4}{4-(j\omega)^2} = \frac{4}{4+\omega^2}$$

7)

Consider the following continuous-time periodic signal

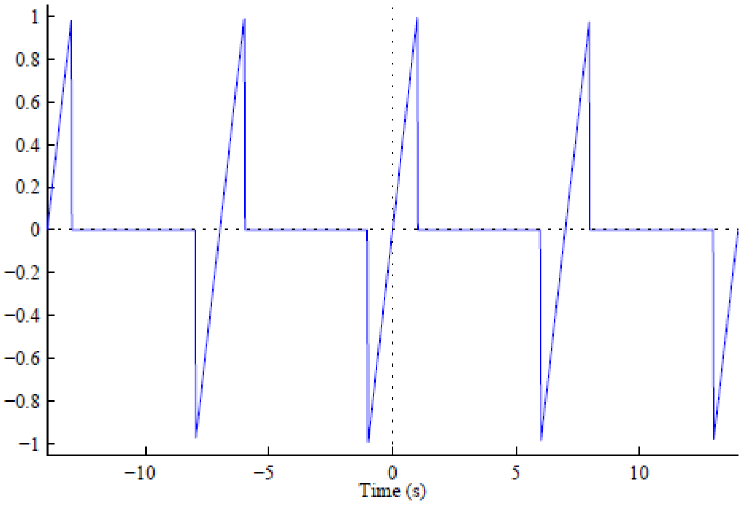
$$x(t) = \begin{cases} t + 7\ell & -1 \leq t + 7\ell \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

where ℓ is any integer.

a. What is the fundamental period of this signal.
   $T = 7$

b. Does the signal have even or odd symmetry?
   Yes, the signal is odd.

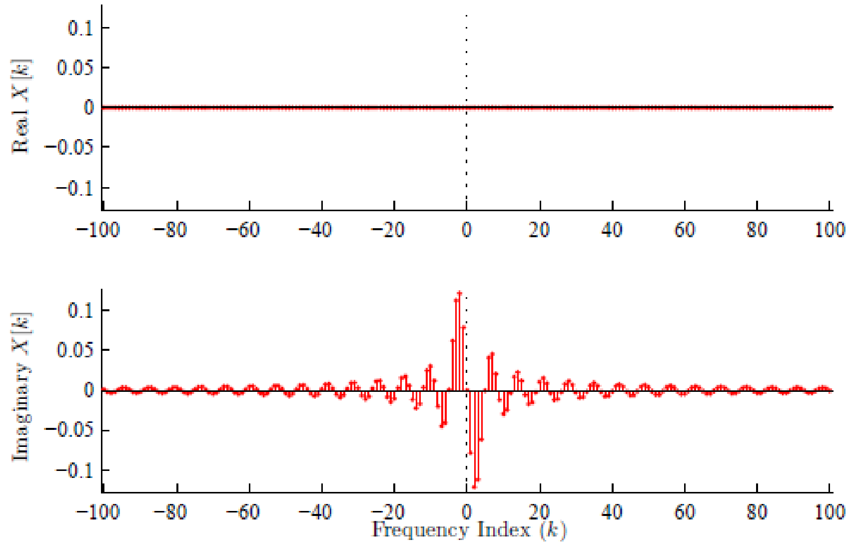
c. Plot four fundamental periods of the signal.



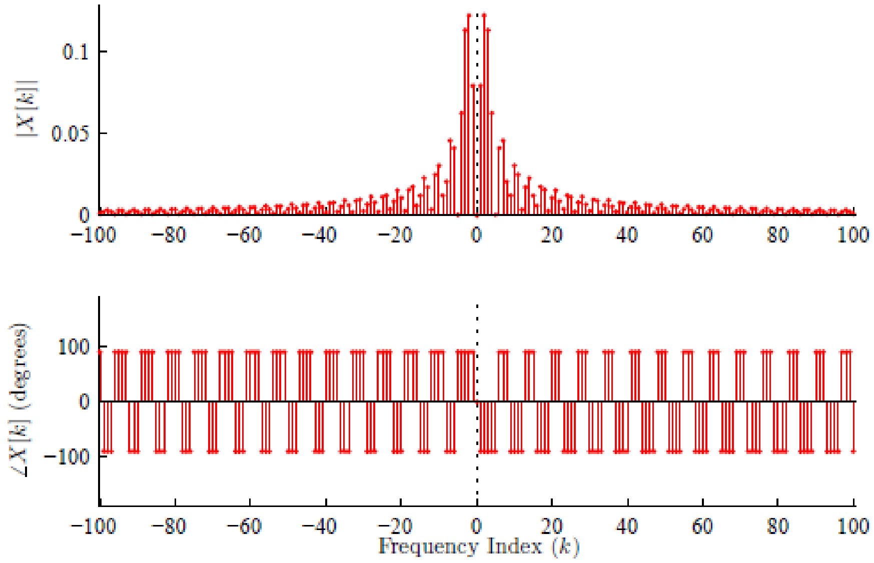
d. Solve for the Fourier series coefficients. Simplify your expression as much as possible.
Hint: $\int t\mathrm{e}^{at}\,\mathrm{d}t = \frac{1}{a^2}\mathrm{e}^{at}(at-1)+C$.

$$
\begin{aligned}
X[k] &= \frac{1}{T}\int_T x(t)\mathrm{e}^{-jk\omega t}\,\mathrm{d}t \\
&= \frac{1}{T}\int_{-1}^{1} t\mathrm{e}^{-jk\omega t}\,\mathrm{d}t \\
&= \frac{1}{T}\frac{1}{(-jk\omega)^2}\,\mathrm{e}^{-jk\omega t}(-jk\omega t-1)\Big|_{-1}^{1} \\
&= \frac{1}{T(jk\omega)^2}\left[\mathrm{e}^{-jk\omega}(-jk\omega-1)-\mathrm{e}^{jk\omega}(jk\omega-1)\right] \\
&= \frac{1}{T(jk\omega)^2}\left[-jk\omega\left(\mathrm{e}^{jk\omega}+\mathrm{e}^{-jk\omega}\right)+\left(\mathrm{e}^{jk\omega}-\mathrm{e}^{-jk\omega}\right)\right] \\
&= \frac{1}{-T(k\omega)^2}\left[-j2k\omega\cos(k\omega)+j2\sin(k\omega)\right] \\
&= \frac{j2k\omega\cos(k\omega)-j2\sin(k\omega)}{T(k\omega)^2} \\
&= \frac{j2k\frac{2\pi}{T}\cos(k\frac{2\pi}{T})-j2\sin(k\frac{2\pi}{T})}{T(k\frac{2\pi}{T})^2} \\
&= \frac{j4\pi k\cos(k\frac{2\pi}{T})-j2T\sin(k\frac{2\pi}{T})}{(2\pi k)^2}
\end{aligned}
$$

e. Plot the real and imaginary parts of the Fourier series coefficients $X[k]$.



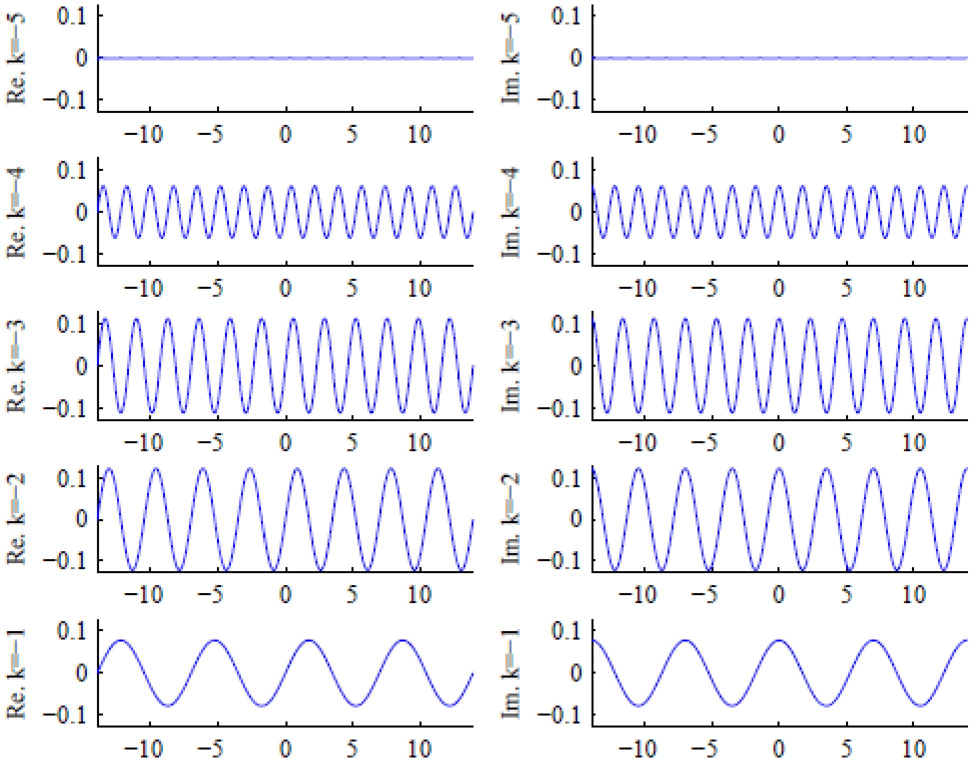
f. Plot the magnitude and phase of the Fourier series coefficients $X[k]$.

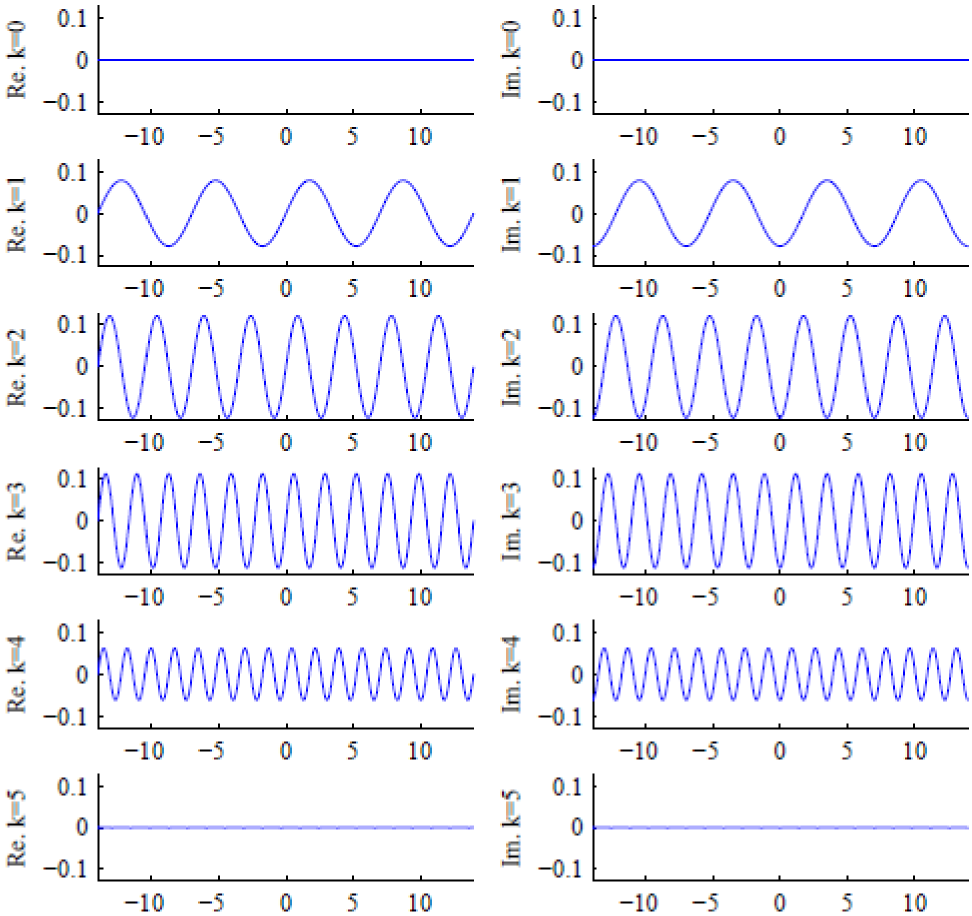


g. What type of symmetry do the coefficients have?
Complex conjugate symmetry: $X[-k] = X^*[k]$. In this case the coefficients are imaginary, so they also have odd symmetry: $X[-k] = -X[k]$.

h. What is the fundamental period of the coefficients?
The coefficients are not periodic and do not have a fundamental period.

i. Plot the real and imaginary parts of the CT complex sinusoids that comprise the continuous-time signal for $k = -5$ to $k = 5$.

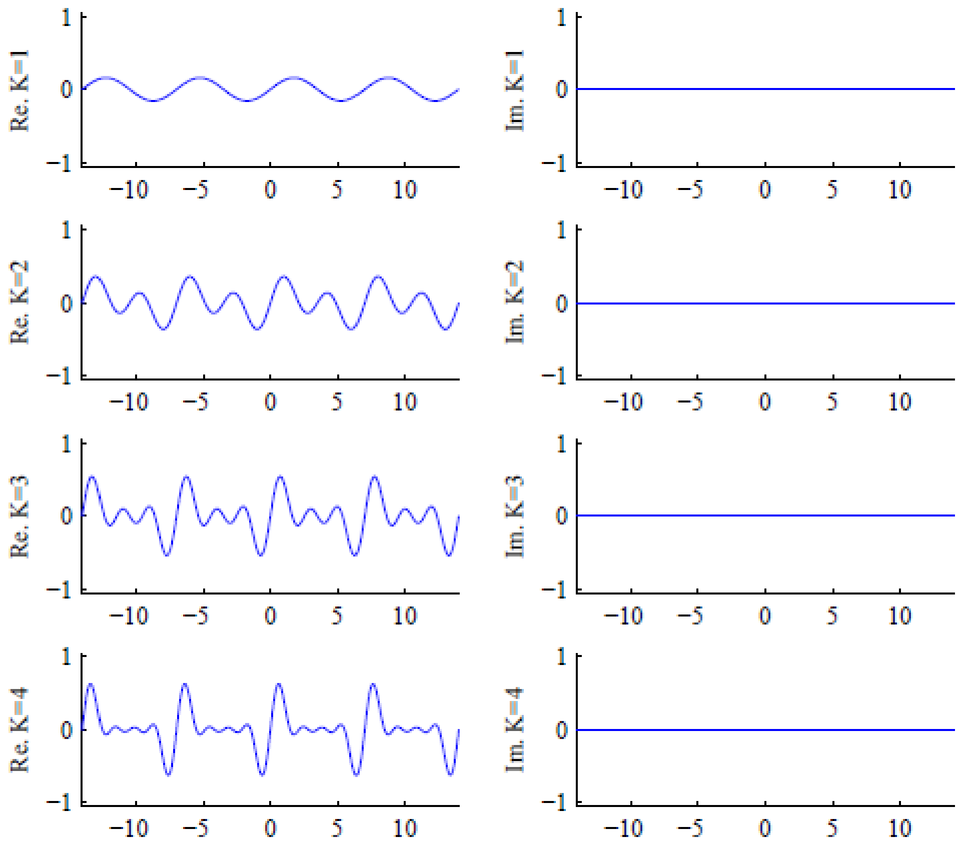


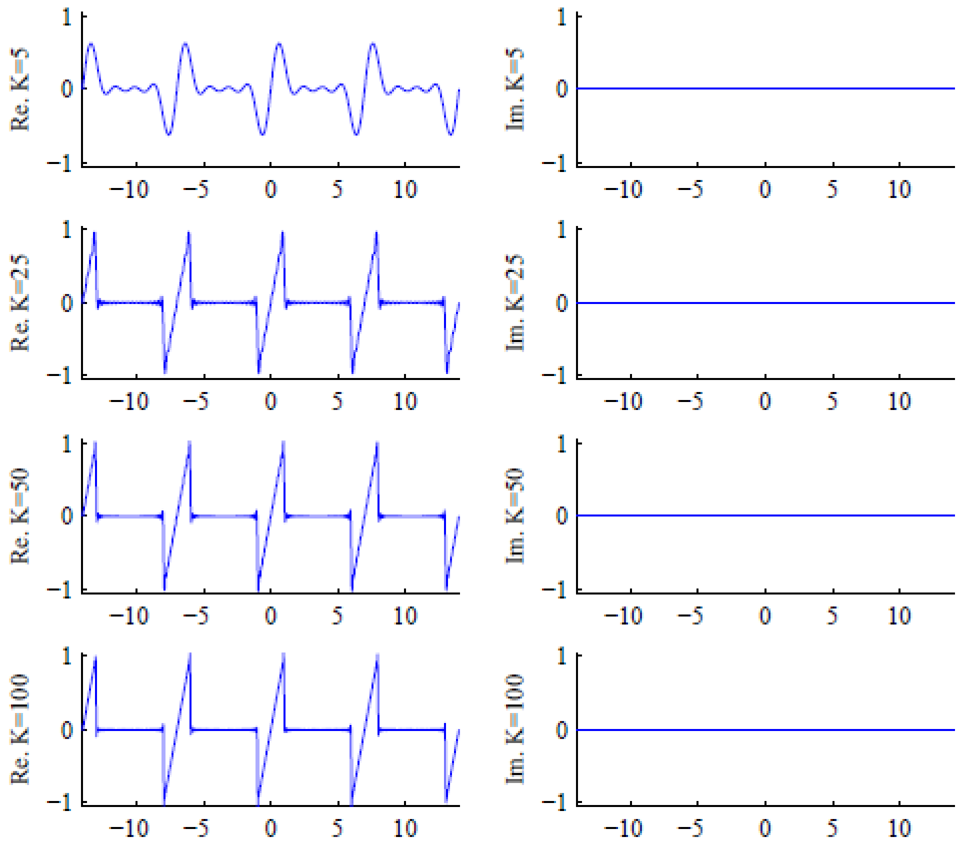


j. Plot the partial sums of the analysis equation,

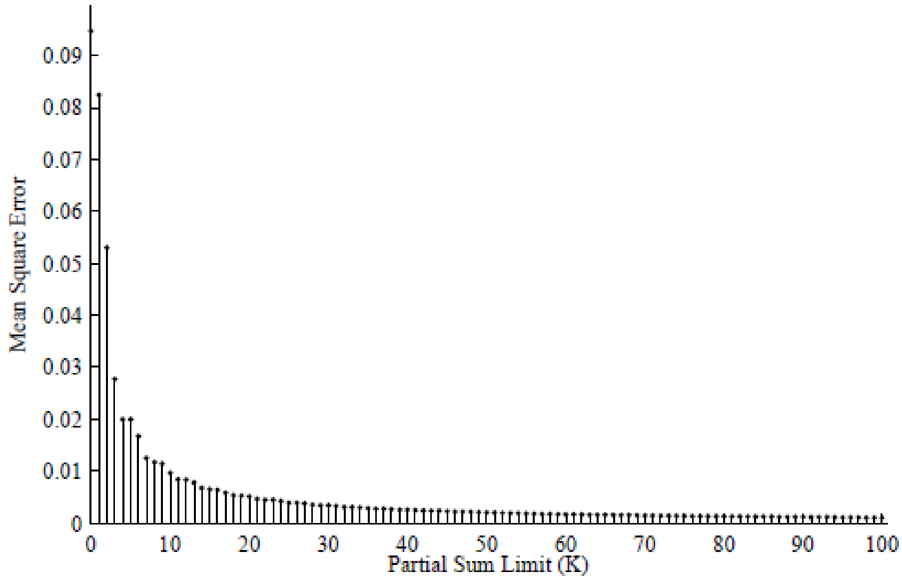
$$\hat{x}_K(t) = \sum_{k=-K}^{K} X[k] e^{jk\omega t}$$

of the CT complex sinusoids that comprise the continuous-time signal for $K = 1, 2, 3, 4, 5, 25, 50, 100$.





k. Plot the mean square error of the partial sums for $K = 0, \ldots, 100$.



l. How do the partial sums of the complex sinusoids compare to the original signal?
They are approximations of the original signal. The approximations with a greater number of terms are more accurate.

m. In the limit, does the following equality hold?

$$\lim_{K\to\infty} \hat{x}_K(t) = x(t)$$

No. This signal contains a discontinuity. At the times of the discontinuities the signal and the Fourier series approximations will not be equal, in general. However, the mean square error will approach zero as $K \to \infty$.

```
function [] = CTFourierSeries

T = 7;                                               % Fundamental period
omega = 2*pi/T;                                      % Fundamental frequency (rad/sec)
t = linspace(-2*T,2*T,1000);                         % Time indices
tmod = mod(t,T);
x = -(T-tmod).*(-1+T<=tmod & tmod<=0+T) + tmod.*(tmod<=1); % Signal

kn = -100:-1;
Xn = (j*4*pi*kn.*cos(kn*2*pi/T)-j*2*T*sin(kn*2*pi/T))./(2*pi*kn).^2;

kp = 1:100;
Xp = (j*4*pi*kp.*cos(kp*2*pi/T)-j*2*T*sin(kp*2*pi/T))./(2*pi*kp).^2;

k = [kn 0 kp];
X = [Xn 0 Xp];


   %====================================================================
   % Plot of the Signal
   %====================================================================
   figure;
   FigureSet(1,5);
   h = plot(t,x,'b');
   xlim([t(1) t(end)]);
   ylim([-1.05 1.05]);
   xlabel('Time (s)');

   box off;
   AxisLines;
   AxisSet(8);
   print('CTFourierSeries-Signal','-depsc');
```

```
%========================================================================
% Plot of the Fourier Series Coefficients (Rectangular form)
%========================================================================
figure;
FigureSet(1,5);
subplot(2,1,1);
    h = stem(k,real(X),'r');
    set(h,'Marker','.');
    xlim([k(1)-0.5 k(end)+0.5]);
    ylim(1.05*max(abs(X))*[-1 1]);
    h = get(gca, 'YLabel');
    set(h,'Interpreter','LaTeX');
    ylabel('Real $X[k]$');
    box off;
    AxisLines;
subplot(2,1,2);
    h = stem(k,imag(X),'r');
    set(h,'Marker','.');
    xlim([k(1)-0.5 k(end)+0.5]);
    ylim(1.05*max(abs(X))*[-1 1]);
    h = get(gca,'YLabel');
    set(h,'Interpreter','LaTeX');
    h = get(gca,'XLabel');
    set(h,'Interpreter','LaTeX');
    ylabel('Imaginary $X[k]$');
    xlabel('Frequency Index ($k$)');
    box off;
    AxisLines;
AxisSet(8);
print('CTFourierSeries-FourierSeriesRectangular','-depsc');
```

```
%========================================================================
% Plot of the Fourier Series Coefficients (Rectangular form)
%========================================================================
figure;
FigureSet(1,5);
subplot(2,1,1);
    h = stem(k,abs(X),'r');
    set(h,'Marker','.');
    xlim([k(1)-0.5 k(end)+0.5]);
    ylim([0 1.05*max(abs(X))]);
    h = get(gca, 'YLabel');
    set(h,'Interpreter','LaTeX');
    ylabel('$|X[k]|$');
    box off;
    AxisLines;
subplot(2,1,2);
    h = stem(k,angle(X)*180/pi,'r');
    set(h,'Marker','.');
    xlim([k(1)-0.5 k(end)+0.5]);
    ylim(190*[-1 1]);
    h = get(gca,'YLabel');
    set(h,'Interpreter','LaTeX');
    h = get(gca,'XLabel');
    set(h,'Interpreter','LaTeX');
    ylabel('$\angle X[k]$ (degrees)');
    xlabel('Frequency Index ($k$)');
    box off;
    AxisLines;
AxisSet(8);
print('CTFourierSeries-FourierSeriesPolar','-depsc');
```

```
%======================================================================
% Plot of the Fourier Series Components
%======================================================================
figure;
FigureSet(1,5,9);
cPlots = 1;                                                  % Plot counter
s = zeros(size(t));                                          % Memory allocation for the sum
for c1=-5:5,
    ik = find(k==c1);
    c  = X(ik)*exp(j*c1*omega*t);                            % Component
    s  = s + c;
    subplot(5*2+1,2,cPlots);
        h = plot(t,real(c),'b');
        cPlots = cPlots + 1;
        xlim([t(1) t(end)]);
        ylim(1.05*max(abs(X))*[-1 1]);
        ylabel(sprintf('Re. k=%d',c1));
        box off;
    subplot(5*2+1,2,cPlots);
        h = plot(t,imag(c),'b');
        cPlots = cPlots + 1;
        xlim([t(1) t(end)]);
        ylim(1.05*max(abs(X))*[-1 1]);
        ylabel(sprintf('Im. k=%d',c1));
        box off;
end
AxisSet(8);
print('CTFourierSeries-FourierSeriesComponents','-depsc');


%======================================================================
% Plot of the Partial Sums
%======================================================================
figure;
FigureSet(1,5,9);
K = [1,2,3,4,5,25,50,100];

cPlots = 1;                                                  % Plot counter
s = zeros(size(t));                                          % Memory allocation for the sum
mse = zeros(max(K),1);
for c1=0:max(K),
    ip = find(k== c1);                                       % Index of positive coefficient
    in = find(k==-c1);                                       % Index of negative coefficient
    c  = X(ip)*exp(j*c1*omega*t) + X(in)*exp(-j*c1*omega*t); % Component
    s  = s + c;                                              % Partial sum, k=-c1 to c1
    mse(c1+1) = mean(abs(x-s).^2);
    if any(c1==K),
        subplot(length(K),2,cPlots);
            h = plot(t,real(s),'b');
            cPlots = cPlots + 1;
            xlim([t(1) t(end)]);
            ylim([-1.05 1.05]);
            ylabel(sprintf('Re. K=%d',c1));
            box off;
        subplot(length(K),2,cPlots);
            h = plot(t,imag(s),'b');
            cPlots = cPlots + 1;
            xlim([t(1) t(end)]);
            ylim([-1.05 1.05]);
            ylabel(sprintf('Im. K=%d',c1));
            box off;
    end
end
AxisSet(8);
print('CTFourierSeries-PartialSums','-depsc');
```

```
%=====================================================================
% Plot of the Mean Square Error of the Partial Sums
%=====================================================================
figure;

FigureSet(1,5);
h = stem(0:max(K),mse,'k');
set(h,'Marker','.');
xlim([-0.05 max(K)+0.5]);
ylim([0 max(mse)*1.05]);
xlabel('Partial Sum Limit (K)');
ylabel('Mean Square Error');
box off;
AxisSet(8);
print('CTFourierSeries-MSE','-depsc');
```

8)

Transform examples. Find the Fourier transforms of the following signals. Which of these signals have Fourier transforms that converge? Which of these signals have Fourier transforms that are real? imaginary?

a. $x(t) = \cos(1000t)$.

b. $x(t) = 3\delta(t)$.

c. $x(t) = 13\cos(100t) - 7\sin(500t)$

d. $x(t) = \begin{cases} 1 & -50 \le t \le 50 \\ 0 & otherwise \end{cases}$

(a)

$$x(t) = \frac{1}{2}\left[e^{j1000t} + e^{-j1000t}\right]$$
$$e^{j\omega_o t} \overset{\mathcal{FT}}{\Longleftrightarrow} 2\pi\delta(\omega - \omega_o)$$
$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \overset{\mathcal{FT}}{\Longleftrightarrow} \alpha_1 X_1(j\omega) + \alpha_2 X_2(j\omega)$$
$$x(t) \overset{\mathcal{FT}}{\Longleftrightarrow} \pi\delta(\omega - \omega_o) + \pi\delta(\omega + \omega_o)$$

This CTFT did not converge (the CTFT is infinite at some frequencies). This is consistent with the knowledge that the signal is a power signal. The signal is even, so the transform is real.

(b)

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{\infty} 3\delta(t)e^{-j\omega t}\,dt \\ &= 3e^{-j\omega 0} \\ &= 3 \end{aligned}$$

The CTFT did converge (is finite at all frequencies). The signal is even and the transform is real.

(c)

$$\cos(100t) \overset{\mathcal{FT}}{\Longleftrightarrow} \pi\delta(\omega - 100) + \pi\delta(\omega + 100)$$
$$\sin(500t) \overset{\mathcal{FT}}{\Longleftrightarrow} j\pi\left[\delta(\omega + 500) - \delta(\omega + 500)\right]$$
$$X(j\omega) = 13\pi\left[\delta(\omega + 100) + \delta(\omega - 100)\right] - j7\pi\left[\delta(\omega + 500) - \delta(\omega + 500)\right]$$

The CTFT did not converge, is not real, and is not imaginary.

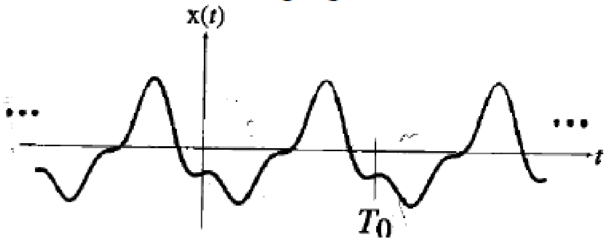
(d)

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-j\omega t}\,\mathrm{d}t \\
&= \int_{-50}^{50} \mathrm{e}^{-j\omega t}\,\mathrm{d}t \\
&= \frac{1}{-j\omega}\mathrm{e}^{-j\omega t}\Big|_{-50}^{50} \\
&= \frac{1}{-j\omega}\left[\mathrm{e}^{-j\omega 50} - \mathrm{e}^{j\omega 50}\right] \\
&= \frac{-2j\sin(50\omega)}{-j\omega} \\
&= \frac{2\sin(50\omega)}{\omega}
\end{aligned}$$

The CTFT did converge and is real. This is consistent with the knowledge that the signal has finite energy and is even, respectively.

9)

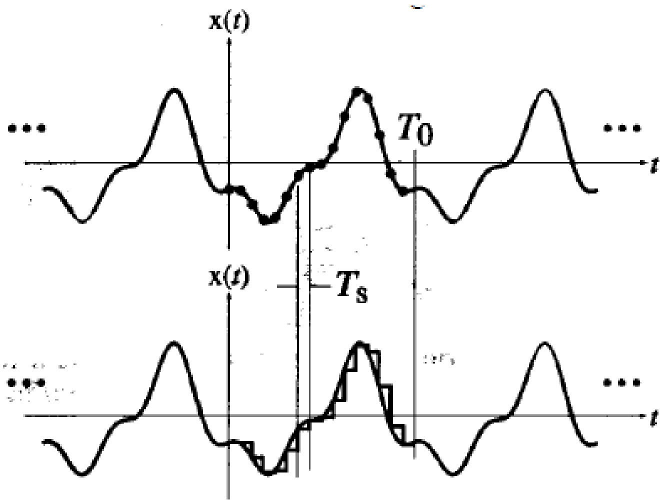
In this problem, we learn how to numerically compute the Fourier series of an arbitrary periodic signal. Let's consider the following signal:



This signal presents some problems. It is not at all obvious how to describe it. Up to this time in our study, we needed a mathematical description of a signal so that we can symbolically compute its Fourier series coefficient:

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)\exp\left(-jk\omega_0 t\right) dt$$

There is a better way – if we have a set of N samples of the signal taken from one period, we can estimate c_k numerically. The larger the N, the better the approximation becomes. Let's say, the sampling interval is T_S. We can approximate the signal as a piece-wise flat function as we have done in class earlier in showing the CT convolution formula.



The Fourier series coefficient can then be approximated as a sum of the area of rectangles:

$$\begin{aligned} c_k &\approx \frac{1}{T_0}\sum_{n=0}^{N-1} x(nT_S)\exp(-jk\omega_0 nT_S)T_S \\ &= \frac{1}{N}\sum_{n=0}^{N-1} x(nT_S)\exp\left(-j\frac{k2\pi}{N}\right) \end{aligned}$$

The last line uses the simple relationship T_0=NT_S (N samples in one period) and ω_0 = $2\pi/T_0 = 2\pi/(NT_S)$. The summation $\sum_{n=0}^{N-1} x(nT_S)\exp\left(-j\frac{k2\pi}{N}\right)$ turns out to be a very important Fourier representation called Discrete Fourier Transform (DFT), and it can be efficiently computed using Fast Fourier Transform (FFT). Thus, if x is a N-sample version of x(t) in MATLAB, we can use the following to compute the Fourier series coefficient:

```
>> c = fft(x)/length(N)
```

As it turns out, only the first half of the vector c represents the first N/2 Fourier series coefficients of x(t). The remaining half represents a reflected version of the conjugate of the same set of coefficients which we can ignore.

Okay, it's your turn to do something. Use the above method to find the magnitude and phase spectrums of the first 64 Fourier series coefficients of the periodic signal x(t), one period of which is described by

$$x(t) = \sqrt{1-t^2} \quad -1 < t < 1$$

Here are some hints to get you started. The fundamental period of the signal is 2. Use N=128 to sample the signal between 0 and 2. To do that, we first define the time instances to do the sampling:

```
>> n = 0:127
>> Ts = 2/128;
>> t = Ts*n;
```

Then, we can do the sampling.

```
>> x = [sqrt(1-t(1:64).^2) sqrt(1-(t(65:128)-2).^2)];  % Can
you explain?
```

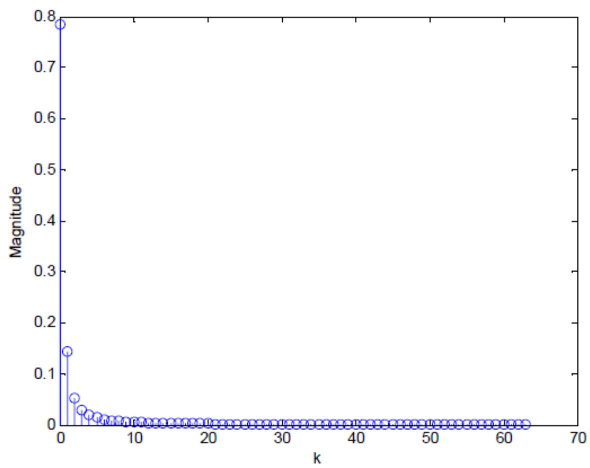
Finally, apply the FFT and use the stem plots to show the magnitude (with command `abs`) and phase (with command `angle`) of the harmonics.

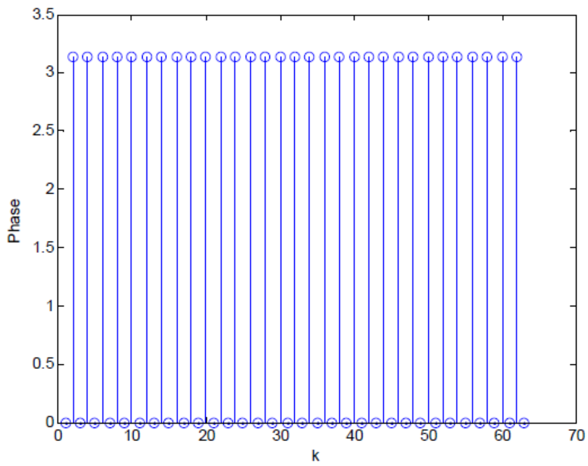
Solution:

MATLAB code:

```
clear;
close all;
n = 0:127;
Ts = 2/128;
t = Ts*n;
x = [sqrt(1-t(1:64).^2) sqrt(1-(t(65:128)-2).^2)];
figure;
plot(t, x);
c = fft(x)/length(n);
figure;

stem(n(1:length(n)/2), abs(c(1:length(n)/2)));
xlabel('k');
ylabel('Magnitude');
figure;
stem(n(1:length(n)/2), angle(c(1:length(n)/2)));
xlabel('k');
ylabel('Phase');
```





10) Starting from the Fourier Series representation, derive Fourier Transform representation. (Hint: Look through lecture slides).

11) Given that Fourier transform equations are as follows,

$$\boxed{\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \\ f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega \end{aligned}}$$

derive the Fourier Transform of following functions by (i) first deriving it from scratch, that is just by using the above integral equations, and (ii) try deriving it using the known Fourier transforms of simpler functions, then (iii) plot amplitude and phase spectrums of the Fourier transforms :

a) 1
b) $\delta(x)$
c) e^{iax}
d) $\cos(ax)$
e) $\sin(ax)$
f) $u(x)$
g) $\sum_{n=-\infty}^{\infty} \delta(x - nT)$
h) $e^{j\omega_0 t}$
i) $rect_a(t) = u(t+a) - u(t-a) = \begin{cases} 1 & -a \leq t < a \\ 0 & \text{else} \end{cases}$

j) $x(t) = triangle(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & |t| \geq 1 \end{cases}$