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On the dynamical behaviour of FitzHugh–Nagumo systems: Revisited☆

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ABSTRACT

The purpose of this paper is to analyse a general form of the FitzHugh–Nagumo model as completely as possible. The main result is that no more than two limit cycles can be bifurcated from the unique fixed point via Hopf bifurcation, and there exist parameters such that this upper bound is attained. For these parameters, the stability of the inner and outer cycle, together with the unique fixed point is also established. The results are approached through Lyapunov coefficients and rely on a theorem by Andronov and Aleksandrovic [A.A. Andronov, A.A. Aleksandrovic, Theory of Bifurcations of Dynamical System on a Plane, Wiley, 1971]. Based on singular perturbation theory a sufficient condition for existence of a unique stable limit cycle is given under certain assumptions.

1. Introduction

Stability of nonlinear systems is one of the central topics in dynamic systems and control theory. For example asymptotic stabilisation plays an important role in control theory. Most work in control theory is on stabilisation of nonlinear systems around a fixed point, see e.g. [1] where necessary and sufficient conditions for feedback stabilisation using continuous feedback is given for two-dimensional control systems. Quite recently attention has also been brought to the stabilisation of limit cycles, e.g. [2]. This amounts to a study of the stability of limit cycles. Consequently it becomes important to know the existence and numbers of limit cycles.

In this paper we consider the FitzHugh–Nagumo class model

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -Cy - Ax(x-B)(x-\lambda) + I \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \varepsilon(x - \delta y), \end{cases} \tag{1}$$

with non-zero parameters $A, B, C, \delta, \varepsilon, \lambda$, where I is en external force, commonly referred to as the magnitude of the stimulating current, which can be a function of t.

Our motivation for studying this system is that this system is frequently used, e.g. in brain research, and to some extent in modelling cardiac movements; it is of great importance to get a good understanding of its dynamics. This is reflected in the number of articles written in this area, see for example [3–11] and the references therein. A short but rather complete description of the physiology behind the biological neuron and the corresponding derivation of the Hodgkin-Huxley model and its simplified version, the FitzHugh–Nagumo model, can be found in [9].

This paper attempts to describe the different dynamical behavior exhibited by (1), in terms of the six parameters, as completely as possible. We hope to gain an analytic insight of this widely used system as much as possible. Although

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the FitzHugh–Nagumo system is a well-studied object (see e.g. [12,13,11,14,14,15]), there are several other reasons that motivate the current study.

It is of mathematical interest to know the number of limit cycles for a polynomial system although we do not have the ambition to solve this problem completely. A classical solved example is the van der Pol equation without external force for which we know that there is only one stable limit cycle. It was claimed in [16] that the number of limit cycles of a special setting of the FitzHugh–Nagumor system would be bifurcated from a fixed point in a manner of 2^k. In this paper we prove that such a number is 2.

It is also of interest to give a satisfactory picture of how bifurcation takes place and which parameters (of the six) play a role in the bifurcation. According to our knowledge, the parameters are essential in biology and different application areas need different parameters. This is one of the reasons we do not *a priori* assume the values of some parameters. We want to point out that there is still no definite answer, in some parameter settings, to the question that the solution of FitzHugh–Nagumo equations converges to a fixed point or to a limit cycle [13].

Since the FitzHugh–Nagumo model, defined by (1), is used for the investigation of a single neuron, in reality we have to study the interconnection and coupling of neurons. In other words, a chain of such systems will be used in more realistic models for instance, [17]. Therefore we believe that a full description of dynamical behavior of the FitzHugh-nagumo system will benefit to understanding more complicated systems based on the FitzHugh–Nagumo system.

It is often and popular in doing dynamical analysis using computer simulations (even with little analysis behind), and research areas such as computational biology is well-established nowadays. Nevertheless some precautions have to be kept in mind. In [13], it was noted that a certain bifurcation could take place in a very narrow interval (of the magnitude of 10^{-7} of values of I). It is clear that a rigorous mathematical analysis, if possible, is highly demanded in such a case.

This paper is organized as follows: The second section contains some well-known classical results, covering the areas of singular perturbation theory (e.g. [18]) and Lyapunov coefficients (e.g. [19]). Since our interest is on bounded solutions we prove the boundedness of solutions in the end of Section 2. In Section 3 conditions on some of the parameters are presented ensuring existence of a unique limit cycle as a result of singular perturbation theory. The main results in Section 4 is an upper bound of the maximal number of limit cycles that can be bifurcated from the origin via Hopf bifurcation and determination of the sign of the second Lyapunov coefficient.

2. Preliminaries

In this section we first collect some notions and results and reformulation of these on discontinuous periodic solutions based on singular perturbation theory and Lyapunov coefficients [24,25]. Then we study the boundedness of solutions to the (1).

2.1. Singular perturbation analysis and discontinuous periodic solutions

This section is concerned with periodic solutions of planar systems of the form

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x, y) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y), \end{cases} \tag{2}$$

where $0 < \varepsilon \ll 1$. In the case where $\varepsilon \gg 1$ the rescaling of time $t = \varepsilon\tau$ transforms (2) into

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x, y) \\ \theta \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y), \end{cases} \tag{3}$$

where $\theta = \frac{1}{\varepsilon} \ll 1$, and thus is not necessary to analyse this case separately. Before further discussions and giving a definition of *discontinuous periodic solutions*, DPS, and the possible existence of limit cycles of system (2) we start with a sketch of the behaviour of its solutions. Assume that we start at a point (x_0, y_0) such that $f(x_0, y_0) \neq 0$. Since ε is very small, the trajectories $(x(t), y(t))$, of (2) will move with almost zero velocity in the y-direction in comparison to the velocity in the x-direction. This type of motion will continue until a point $(x(t_1), y(t_2))$ is reached such that $f(x(t_1), y(t_2))$ is of the same order as ε. Thus for $t \in [0, t_1]$ the following approximation of (2) makes sense.

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x, y_0) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 0. \end{cases} \tag{4}$$

The fixed points of the approximated system are given by the set $F_{\tilde{y}} = \{(x, \tilde{y}) \in \mathbb{R}^2 | f(x, \tilde{y}) = 0\}$ and the stability is determined by the sign of $\frac{\partial f}{\partial x}$ along S where S is given by

$$S = \bigcup_{\tilde{y} \in \mathbb{R}} F_{\tilde{y}}.$$

Thus the trajectories of (4) will approach points of S such that $\frac{\partial f(x,y)}{\partial x} < 0$. Close to such points the velocities of $x(t)$ and $y(t)$ will be of the same order and thus the position of the fixed points of (4) will start to move, i.e the fixed points will change continuously between different branches of the graph $F_{\tilde{y}}$ of S. Therefore, the trajectories will lie in a small neighbourhood of S until a point $(x(t_2), y(t_2))$ is reached where $\frac{\partial f(x(t_2),y(t_2))}{\partial x} = 0$ and $\frac{\partial^2 f(x(t_2),y(t_2))}{\partial x^2} \neq 0$. For $t \in [t_1, t_2]$ the following approximation, also referred to as the reduced system, of (2) is valid,

$$\begin{cases} f(x, y) = 0 \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y). \end{cases} \tag{5}$$

At the so-called *break point* $(x(t_2), y(t_2))$, the stability is disturbed and the trajectories of (2) are again approximated by (4) and everything is repeated until the trajectories either reach a fixed point of (2) or there are no break points and the trajectories are trapped in a small neighbourhood of the set S. The motion of system (4) is called *fast flow* while the motion of the system (5) is referred to as *slow flow*. Now it should come with no surprise that under certain conditions, system (2) has a family of periodic solutions, L_ε, parametrised by ε. In order to state the major theorem of this section we first need to define the concept of *discontinuous periodic solutions*, DPS. We start by introducing the following sets, already mentioned above:

- $S = \{(p, q) \in \mathbb{R}^2 | f(p, q) = 0\}$,
- $K = \{(p, q) \in S | \frac{\partial f(p,q)}{\partial x} = 0\}$,
- $L = \{(p, q) \in S | \frac{\partial f(p,q)}{\partial x} < 0\}$,

For the sake of exposition we make the following remarks about solutions to the approximated systems (4) and (5) respectively.

Remark 1. Given a point $(x_0, y_0) \in \mathbb{R}^2 \setminus S$, there exists a unique solution $x_{FF}(t)$ to the approximation (4) satisfying $(x_{FF}(0), y_0) = (x_0, y_0)$ and

$$\lim_{t\to\infty} (x_{FF}(t), y_0) \in L.$$

Remark 2. By assumption, the Implicit Function Theorem can be applied to the equation $f(x, y) = 0$ in the above approximation. This gives the existence of a function

$$h : U \to V$$

such that $x = h(y)$, where $U \subset \mathbb{R}$ and $\subset \mathbb{R}$. Thus Eq. (5) may be rewritten as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(h(y), y). \tag{6}$$

For $(x_0, y_0) \in L$ there exists a unique solution, $y_{SF}(t)$, to Eq. (6) for $t \in [0, T_{(x_0,y_0)})$ where $T_{(x_0,y_0)}$ satisfies the condition

$$\lim_{t\to T_{(x_0,y_0)}} (h(y_{SF}(t)), y_{SF}(t)) \in K$$

and $(h(y_{SF}(0)), y_{SF}(0)) = (x_0, y_0)$.

With these preparations, the definition of DPS is rather straightforward although rather lengthy.

Definition 1. Construct a sequence $\{p^k_{(x,y)}\}$ of points in $\mathbb{R}^2$ in the following way.

(i) Let $p^0_{(x,y)} = (x, y) \in \mathbb{R}^{\setminus} S$.

(ii) Let $p^1_{(x,y)} = (x_1, y) \in L$, where

$$x_1 = \lim_{t\to\infty} x_{FF}(t)$$

and $x_{FF}(t)$ is the solution to the approximated system (4) with initial condition $x_{FF}(0) = x$, (see Remark 1). Define

$$FF_1 = \{(x_{FF}(t), y) | t \in [0, \infty)\}.$$

(iii) Let $p^2_{(x,y)} = (h(y_2), y_2) \in K$, where

$$y_2 = \lim_{t\to T} y_{SF}(t)$$

and $y_{SF}(t)$ is the solution to the approximated system (5) with initial condition $y_{SF}(0) = y$, (see Remark 2). Define

$$SF_1 = \{(h(y_{SF}(t)), y_{SF}(t))|t \in [0, T)\}.$$

(iv) Let $p^3_{(x,y)} = (x_3, y_2) \in L$, where

$$x_3 = \lim_{t\to\infty} x_{FF}(t)$$

and $x_{FF}(t)$ is the solution to the approximated system (4) which has the property that

$$\lim_{t\to-\infty} x_{FF}(t) = h(y_2)$$

(see Remark 1). Define

$$FF_2 = \{(x_{FF}(t), y_2)|t \in \mathbb{R}\} \cup \{h(y_2), y_2\}.$$

The sequence $\{p^k_{(x,y)}\}$ is now continued in exactly the same way. System (2) is said to have a DPS, Γ_0, if there exist $m, n \in \mathbb{N}^+$ such that $m < n$ and

$$p^m_{(x,y)} = p^n_{(x,y)}$$

for some $(x, y) \in \mathbb{R}^2$. Furthermore, this DPS is given by

$$\Gamma_0 = \bigcup_{k=m}^{n} (FF_k \cup SF_k).$$

Finally all preparations have been made in order to state the following theorem by Mishchenko and Rosov, Theorem 14 in [20].

Theorem 1. *Let*

$$\begin{cases} \varepsilon \dfrac{dx}{dt} = f(x, y) \\ \dfrac{dy}{dt} = g(x, y) \end{cases} \tag{7}$$

be a dynamical system on the plane and assume that the following conditions are satisfied:

(i) *All second derivatives of f and g are continuous at each point in the plane.*
(ii) *At all points of S it holds true that*

$$f_x^2(x, y) + f_y^2(x, y) > 0. \tag{8}$$

(iii) *For all points in K it holds that*

$$\frac{\partial^2 f(p, q)}{\partial x^2} \neq 0. \tag{9}$$

(iv) *There are no fixed points in the set $K \cup L$.*
(v) *System (7) has a DPS, denoted L_0.*

Then for each sufficiently small ε system (7) has a unique stable limit cycle, L_ε, and furthermore

$$\lim_{\varepsilon\to 0} L_\varepsilon = L_0. \tag{10}$$

2.2. Lyapunov coefficients

Now we turn to a theoretical background to the theory of Lyapunov coefficients that will be used in Section 4 for analysis of our specific class of FitzHugh–Nagumo systems. As a starting point we assume that a dynamical system is given in the form

$$\begin{cases} \dfrac{dx}{dt} = f(x, y, \mu) \\ \dfrac{dy}{dt} = g(x, y, \mu), \end{cases} \tag{11}$$

where f and g are real valued analytic functions on $U \times V \times W \subset \mathbb{R}^3$ and that $(0, 0, \mu_0)$ is a fixed point of (11). Associated with (11) and this point is the system matrix, $\mathcal{A}$, defined by

$$\mathcal{A} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where all derivatives are calculated at the fixed point.

Remark 3. The eigenvalues of $\mathcal{A}$ solves the equation

$$s^2 - \sigma(\mu_0)s + \Delta(\mu_0) = 0,$$

where $\sigma(\mu_0) = \text{trace}(\mathcal{A})$ and $\Delta(\mu_0) = \det(\mathcal{A})$.

It is a well known fact that the origin is structurally stable iff $\Re(s_i(\mu_0)) \neq 0$, the real part of $s_i(\mu_0)$, where $s_i(\mu_0)$ are the eigenvalues of $\mathcal{A}$ for $i = 1, 2$. To put it in another way, for all values of μ close to μ_0 the solutions of the perturbed systems restricted to a small neighbourhood of the origin are topologically equivalent if $\Re(s_i(\mu_0)) \neq 0$.

Definition 2. Given a real analytic dynamical system

$$\mathcal{B} : \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = f(x, y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = g(x, y) \end{cases}$$

we say that another system

$$\tilde{\mathcal{B}} : \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \tilde{f}(x, y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{g}(x, y) \end{cases}$$

is δ close up to order k to system $\mathcal{B}$ in $G \subset \mathbb{R}^2$ if

$$D_i < \delta, \quad i = 1, 2 \tag{12}$$

where

$$D_1 = \max_{(x,y)\in G} |f^{r+s}_{x^r y^s} - \tilde{f}^{r+s}_{x^r y^s}|, \quad r + s = 0, 1, \ldots, k \tag{13}$$

and

$$D_2 = \max_{(x,y)\in G} |g^{r+s}_{x^r y^s} - \tilde{g}^{r+s}_{x^r y^s}|, \quad r + s = 0, 1, \ldots, k. \tag{14}$$

Thus bifurcation can only occur when at least one of the eigenvalues has zero real part. We are going to study the case where $s_i(\mu_0) = \pm\imath\Delta(\mu_0)$, where $\imath^2 = -1$, with $\Delta(\mu_0) > 0$. To this end we assume that the eigenvalues of $\mathcal{A}$ are given by $s(\mu) = \alpha(\mu) \pm \mathrm{i}\beta(\mu)$ and that

$$\begin{cases} \sigma(\mu_0) = 0 \\ \Delta(\mu_0) > 0 \end{cases} \tag{15}$$

for some parameter value $\mu = \mu_0$. By Remark 3 and assumption (15) $\beta(\mu) > 0$ for all μ in a small neighbourhood of μ_0 since $\beta(\mu) = \sqrt{4\Delta(\mu) - \sigma^2(\mu)}$ and the determinant and trace functions are continuous. With these assumptions, the system defined by (11) can be put in the canonical form (see proof in [21])

$$\begin{cases} \frac{\mathrm{d}\xi}{\mathrm{d}t} = \alpha(\mu)\xi - \beta(\mu)\eta + f(\xi, \eta, \mu) \\ \frac{\mathrm{d}\eta}{\mathrm{d}t} = \beta(\mu)\xi + \alpha(\mu)\eta + g(\xi, \eta, \mu) \end{cases} \tag{16}$$

by the real non-singular transformation

$$\begin{cases} \xi = (a - \alpha(\mu))y + cx \\ \eta = \beta(\mu)y. \end{cases} \tag{17}$$

By introducing a complex variable $z = \xi + \mathrm{i}\eta$ system (16) is transformed into a single complex differential equation

$$\dot{z} = (\alpha(\mu) + \mathrm{i}\beta(\mu))z + f\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2\mathrm{i}}\right) + \mathrm{i}g\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2\mathrm{i}}\right) := s(\mu)z + H(z, \bar{z}, \mu) \tag{18}$$

where H must be analytic since both f and g are.

The following two lemmas show that in a neighbourhood of the origin (18) can be put in a canonical form by a smooth change of coordinates.

Lemma 2. *Let*

$$\dot{z} = s(\mu)z + c_1(\mu)z^2\bar{z} + \cdots c_k(\mu)z^{k+1}\bar{z}^k + h(z, \bar{z}, \mu), \tag{19}$$

where

$$h(z, \bar{z}, \mu) = \sum_{r+s\geq 2(k+1)} h_{rs}(\mu)\frac{z^r\bar{z}^s}{r!s!}. \tag{20}$$

Then there exists a smooth change of variables

$$z = w + \psi(w, \bar{w}, \mu) \tag{21}$$

such that

$$\dot{w} = s(\mu)w + c_1(\mu)w^2\bar{w} + \cdots + c_k(\mu)w^{k+1}\bar{w}^k + o(|w|^{2(k+1)+1}), \tag{22}$$

where

$$\psi(w, \bar{w}, \mu) = \sum_{r+s=2(k+1)} \psi_{rs}(\mu)\frac{w^r\bar{w}^s}{r!s!}. \tag{23}$$

Proof. We assume that there is a change of coordinates satisfying (21) and (22). Using (19) and (21) we obtain

$$\begin{aligned}\dot{z} &= s(\mu)(w + \psi(w, \bar{w}, \mu)) + c_1(\mu)(w + \psi(w, \bar{w}, \mu))^2(\bar{w} + \bar{\psi}(w, \bar{w}, \mu)) + \cdots \\ &\quad + c_k(\mu)(w + \psi(w, \bar{w}, \mu))^{k+1}(\bar{w} + \bar{\psi}(w, \bar{w}, \mu))^k \\ &\quad + \sum_{r+s\geq 2(k+1)} h_{rs}(\mu)\frac{(w + \psi(w, \bar{w}, \mu))^r(\bar{w} + \bar{\psi}(w, \bar{w}, \mu))^s}{r!s!} \\ &= s(\mu)w + c_1(\mu)w^2\bar{w} + \cdots + c_k(\mu)w^{k+1}\bar{w}^k + \sum_{r+s=2(k+1)} (s(\mu)\psi_{rs}(\mu) + h_{rs}(\mu))\frac{w^r\bar{w}^s}{r!s!} + o(|w|^{2(k+1)+1}).\end{aligned} \tag{24}$$

On the other hand taking derivatives of (21) and using (22) yields

$$\begin{aligned}\dot{z} &= \dot{w} + \psi_w(\mu)(w, \bar{w})\dot{w} + \psi_{\bar{w}}(\mu)(w, \bar{w})\dot{\bar{w}} \\ &= s(\mu)w + c_1(\mu)w^2\bar{w} + \cdots + c_k(\mu)w^{k+1}\bar{w}^k + \sum_{r+s=2(k+1)} \psi_{rs}(\mu)(rs(\mu) + s\bar{s}(\mu))\frac{w^r\bar{w}^s}{r!s!} + o(|w|^{2(k+1)+1}).\end{aligned} \tag{25}$$

Comparing (24) with (25) gives following expression for $\psi_{rs}(\mu)$

$$\psi_{rs}(\mu) = \frac{h_{rs}(\mu)}{(r-1)s(\mu) + s\bar{s}(\mu)}. \tag{26}$$

Since

$$(r-1)s(\mu_0) + s\bar{s(\mu_0)} = \mathrm{i}\beta(\mu_0)(r-1-s) = \mathrm{i}\beta(\mu_0)(2(k+1) - (2s+1)) \neq 0 \tag{27}$$

the change of variables is smooth for $|\mu - \mu_0|$ small. Eq. (22) now follows from (26) and (23). $\square$

Lemma 3. *Let*

$$\dot{z} = s(\mu)z + c_1z^2\bar{z} + \cdots + c_kz^{k+1}\bar{z}^k + \tilde{h}(z, \bar{z}), \tag{28}$$

where

$$\tilde{h}(z, \bar{z}) = \sum_{r+s\geq 2(k+1)+1} \tilde{h}_{rs}\frac{z^r\bar{z}^s}{r!s!}. \tag{29}$$

Then there exists a smooth change of variables

$$z = w + \phi(w, \bar{w}) \tag{30}$$

such that

$$\dot{w} = s(\mu)w + c_1 w^2\bar{w} + \cdots + c_k w^{k+1}\bar{w}^k + c_{k+1}w^{k+2}\bar{w}^{k+1} + O(|w|^{2(k+2)}), \tag{31}$$

where

$$\phi(w, \bar{w}) = \sum_{r+s=2(k+1)+1} \phi_{rs}\frac{w^r\bar{w}^s}{r!s!}. \tag{32}$$

Proof. Using the same method as in the preceding lemma gives

$$\phi_{rs} = \frac{\tilde{h}_{rs}}{(r-1)s(\mu) + s\bar{s}(\mu)}. \tag{33}$$

The big different here is that the denominator is zero for $r = k+2$ and $s = k+1$. In order to get a smooth change of variables we put $\phi_{k+2,k+1} = 0$. This results in

$$c_{k+1} = \frac{\tilde{h}_{k+2,k+1}}{(k+2)!(k+1)!} \tag{34}$$

proving the lemma. □

Note that by using Lemmas 2 and 3 alternately all Lyapunov coefficients can be calculated via Eq. (34).

When determining the number of limit cycles of system (16) in a small neighbourhood of the origin the following lemma is useful since it shows that the number remains the same if the higher order terms are dropped.

Lemma 4. *The system*

$$\dot{w} = s(\mu)w + c_1(\mu)w^2\bar{w} + \cdots + c_k(\mu)w^{k+1}w^k + O(|w|^{2(k+1)}) \tag{35}$$

has the same number of limit cycles as

$$\dot{w} = s(\mu)w + c_1(\mu)w^2\bar{w} + \cdots + c_k(\mu)w^{k+1}w^k \tag{36}$$

in a small neighbourhood of the origin.

Proof. Introduce the notation $c_r(\alpha) = a_r(\alpha) + \mathrm{i}b_r(\alpha)$ and convert (35) into a system of differential equations in polar coordinates

$$\begin{cases} \dot{r} = \alpha r + a_1 r^3 + \cdots a_k r^{2k+1} + \Phi(r,\theta) \\ \dot{\theta} = \beta + b_1 r^2 + \cdots b_k r^{2k} + \Psi(r,\theta), \end{cases} \tag{37}$$

where $\Phi(r,\theta) = O(|r|^{2(k+1)})$ and $\Psi(r,\theta) = O(|r|^{2k+1})$. Eliminating the time dependence we get

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{\alpha r + a_1 r^3 + \cdots a_k r^{2k+1} + \Phi(r,\theta)}{\beta + b_1 r^2 + \cdots b_k r^{2k} + \Psi(r,\theta)} \\ &= \frac{\alpha r + a_1 r^3 + \cdots a_k r^{2k+1} + \Phi(r,\theta)}{\beta} \frac{1}{1 + \frac{r^2}{\beta}(b_1 + \cdots b_k r^{2(k-1)} + \frac{\Psi(r,\theta)}{r^2})}. \end{aligned} \tag{38}$$

Since we are interested in the behaviour in a small neighbourhood of the origin r is assumed to be very small. Furthermore, we have by assumption that $\beta \neq 0$. Thus we can use the formula for geometric series on the last term in (38).

$$\begin{aligned} &\frac{1}{1 + \frac{r^2}{\beta}(b_1 + \cdots b_k r^{2(k-1)} + \frac{\Psi(r,\theta)}{r^2})} \\ &\quad = 1 + \frac{r^2}{\beta} \sum_{\substack{n_1,\ldots,n_{k+1} \\ n_1+\cdots+n_{k+1}=2}} \binom{2}{n_1,\ldots,n_{k+1}} b_1^{n_1}(b_2 r^2)^{n_2}\cdots(b_k r^{2(k-1)})^{n_k}\left(\frac{\beta\Psi(r,\theta)}{r^2}\right)^{n_{k+1}} \\ &\qquad + \cdots + \left(\frac{r^2}{\beta}\right)^k \sum_{\substack{n_1,\ldots,n_{k+1} \\ n_1+\cdots+n_{k+1}=k}} + \binom{k}{n_1,\ldots,n_{k+1}} b_1^{n_1}(b_2 r^2)^{n_2}\cdots(b_k r^{2(k-1)})^{n_k}\left(\frac{\beta\Psi(r,\theta)}{r^2}\right)^{n_{k+1}} \\ &\qquad + O(r^{2(k+1)}). \end{aligned} \tag{39}$$

Inserting the above expression in (38) gives

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{1}{\beta}\left(\alpha r + \gamma_1 r^3 + \cdots + \gamma_k r^{2k+1} + R(r,\theta)\right), \tag{40}$$

where the coefficients γ_s neither depend on Ψ nor Φ and $R(r,\theta) = O(r^{2(k+1)})$. Let $r(\theta; r_0)$ be the solution to (40) with initial value r_0. Since the origin is assumed to be a fixed point $r(\theta; 0) = 0$. Thus the Taylor expansion of $r(\theta; r_0)$ around the origin is given by

$$r(\theta, r_0) = u_1(\theta)r_0 + \cdots + u_{2k+1}(\theta)r_0^{2k+1} + O(|r_0|^{2(k+1)}). \tag{41}$$

Differentiating (41) and using (40) shows that the differential equations for $u_s(\theta)$ is independent of the term $R(r,\theta)$. The number of periodic orbits can now be determined as the number of positive real fixed points, r_0, of $r(2\pi, r_0)$. Since the functions $u_s(\theta)$ are independent of $R(r,\theta)$ the number of small positive real fixed points will be the same when the term $O(|r_0|^{2(k+1)})$ is dropped in Eq. (35). □

Remark 4. From (37) we see that the trajectories will move in a counter-clockwise direction since $\beta(\mu) > 0$ for all μ close to μ_0 by assumption and r is assumed to be small.

The following corollary, saying that the behaviour of the system is determined by the the functions $c_r(\mu)$ at $\mu = \mu_0$ and further that it is only necessary to compute the first nonzero coefficient, is useful in applications.

Corollary 1. *In order to study the bifurcations of a general system*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(x, y, \mu) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, \mu) \end{cases} \tag{42}$$

in a neighbourhood of the bifurcation value $\mu = \mu_0$ it is sufficient to calculate the first nonzero coefficient $c_k(\mu_0)$.

Proof. Since all changes of variables transforming (11) into

$$\begin{cases} \dot{r} = \alpha(\mu)r + a_1(\mu)r^3 + \cdots a_k(\mu)r^{2k+1} + \Phi(r,\theta,\mu) \\ \dot{\theta} = \beta(\mu) + b_1(\mu)r^2 + \cdots b_k(\mu)r^{2k} + \Psi(r,\theta,\mu) \end{cases} \tag{43}$$

are smooth the functions $c_r(\mu) = a_r(\mu) + \mathrm{i}b_r(\mu)$ are also smooth. Thus for μ close to μ_0 the leading terms in the expressions for $\dot{r}$ and $\dot{\theta}$ are given by $a_k(\mu_0)$ and $b_k(\mu_0)$ respectively where $c_k(\mu)$ is the first non-vanishing coefficient at $\mu = \mu_0$.

Definition 3. Let $c_m(\mu_0) = a_m(\mu_0) + \mathrm{i}b_m(\mu_0)$. Then the real number $a_m(\mu_0)$ is called the m-th Lyapunov coefficient often denoted by L_m.

Note that this definition is not the exact notion by Andronov em et al. [19]. Nevertheless, we can show that they are the same modulo a positive multiplicative constant by straightforward yet tedious computations. Thus the sign of the Lyapunov coefficients are preserved, using different form of definitions.

Definition 4. The origin is said to be a focus of multiplicity m of system (16) if L_m is the first non-vanishing Lyapunov coefficient.

Let us now turn to the question about how many limit cycles that can bifurcate from a focus of a certain multiplicity. This question is answered by the following two theorems that can be found in [19].

Theorem 5. *If $O(0,0)$ is a multiple focus of multiplicity k ($k \geq 1$) of a dynamic system ($\mathcal{A}$) of class $N \geq 2 + 1$ or of analytical class, then*

1. *there exist $\varepsilon_0 > 0$ and $\delta_0 >$ such that any system $(\tilde{\mathcal{A}})\delta_0$-close to rank $2k+1$ to system ($\mathcal{A}$) has at most k closed paths in $U_{\varepsilon_0}(O)$;*
2. *for any $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$, there exists a system ($\tilde{\mathcal{A}}$) of class N or (respectively) of analytical class which is δ-close to rank $2k+1$ to ($\mathcal{A}$) and has k closed paths in $U_\varepsilon(O)$.*

Theorem 6. *Let $O(0,0)$ be multiple focus of multiplicity k of a dynamic system ($\mathcal{A}$) of class $N \geq 2k+1$ or of analytical class, and let ε_0 and δ_0 be positive numbers defined by the first part of Theorem 5 and such that any system $(\tilde{\mathcal{A}})\delta_0$-close to ($\mathcal{A}$) has a single equilibrium state in $U_{\varepsilon_0}(O)$ which is a focus. Then*

1. *for any ε and δ, $0 < \varepsilon \leq \varepsilon_0$, $0 < \delta \leq \delta_0$, and for any s, $1 \leq s \leq k$, there exists a system ($\mathcal{B}$) of class N (or respectively, analytical) which is δ-close to rank $2k+1$ to system ($\mathcal{A}$) and has in $U_\varepsilon(O)$ precisely s closed paths.*

Table 1
Hopf bifurcation table for system (11).

	$\mu < \mu_0$	$\mu = \mu_0$	$\mu > \mu_0$
$L_1 > 0, \sigma'(\mu_0) > 0$	Origin stable Unstable L.C	Origin unstable No L.C.	Origin unstable No L.C.
$L_1 > 0, \sigma'(\mu_0) < 0$	Origin unstable No L.C.	Origin unstable No L.C.	Origin stable Unstable L.C.
$L_1 < 0, \sigma'(\mu_0) > 0$	Origin stable No L.C.	Origin stable No L.C.	Origin unstable Stable L.C.
$L_1 < 0, \sigma'(\mu_0) < 0$	Origin unstable Stable L.C.	Origin stable No L.C.	Origin stable No L.C.

2. *if system $(\mathcal{B})$ is δ_0-close to rank $2k+1$ to system $(\mathcal{A})$ and has k limit cycles in U_{ε_0}, all these cycles, and likewise the focus of system $(\mathcal{B})$ lying in U_{ε_0}, are structurally stable (simple) i.e. they are either stable or unstable.*

Applying the above theorems to (16) gives the bifurcation table, see Table 1.

We have seen that the Lyapunov coefficients play a crucial role for the behaviour of solutions of a system with a pair of purely imaginary eigenvalues. Explicit formulae are given in [22] and are shown in Appendix, due to access limitation of the reference.

Remark 5. The exact expressions of the Lyapunov coefficients can be different by different transformations of coordinates. However the signs are invariant.

2.3. Boundedness of solutions

In this section we are going to present necessary conditions under which our system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -Cy - Ax(x-B)(x-\lambda) + I \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \varepsilon(x - \delta y) \end{cases} \tag{44}$$

has bounded solutions. This analysis depends on the construction of a Lyapunov function. We will restrict to the case $I = 0$. By making the change of variables

$$\begin{cases} u = y \\ v = \varepsilon(x - \delta y), \end{cases} \tag{45}$$

the system can be written in the Liénard form

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = v \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -vf(u,v) - g(u), \end{cases} \tag{46}$$

where

$$\begin{cases} f(u,v) = \varepsilon A \left(\dfrac{v^2}{\varepsilon^3} + \dfrac{3\delta u - (B+\lambda)}{\varepsilon^2} \right) v + \dfrac{3\delta^2}{\varepsilon} u^2 - \dfrac{2\delta(B+\lambda)}{\varepsilon} u + \dfrac{\varepsilon\delta + AB\lambda}{A\varepsilon} \\ g(u) = -A\varepsilon u(\delta^3)u^2 - \delta^2(B+\lambda)u + \dfrac{C + AB\delta\lambda}{A}. \end{cases} \tag{47}$$

Define

$$V(u,v) = \frac{v^2}{2} + G(u), \tag{48}$$

where

$$G(u) = \int_0^u g(s)\mathrm{d}s. \tag{49}$$

Then, with some restrictions on the parameters, V is a Lyapunov function on the whole $\mathbb{R}^2$ except for a bounded set by the following lemma.

Lemma 7. *Let*

(i) $A > 0$
(ii) $\varepsilon > 0$
(iii) $\delta > 0$

Then $V(u, v) = \frac{v^2}{2} + G(u)$ *is a Lyapunov function on* $\mathbb{R}^2$ *except for a bounded set.*

Proof. Using the given assumptions, the proof of Corollary 1 in [23] goes through.

Corollary 2. *The solutions to system (44) are bounded if* $A > 0, \varepsilon > 0, \delta > 0$.

3. Analysis of a class of FitzHugh–Nagumo systems with respect to singular perturbation theory

In this section we are going to find conditions on the parameters such that Theorem 1 is applicable. In order to transform our system to the canonical form in the previous section we make the time scaling $\tau = \varepsilon t$ and assume that $0 < \varepsilon \ll 1$. This leads to

$$\begin{cases} \varepsilon \dfrac{dx}{d\tau} = -Cy + Ax(B - x)(x - \lambda) := f(x, y) \\ \dfrac{dy}{d\tau} = x - \delta y := g(x, y). \end{cases} \tag{50}$$

In the sequel the symbol τ will be replaced by t. The verification of the first two conditions of Theorem 1 is easy since

(i) The functions f and g are polynomials and therefore have continuous second derivatives.
(ii)

$$f_x^2(x, y) + f_y^2(x, y) = A^2(x^2 + 4(\lambda + B)^2) > 0. \tag{51}$$

Thus in order to apply Theorem 1 it remains to find conditions on the parameters such that the remaining conditions are satisfied. For this particular system the sets S, K and L are given by

$$\begin{cases} S = \left\{(x, y) \in \mathbb{R}^2 | y = -\dfrac{A}{C}x(x - B)(x - \lambda) := F(x)\right\} \\ K = \{(x_1, F(x_1)), (x_2, F(x_2))\} \\ L = (L_{A^-} \cup L_{A^+}), \end{cases} \tag{52}$$

where x_i are the real roots of $F'(x) = 0$ for $i = 1, 2$ and

$$L_{A^-} = \{(x, y) \in \mathbb{R}^2 | y = F(x), x_1 < x < x_2, A < 0\}$$

and

$$L_{A^+} = \{(x, y) \in \mathbb{R}^2 | y = F(x), x < x_1 \vee x > x_2, A > 0\}.$$

The approximation (5) of system (50) can be written as

$$\frac{dx}{dt} = \frac{x - \delta F(x)}{F'(x)} \tag{53}$$

for $(x, y) \in L$ and where $F(x)$ is defined in the expression of S.

The following lemma gives necessary conditions for system (50) to have a DPS.

Lemma 8. *If system (50) has a DPS then*

(i) $(B - \lambda)^2 + B\lambda > 0$
(ii) $A > 0$
(iii) $\frac{dx}{dt} > 0, \; x < x_1$
(iv) $\frac{dx}{dt} < 0, \; x > x_2$

Proof. Let Γ_0 be a DPS of (50). Since the minimum number of sections a DPS can be composed of is four, two slow and two fast, we conclude that system (50) must have at least two breakpoints, i.e. the cardinality of K, $|K| \geq 2$. But from the definition of K we know that $|K| \leq 2$ and thus we have shown that $|K| = 2$. This however is equivalent to the first condition. From the fact that $|K| = 2$ we know that Γ_0 is composed of exactly two sections of slow flow and two sections of fast flow. From the definition of DPS we know that these are disjoint sets and that the sections of slow flow are subsets of L. If $A < 0$ then L consists of only one connected component. This contradicts that the sections of slow flow are disjoint and separated by at least one breakpoint. Thus by assumption that all parameters are nonzero we deduce that $A > 0$. Since $F(x_1) < F(x_2)$ for $A > 0$ and Γ_0 is the DPS the remaining two statements hold true. □

That the above conditions are not sufficient can be seen from the fact that there might be a fixed point of (50) on L making a DPS impossible. A more exact answer is given by the following lemma.

Lemma 9. *Let α_i be the roots of $x - \delta F(x)$, x_i be roots of $F'(x)$ and let p_j satisfy $F(x_i) = F(p_i)$. Then system (50) has a DPS iff*

(i) $\alpha_i \notin [p_2, x_1]$
(ii) $\alpha_i \notin [x_2, p_1]$
(iii) $(B-\lambda)^2 + B\lambda > 0$
(iv) $A > 0$
(v) $\frac{dx}{dt} > 0, x < x_1$
(vi) $\frac{dx}{dt} < 0, x > x_2$

Proof. By Lemma 8 conditions (iii)–(vi) are necessary for existence of a DPS. With the extra conditions (i)–(ii) a DPS is given by

$$\Gamma_0 = \bigcup_{k=1}^{2} (SF_k \cup FF_k)$$

where

$$SF_1 = \{(x, y) \in \mathbb{R}^2 | y = F(x), p_2 \leq x < x_1\} \cap L$$
$$FF_1 = \{(x, y) \in \mathbb{R}^2 | x_1 \leq x < x_2, y = F(x_1)\}$$
$$SF_2 = \{(x, y) \in \mathbb{R}^2 | y = F(x), x_2 < x \leq p_1\} \cap L$$
$$FF_2 = \{(x, y) \in \mathbb{R}^2 | x_1 < x \leq p_2, y = F(x_2)\}.$$

The SF_i are the sections of slow flow and the SF_i are the sections of fast flow for $i = 1, 2$. □

3.1. A unique fixed point

When the origin is a unique fixed point of (50), i.e

$$(B-\lambda)^2 - \frac{4C}{A\delta} < 0 \tag{54}$$

the calculations are quite straightforward. From Lemma 9 we know that the sign of $\dot{x}$ is crucial for the existence of DPS and from (53)

$$\operatorname{sign}\left(\frac{dx}{dt}\right) = \operatorname{sign}(x - \delta F(x))\operatorname{sign}(F'(x)). \tag{55}$$

An analysis of the separate parts of the above expression shows that

$$\operatorname{sign}(x - \delta F(x)) = \begin{cases} -\operatorname{sign}\left(\dfrac{\delta A}{C}\right) & \text{for } x < 0 \\ \operatorname{sign}\left(\dfrac{\delta A}{C}\right) & \text{for } x > 0, \end{cases} \tag{56}$$

and

$$\operatorname{sign}(F'(x)) = \begin{cases} -\operatorname{sign}\left(\dfrac{A}{C}\right) & \text{for } x < x_1, \text{ or } x > x_2 \\ \operatorname{sign}\left(\dfrac{A}{C}\right) & \text{for } x_1 < x < x_2. \end{cases} \tag{57}$$

Combining (57) and (56)we end up with the following expression for $\operatorname{sign}(\frac{dx}{dt})$:

$$\operatorname{sign}\left(\frac{dx}{dt}\right) = \begin{cases} \operatorname{sign}(\delta) & \text{for } p_2 < x < x_1 \\ -\operatorname{sign}(\delta) & \text{for } x_2 < x < p_1. \end{cases} \tag{58}$$

By application of Lemma 9 we have thus shown the following Theorem:

Theorem 10. *Let the origin be a unique fixed point of system (50). Then it has a DPS iff the following conditions are fulfilled*

(i) *Condition* (iii)–(iv) *of Lemma* 9.
(ii) $x_1 < 0 < x_2$
(iii) $\delta > 0$.

3.2. Three fixed points

Now we shall determine conditions under which (50) has DPS while at the same time it has three distinct fixed points. This assumption requires an extended analysis of $\operatorname{sign}(x_0 - \delta F(x_0))$. Let $I_k = (a_k, b_k)$ where

$$a_k = \begin{cases} -\infty & k = 1 \\ \alpha_{k-1} & k = 2, 3, 4 \end{cases} \tag{59}$$

and

$$b_k = \begin{cases} \alpha_k, & k = 1, 2, 3 \\ \infty, & k = 4. \end{cases} \tag{60}$$

Then

$$\operatorname{sign}(x - \delta F(x)) = (-1)^k \operatorname{sign}\left(\frac{\delta A}{C}\right), \quad x \in I_k. \tag{61}$$

Lemma 11. *Define p_1 and p_2 by $f(x_i) = f(p_i)$ for $i = 1, 2$ respectively. Assume further that all elements in $\mathcal{S} = \{x_1, x_2, \alpha_1, \alpha_2, \alpha_3\}$ are different where α_i are the zeros of $x - \delta F(x)$. A necessary condition for (50) to have a DPS is that the elements of $\mathcal{S}$ divide the interval $[p_2, p_1]$ into an even number of subintervals.*

Proof. First note that $\dot{x}$ only changes sign at points of $\mathcal{S}$. Since they are different by assumption, corresponding to each of these elements there is a change in sign for exactly one of $(x - \delta F(x))$ and $F'(x)$. By Lemma 9 it is necessary that $\dot{x}$ have opposite sign in the intervals $[p_2, x_1]$ and $[x_2, p_1]$. Thus $\dot{x}$ must shift sign an odd number of times. But this is equivalent to $\mathcal{S}$ dividing $[p_2, p_1]$ into an even number of subintervals. □

Using Lemma 11 yields four possibilities for the existence of a DPS emerging from the cases of three and five elements of $\mathcal{S}$ lying in the interval $[p_2, p_1]$. These are

C_1: $x_1 < \alpha_1 < x_2$ and $\alpha_2, \alpha_3 > p_1$;
C_2: $x_1 < \alpha_2 < x_2$ and $\alpha_1 < p_2,\ \alpha_3 > p_1$;
C_3: $x_1 < \alpha_3 < x_2$ and $\alpha_1, \alpha_2 < p_2$;
C_4: $x_1 < \alpha_1 < \alpha_2 < \alpha_3 < x_2$.

In these cases $\operatorname{sign}(\dot{x})$ can be described as follows

$$\operatorname{sign}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \begin{cases} (-1)^{\mathrm{i}(C_k)-1}\operatorname{sign}(\delta) & \text{for } x \in [p_2, x_1] \\ (-1)^{\mathrm{i}(C_k)}\operatorname{sign}(\delta) & \text{for } x \in [x_2, p_1] \end{cases} \tag{62}$$

where $\mathrm{i}(C_k) := k$. Thus we see that δ has to be greater than zero in the cases with an even number and less than zero for the odd numbered cases in order for a DPS to exist. Therefore we have shown the following theorem.

Theorem 12. *Assume that system (50) has three fixed points. Then it has a DPS iff the following conditions are fulfilled*

(i) *Condition* (iii)–(iv) *of Lemma 9.*
(ii) *One of the cases C_k holds*
(iii) *$\delta < 0$ for k even and $\delta > 0$ for k odd*

3.3. Unstable periodic solutions

Letting $\tau = -t$ in (50) leads to

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = Cy - Ax(B - x)(x - \lambda) := \tilde{f}(x, y) = -f(x, y) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + \delta y := \tilde{g}(x, y) = -g(x, y). \end{cases} \tag{63}$$

Proceeding as earlier we get the approximation (5) of system (63) to be

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -\frac{x - \delta F(x)}{F'(x)} = -\frac{\mathrm{d}x}{\mathrm{d}t}, \tag{64}$$

for $(x, y) \in L$. In this case

$$\frac{\partial \tilde{f}}{\partial x} = -\frac{\partial f}{\partial x} \Rightarrow \operatorname{sign}\left(\frac{\partial \tilde{f}}{\partial x}\right) = \begin{cases} \operatorname{sign}(A), & x < x_1, x > x_2 \\ -\operatorname{sign}(A), & x_1 < x < x_2. \end{cases} \tag{65}$$

Using that

$$\operatorname{sign}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right) = -\operatorname{sign}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)$$

Eq. (58) yields that

$$\operatorname{sign}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right) = \begin{cases} -\operatorname{sign}(\delta) & \text{for } p_2 < x < x_1 \\ \operatorname{sign}(\delta) & \text{for } x_2 < x < p_1 \end{cases} \tag{66}$$

if the origin is the unique fixed point. Furthermore, applying (62) to this system gives that

$$\operatorname{sign}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right) = \begin{cases} (-1)^{\mathrm{i}(c_k)}\operatorname{sign}(\delta), & x \in [p_2, x_1] \\ (-1)^{\mathrm{i}(c_k)+1}\operatorname{sign}(\delta), & x \in [x_2, p_1], \end{cases} \tag{67}$$

if there are three fixed points. In analogy with Theorems 10 and 12 the following conclusions hold true:

Theorem 13. *Let the origin be a unique fixed point of system (63). Then it has a DPS iff the following conditions hold true*

(i) $(B-\lambda)^2 + B\lambda > 0$
(ii) $A < 0$
(iii) $\delta < 0$

Theorem 14. *Assume that system (63) has three fixed points. Then it has a DPS iff the following conditions hold true*

(i) $(B-\lambda)^2 + B\lambda > 0$
(ii) $A < 0$
(iii) *One of the cases C_k holds*
(iv) $\delta > 0$ *for k even and* $\delta < 0$ *for k odd*

Remark 6. Since A has to be greater than zero for positive time and less than zero for negative time, stable and unstable periodic solutions to (50) emerging from DPS cannot coexist.

3.4. Analysis for large values of ε

What about the case $\varepsilon \gg 1$? A scaling of time, $t = \varepsilon\tau$, transforms (50) to

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = -Cy + Ax(B-x)(x-\lambda) := f(x, y) \\ \theta\dfrac{\mathrm{d}y}{\mathrm{d}t} = x - \delta y := g(x, y) \end{cases} \tag{68}$$

where $\theta = \frac{1}{\varepsilon} \ll 1$. For this system the sets S, L and K are given by

$$\begin{cases} S = \left\{(x, y) \in \mathbb{R}^2 | y = \dfrac{x}{\delta} := \tilde{F}(x)\right\} \\ K = \emptyset \\ L = \{(x, y) \in S | \delta > 0\}. \end{cases} \tag{69}$$

Remark 7. Since K is empty there cannot exist any DPS.

3.5. A unique limit cycle

After the investigation of occurrences of DPS for the system, if we put the pieces together we arrive at the following theorems for one and three fixed points respectively.

Theorem 15. *Let the assumptions of Theorem 10 hold. Then for ε sufficiently small, system (50) has a family of limit cycles L_ε. Furthermore, these limit cycles are unique for every such ε and*

$$\lim_{\varepsilon \to 0} L_\varepsilon = L_0 \tag{70}$$

where L_0 is the DPS of the system.

Theorem 16. *Assume that system (50) has three fixed points and that*

(i) *Condition* (iii)–(iv) *of Lemma 9.*

(ii) $x_1 < \alpha_1 < \alpha_2 < \alpha_3 < x_2$
(iii) $\delta < 0$

Then for ε sufficiently small, system (50) has a family of limit cycles L_ε. Furthermore these limit cycles are unique for every such ε and

$$\lim_{\varepsilon \to 0} L_\varepsilon = L_0 \tag{71}$$

where L_0 is the DPS of the system.

Both theorems are proved by verification of the conditions in Theorem 1. Notice that the possible cases C_1, C_2 and C_3 for DPS do not satisfy the fourth condition in Theorem 1. Thus they have to be pulled out. Finally, observe also that the conditions on A and δ in the above theorems are very natural since they ensure boundedness of the solutions by Corollary 2.

4. Hopf and Bautin bifurcation

In the previous section we deduced conditions ensuring a unique limit cycle. Although this is a very nice result it has the drawback that ε was assumed to be very small. Natural questions to ask are if there exist limit cycles not occurring as a result of singular perturbation and how many there are and their stability. A standard method frequently used to answer the first question is the Hopf Bifurcation Theorem. In order to find some answers to the remaining ones it is very useful to use the theory of Lyapunov coefficients introduced in Section 2. It should be remarked that all analysis in this section is focused on the behaviour in a neighbourhood of the origin.

4.1. $I = 0$

The class of FitzHugh–Nagumo systems that we are interested in are, as stated before, of the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -Cy - Ax(x-B)(x-\lambda) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \varepsilon(x - \delta y). \end{cases} \tag{72}$$

The system matrix for this system is given by

$$\mathcal{A} = \begin{pmatrix} -AB\lambda & -C \\ \varepsilon & -\delta\varepsilon \end{pmatrix}.$$

First of all one verifies that $\sigma = -AB\lambda - \varepsilon\delta$ and $\Delta = AB\delta\varepsilon\lambda + \varepsilon C$. Since we are interested in the case when (72) has a pair of purely imaginary eigenvalues, the following restriction on the parameters must hold

$$\begin{cases} \delta = \delta^* = -\dfrac{AB\lambda}{\varepsilon} \\ \Delta = -A^2B^2\lambda^2 + C\varepsilon > 0. \end{cases} \tag{73}$$

An application of Hopf Bifurcation Theorem shows that our system has a periodic solution for all sets of nonzero parameters $(A, B, C, \delta, \varepsilon, \lambda)$ satisfying (73) and $\varepsilon \neq 0$. For a more thorough investigation we use the theory of Lyapunov coefficients in Section 2. The first Lyapunov coefficient is given by

$$L_1(\delta*) = \frac{-A\pi}{4\Delta^{\frac{3}{2}}}[2A^2B\lambda^3 + A^2B^2\lambda^2 + 2A^2B^3\lambda + 3C\varepsilon] = \frac{-A\pi}{4\Delta^{\frac{3}{2}}} p_1(\lambda). \tag{74}$$

An easy computation shows that $\frac{\mathrm{d}}{\mathrm{d}\lambda}(p_1(\lambda))$ does not have real zeros since $-11A^4B^4 < 0$ for all non-zero parameters A, B, C, ε. Thus, for every ε, C, there exists a unique real zero $\bar{\lambda} = \bar{\lambda}(\varepsilon, C)$ to the equation $L_1 = 0$ for every ε and C. From (74) the sign of L_1 can be decided by

$$\mathrm{sign}(L_1) = -\mathrm{sign}(A)\mathrm{sign}(B)\mathrm{sign}(\lambda - \bar{\lambda}) \tag{75}$$

and thus there are eight possibilities:

I. $A > 0, B > 0$ and $\lambda < \bar{\lambda}$;
II. $A > 0, B < 0$ and $\lambda > \bar{\lambda}$;
III. $A < 0, B > 0$ and $\lambda > \bar{\lambda}$;
IV. $A < 0, B < 0$ and $\lambda < \bar{\lambda}$;
V. $A < 0, B < 0$ and $\lambda > \bar{\lambda}$;
VI. $A < 0, B > 0$ and $\lambda < \bar{\lambda}$;
VII. $A > 0, B < 0$ and $\lambda < \bar{\lambda}$;

Table 2
Hopf bifurcation table for system (72).

	$\delta < \frac{-AB\lambda}{\varepsilon}$	$\delta = \frac{-AB\lambda}{\varepsilon}$	$\delta > \frac{-AB\lambda}{\varepsilon}$
Cases I–IV, $\varepsilon < 0$	Origin stable Unstable L.C	Origin unstable No L.C.	Origin unstable No L.C.
Cases I–IV, $\varepsilon > 0$	Origin unstable No L.C.	Origin unstable No L.C.	Origin stable Unstable L.C.
Cases V–VIII, $\varepsilon < 0$	Origin stable No L.C.	Origin stable No L.C.	Origin unstable Stable L.C.
Cases V–VIII, $\varepsilon > 0$	Origin unstable Stable L.C.	Origin stable No L.C.	Origin stable No L.C.

VIII. $A > 0, B > 0$ and $\lambda > \bar{\lambda}$.

Application of Table 1 on our system gives the bifurcation table, Table 2. See Figs. 1–7. At the parameter value $\lambda = \bar{\lambda}$, the first Lyapunov coefficient can not be used to draw any conclusions. In this case the second coefficient, L_2, has to be calculated.

$$L_2 = -\frac{\pi A^3 B\bar{\lambda}}{24\Delta^{\frac{5}{2}}}[16A^2B\bar{\lambda}^3 + 23A^2B^2\bar{\lambda}^2 + 16A^2B^3\bar{\lambda} + 9C\varepsilon] = -\frac{\pi A^3 B\bar{\lambda}}{24\Delta^{\frac{5}{2}}}p_2(\bar{\lambda}). \tag{76}$$

Theorem 17. *If $A, B, C, \varepsilon, \lambda \neq 0$, $\delta = \frac{-AB\lambda}{\varepsilon}$ ($\sigma = 0$) and $-A^2B^2\lambda^2 + C\varepsilon > 0$ ($\Delta > 0$) the origin of system (72) is a multiple focus of order not larger than two.*

Proof. The conditions on σ and Δ are equivalent to saying that the origin is a multiple focus. From (74) and (76) and the assumption of nonzero parameters it follows that

$$\begin{cases} L_1 = 0 \\ L_2 = 0 \end{cases} \Leftrightarrow \begin{cases} p_1 = p_1(A, B, C, \varepsilon, \lambda) = A^2B\lambda(2\lambda^2 + B\lambda + 2B^2) + 3C\varepsilon = 0 \\ p_2 = p_2(A, B, C, \varepsilon, \lambda) = A^2B\lambda(16\lambda^2 + 23B\lambda + 16B^2) + 9C\varepsilon = 0. \end{cases} \tag{77}$$

From this we conclude that $L_1 = L_2 = 0 \Rightarrow \lambda = -B$ by elimination of the term $C\varepsilon$. Replacement of λ by $-B$ in the expression for p_1 gives that $A^2B^4 = C\varepsilon$. But this contradicts the assumption that $\Delta > 0$. Thus we conclude that L_1 and L_2 cannot be zero simultaneously and by definition the origin cannot be a multiple focus of order greater than two since $L_1 = 0 \Rightarrow L_2 \neq 0$. □

Corollary 3. *Let the assumption of Theorem 17 hold. If $L_1 = 0$ then $L_2 < 0$ for $A > 0$ and $L_2 > 0$ for $A < 0$.*

Proof. By the Routh-Hurwitz test we can show that the unique real solution $\bar{\lambda}$ of the polynomial equation $p_1(\lambda) = 0$ is negative if $B > 0$ and positive if $B < 0$, that is $\bar{\lambda}B < 0$. Thus the sign of L_2 is determined by the sign of $Ap_2(\lambda)$ by (76). A straightforward calculation shows that $p_2(\bar{\lambda}) = 8p_1(\bar{\lambda}) - 15\Delta = -15\Delta < 0$ since$p_1(\bar{\lambda}) = 0$. Hence $\text{sign}(L_2) = -\text{sign}(A)$.

Theorem 18. *Let the assumptions of Theorem 17 hold and assume that $L_1 = 0$. Then*

(i) *There exist parameters such that two limit cycles bifurcate from the origin and this is the maximal number of bifurcating limit cycles.*
(ii) *Assume the parameters are given such that there exists two limit cycles and also assume that $A > 0$. Then the origin is stable and the inner cycle is unstable while the outer is stable.*
(iii) *Assume the parameters are given such that there exists two limit cycles and also assume that $A < 0$. Then the origin is unstable and the inner cycle is stable while the outer is unstable.*

Remark 8. The existence of parameters fulfilling (ii) or (iii) above is ensured by (i).

Proof. (i) Apply Theorems 17 and 5, together with Theorem 8.2 in [19].
(ii) By Corollary 3 and (37) the origin is stable. Thus the inner cycle is unstable from the inside. Since the parameters are given such that two limit cycles exists the inner cycle is also unstable from the outside by Theorem 6 part 2. Application of Theorem 6 part 2 twice more shows that the outer cycle is stable.
(iii) Imitate the proof of (ii) with the origin being unstable. □

Remark 9. When $L_1 = 0$, the bifurcation is also called Bautin bifurcation or generalized Hopf bifurcation. The points in the parameter space where Bautin bifurcation takes place are said to be Bautin points.

4.2. *I* is a non-zero parameter

Now we turn to study the full system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -Cy - Ax(x-B)(x-\lambda) + I \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \varepsilon(x - \delta y). \end{cases} \tag{78}$$

The equilibrium points of (78) are given by the equations

$$\begin{cases} x = \delta y \\ y = -\dfrac{A}{C}x(x-B)(x-\lambda) + \dfrac{I}{C}, \end{cases} \tag{79}$$

which is equivalent to

$$k(y) := y^3 - \frac{\lambda + B}{\delta}y^2 + \frac{AB\lambda\delta + C}{A\delta^3}y = \frac{I}{A\delta^3}. \tag{80}$$

Restricting to the case of a unique fix point is equivalent to saying that $k'(y)$ does not have any real zeros, i.e. the parameters must satisfy the condition

$$\frac{A\delta(\lambda - B)^2 + AB\lambda\delta - 3C}{A\delta} < 0. \tag{81}$$

Now, let (x_0, y_0) denote the unique fixed point and make the change of variables

$$\begin{cases} \tilde{x} = x - x_0 \\ \tilde{y} = y - y_0. \end{cases} \tag{82}$$

In these coordinates the system takes the form

$$\begin{cases} \dot{\tilde{x}} = -C\tilde{y} - A\phi(\tilde{x}) \\ \dot{\tilde{y}} = \varepsilon(\tilde{x} - \delta\tilde{y}), \end{cases} \tag{83}$$

where

$$\phi(\tilde{x}) = -A(\tilde{x}^3) + \tilde{x}^2(3x_0 - (\lambda + B)) + \tilde{x}(3x_0^2 - 2x_0(B + \lambda) + B\lambda). \tag{84}$$

Depending on the value of x_0 two different cases appear. First assume that

$$x_0 \in (\alpha - \beta, \alpha + \beta), \tag{85}$$

where

$$\begin{cases} \alpha = \dfrac{\lambda + B}{3} \\ \beta = \dfrac{\sqrt{(\lambda - B)^2 + B\lambda}}{3}. \end{cases} \tag{86}$$

In this case there exist γ_1 and γ_2, the zeros of the polynomial ϕ, such that

$$\phi(\tilde{x}) = -A\tilde{x}(\tilde{x} - \gamma_1)(\tilde{x} - \gamma_2) \tag{87}$$

and which are given by

$$\begin{cases} \gamma_1 + \gamma_2 = -3x_0 + (\lambda + B) \\ \gamma_1\gamma_2 = 3x_0^2 - 2x_0(B + \lambda) + B\lambda. \end{cases} \tag{88}$$

Thus (78) can be written as

$$\begin{cases} \dot{\tilde{x}} = -C\tilde{y} - A\tilde{x}(\tilde{x} - \gamma_1)(\tilde{x} - \gamma_2) \\ \dot{\tilde{y}} = \varepsilon(\tilde{x} - \delta\tilde{y}) \end{cases} \tag{89}$$

and therefore the analysis of this system is exactly the same as for the case $I = 0$. The consequence of this observation is summarised as the following theorem.

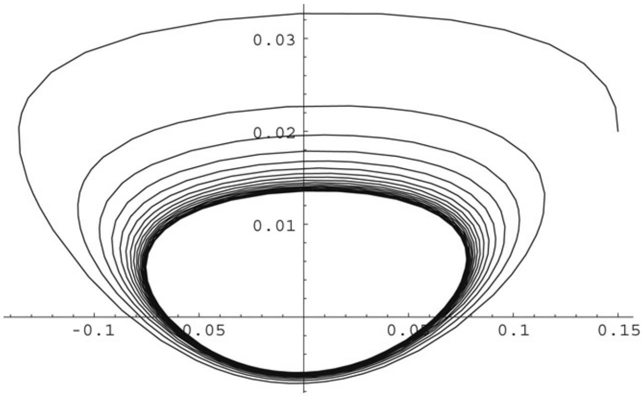

Fig. 1. The parameter values are; $A = B = C = 1$, $\delta = 0.5$, $\varepsilon = 0.015$ and $\lambda = -0.01$. Thus the origin is a unique unstable fixed point, $L_1 < 0$ and $\delta < \delta^*$. Furthermore, $x_0 = 0.15$ and $y_0 = 0.02$.

Theorem 19. *Let A, C, ε be non-zero. Assume that $3x_0^2 - 2x_0(B+\lambda) + B\lambda \neq 0$, $\delta = -\frac{A(3x_0^2-2x_0(B+\lambda)+B\lambda)}{\varepsilon}$, and $-A^2(3x_0^2 - 2x_0(B+\lambda) + B\lambda)^2 + C\varepsilon > 0$. Then, the unique fixed point (x_0, y_0) is a multiple focus of order at most two. Consequently, if the first Lyapunov coefficient is zero, then the second Lyapunov coefficient is less than zero for $A > 0$, and the second Lyapunov coefficient is greater than zero for $A < 0$.*

Furthermore, if the first Lyapunov coefficient is zero, then

(i) *There exist parameters such that two limit cycles bifurcate from the fixed point (x_0, y_0) and this is the maximal number of bifurcating limit cycles.*
(ii) *Assume the parameters are given such that there exist two limit cycles and also assume that $A > 0$. Then the fixed point (x_0, y_0) is stable and the inner cycle is unstable while the outer is stable.*
(iii) *Assume the parameters are given such that there exist two limit cycles and also assume that $A < 0$. Then the fixed point (x_0, y_0) is unstable and the inner cycle is stable while the outer is unstable.*

We leave the the other case for further investigation.

5. Conclusions and further discussions

The main result of this paper is that the FitzHugh–Nagumo system with I being zero has at most two limit cycles that can be bifurcated from origin, the unique fixed point, via Hopf and Boutin bifurcations. In connection to this we have also shown that if the first Lyapunov coefficient is zero then the second one is always less (greater) than zero if $A > 0$ ($A < 0$). A consequence of this is that the origin has to be stable (unstable) if two limit cycles are to bifurcate from it. Further, if there is such a bifurcation the outer (inner) limit cycle is stable while the inner one is unstable (unstable).

These results hold true for non-zero constant I under an assumption on the parameters such that the cubic polynomial, resulting from a translation of the variable changes, has three real zeros.

In the particular case of one special parameter being very small we provided sufficient conditions for the existence of a unique stable limit cycle for I being zero. It is clear that the case of non-zero I might possibly cause the so called Canards. This will be further investigated.

Acknowledgement

We would like to thank Dr H. Thunberg for his insightful comments and discussions.

Appendix

In this Appendix we shall give explicit expressions of the first and the second Lyapunov coefficients described in Section 2 and used in our analysis. We also provide some simulations of the system for eight sets of parameters to illustrate the results in the paper.

The original model

$$\dot{u} = c[w + u - u^3/3 + I],$$
$$\dot{w} = -(u - a + bw)/c$$

was proposed to demonstrate that the Hodgkin–Huxley model belongs to a more general class of systems that exhibit the properties of excitability and oscillations. The van der Pol oscillator was an example of this class to serve as a fundamental

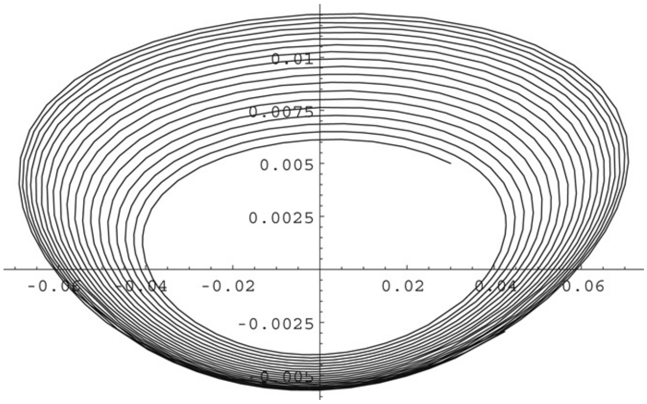

Fig. 2. The parameter values are; $A = B = C = 1$, $\delta = 0.5$, $\varepsilon = 0.015$ and $\lambda = -0.01$. Thus the origin is an unique unstable fixed point, $L_1 < 0$ and $\delta < \delta^*$. Furthermore, $x_0 = 0.03$ and $y_0 = 0.005$.

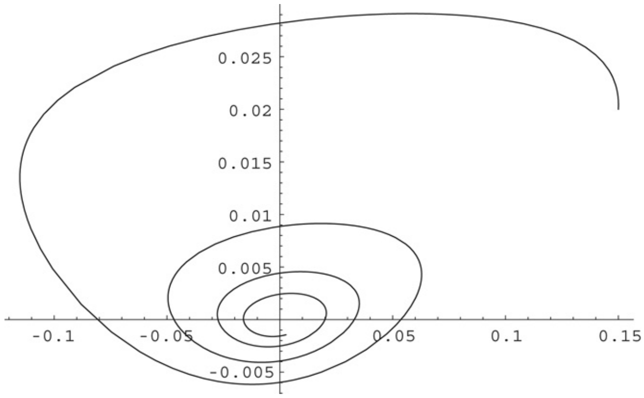

Fig. 3. The parameter values are; $A = B = C = 1$, $\delta = 2$, $\varepsilon = 0.015$ and $\lambda = -0.01$. Thus the origin is a unique stable fixed point, $L_1 < 0$ and $\delta > \delta^*$. Furthermore, $x_0 = 0.15$ and $y_0 = 0.02$.

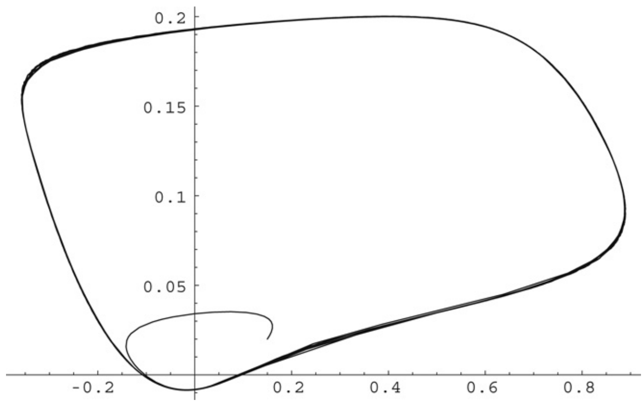

Fig. 4. The parameter values are; $A = B = C = 1$, $\delta = 2$, $\varepsilon = 0.015$ and $\lambda = -0.04$. Thus the origin is a unique unstable fixed point, $L_1 > 0$ and $\delta < \delta^*$. Furthermore, $x_0 = 0.15$ and $y_0 = 0.02$.

prototype. And the FitzHugh–Nagumo system is a suitable modification. In these equations the variable u represents the excitability of the system and could be identified with voltage (membrane potential in the axon); w is a recovery variable, representing combined forces that tend to return the state of the axonal membrane to rest. Finally I is the applied stimulus that leads to excitation (such as input current), or rectangular pulses. In order to obtain suitable behaviour, the following

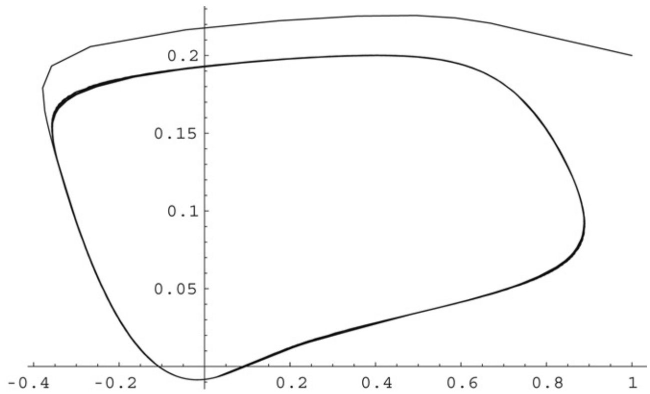

Fig. 5. The parameter values are; $A = B = C = 1$, $\delta = 2$, $\varepsilon = 0.015$ and $\lambda = -0.04$. Thus the origin is a unique unstable fixed point, $L_1 > 0$ and $\delta < \delta^*$. Furthermore, $x_0 = 1$ and $y_0 = 0.2$.

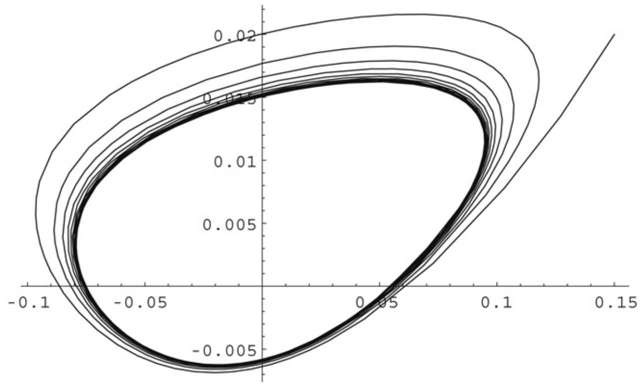

Fig. 6. The parameter values are; $A = B = C = 1$, $\delta = 3$, $\varepsilon = 0.015$ and $\lambda = -0.04$. Thus the origin is a unique stable fixed point, $L_1 > 0$ and $\delta > \delta^*$. Furthermore, $x_0 = 0.15$ and $y_0 = 0.02$ and the time is reversed.

assumptions were made about the constants a, b and c:

$$1 - 2b/3 < a < 1, \quad 0 < b < 1, \quad b < c^2.$$

Today there are variant formulations of this system, used in a variety of biological oscillations. The meaning of the variables and parameters are different from the original FitzHugh–Nagumo's. The formulation we work on is taken from [6] which is widely used in literature.

To obtain the second Lyapunov coefficients it is assumed that the system is in the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y + \tilde{f}(x, y, \mu_0) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + \tilde{g}(x, y, \mu_0). \end{cases}$$

If not, we use the transformation

$$\begin{cases} y_1 = x_1 \\ y_2 = -\dfrac{ax_1 + bx_2}{\sqrt{\Delta}} \end{cases}$$

$$L_1 = \frac{\pi}{4\sqrt{\Delta}}[3(a_{30} + b_{03} + a_{12} + b_{21})] - \frac{\pi}{4\Delta}[2(a_{20}b_{20} - a_{02}b_{02}) - a_{11}(a_{02} + a_{20}) + b_{11}(b_{02} + b_{20})]$$

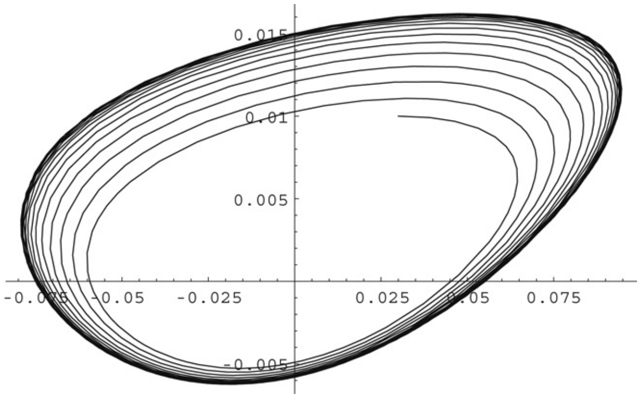

Fig. 7. The parameter values are; $A = B = C = 1$, $\delta = 3$, $\varepsilon = 0.015$ and $\lambda = -0.04$. Thus the origin is a unique stable fixed point, $L_1 > 0$ and $\delta > \delta^*$. Furthermore, $x_0 = 0.03$ and $y_0 = 0.01$ and the time is reversed.

$$
\begin{aligned}
L_2 \;=\; & -\frac{\pi}{24}[a_{02}b_{20}(5a_{02}b_{11} + 10a_{02}a_{20} + 4b_{11}^3 + 11a_{20}b_{11} + 6a_{20}^2 \\
& - 5a_{11}b_{20} - 10b_{20}b_{02} - 4a_{11}^2 - 11a_{11}b_{02} - 6b_{02}^2) + a_{20}b_{02}(6b_{02}^2 \\
& - 5a_{11}b_{02} + 10b_{02}b_{20} - 2a_{11}^2 - 5a_{11}b_{20} + 5a_{20}b_{11} - 6a_{20}^2 - 10a_{20}a_{02} \\
& + 2b_{11}^2 + 5a_{02}b_{11}) + a_{02}b_{02}(5b_{11}^2 - a_{11}^2 - 6a_{11}b_{02}) - a_{20}b_{20}(5a_{11}^2 - b_{11}^2 - 6a_{20}b_{11}) \\
& + a_{11}^3(a_{20} + a_{02}) - b_{11}^3(b_{02} + b_{20}) - 5b_{20}^2(a_{12} + 3b_{03}) \\
& + b_{02}^2(3b_{21} - 6a_{12} - 5a_{30}) + a_{11}^2(a_{12} + a_{30}) + b_{20}b_{02}(5b_{21} - 5a_{12} - 9b_{03} + 5a_{30}) \\
& - b_{20}a_{11}(4a_{12} + 9b_{03} + 5a_{30}) + b_{02}a_{11}(3b_{21} - a_{12}4a_{30}) \\
& - 5a_{02}^2(b_{21} + 3a_{30}) + a_{20}^2(3a_{12} - 6b_{21} - 5b_{03}) + b_{11}^2(b_{21} + b_{03}) \\
& + a_{20}a_{02}(5a_{12} - 5b_{21} - 9a_{30} + 5b_{03}) - a_{02}b_{11}(4b_{21} + 9a_{30} + 5b_{03}) + a_{20}b_{11}(3a_{12} - b_{21} + 4b_{03}) \\
& + 4b_{20}b_{11}(2b_{30} + b_{12}) + b_{02}b_{11}(7b_{20} - a_{21} + 5b_{12} + a_{03}) \\
& + 2a_{11}b_{11}(a_{03} + b_{30}) + 2a_{20}b_{20}(8b_{30} - 5a_{21} - b_{12}) + 2a_{20}b_{02}(4b_{30} \\
& - 5a_{21} - 5b_{12} + 4a_{03}) + a_{20}a_{11}(b_{30} + 5a_{21} - b_{12} + 7a_{03}) \\
& - 2a_{02}b_{20}(a_{21} + b_{12}) + 2a_{02}b_{02}(8a_{03} - 5b_{12} - a_{21}) + 4a_{02}a_{11}(2a_{03} + a_{21}) \\
& + b_{11}(5b_{04} - b_{22} + 2a_{13} - 3b_{40}) + a_{02}(2b_{22} + 20b_{04} + 5a_{13} + 3a_{31}) \\
& + a_{20}(4b_{22} + 22b_{04} + 7a_{13} - 6b_{40} + 9a_{31}) - b_{20}(2a_{22} + 20a_{40} + 5b_{31} + 3b_{13}) \\
& - a_{11}(5a_{40} - a_{22} + 2b_{31} - 3a_{04}) + +3a_{21}(2a_{30} + b_{03} + a_{12}) \\
& - 3b_{12}(2b_{03} + a_{30} + b_{21}) + 3a_{03}(a_{12} + 3b_{03}) - 3b_{30}(b_{21} + 3a_{30}) \\
& - b_{02}(4a_{22} + 22a_{40} + 7b_{31} - 6a_{04} + 9b_{13}) + 3b_{41} + 3b_{23} + 15b_{05} + 15a_{50} + 3a_{32} + 3a_{14}]
\end{aligned}
$$

where the coefficients a_{rs} and b_{rs} in L_1 are the Taylor coefficients of f and g and in L_2 are the Taylor coefficients of f and g after the above change of coordinates, and are given by

$$
\begin{cases}
a_{rs} = \displaystyle\sum_{k=s}^{r+s} \binom{k}{s} \frac{(-1)^k a^{(k-s)} \Delta^{\frac{s}{2}}}{(r+s-k)!k!b^k} f_{r+s-k,k} \\
b_{rs} = \displaystyle\sum_{k=s}^{r+s} \binom{k}{s} \frac{(-1)^{k+1} a^{(k-s)} \Delta^{\frac{s-1}{2}}}{(r+s-k)!k!b^k} (af_{r+s-k,k} + bg_{r+s-k,k}).
\end{cases}
$$

Remark A.1. Plot Figs. 4 and 5 reflect the local property of the Lyapunov coefficients and the conclusions presented in Table 1. The existence of the stable limit cycle indicated in the picture can be verified by Theorem 5 in our earlier paper [23].

Remark A.2. In order to indicate the existence of the claimed unstable limit cycle for the parameters as in plots Figs. 6 and 7 the time is reversed in these plots.

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